THE POSITIVITY PROBLEM FOR A SEQUENCE SATISFYING A LINEAR RECURRENCE RELATION WITH INTEGER COEFFICIENTS

A DISSERTATION

BY

PINTHIRA TANGSUPPHATHAWAT

PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DOCTOR OF PHILOSOPHY IN MATHEMATICS

AT SRINAKHARINWIROT UNIVERSITY

DECEMBER 2012

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AN ABSTRACT

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Consider a kth order linear recurrence relation of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} + \cdots + a_k u_{n-k} \quad (n \geq k),$$

where $a_1, a_2, a_3, \ldots, a_k (\neq 0)$ are integers. The recurrence relation thus defines a unique sequence of integers provided the initial integers $u_0, u_1, u_2, \ldots, u_{k-1}$ are given.

The classical *Positivity Problem* states: Is it possible to decide whether the sequence $(u_n)_{n\geq 0}$ is nonnegative ? Equivalently, is it decidable whether $u_n \geq 0$ for each $n \geq 0$?

The Positivity Problem for sequences satisfying a second order linear recurrence relation was shown to be decidable in 2006 by Halava-Harju-Hirvensalo. The Positivity Problem for sequences satisfying a third order linear recurrence relation was shown to be decidable in 2009. The main objective of this thesis is to prove that the same conclusion holds for each sequence satisfying a fourth order linear recurrence relation with integer coefficients. In order to prove this result, several auxiliary results, such as the classification of roots of an algebraic equation of fourth degree, the distribution of the values of certain cosine functions, and the behavior of a combination of cosine values are obtained.

ปัญหาค่าความเป็นบวกสำหรับลำดับที่สอดคล้องความสัมพันธ์เวียนเกิดเชิงเส้น
ที่มีสัมประสิทธิ์เป็นจำนวนเต็ม

บทคัดย่อ

ของ

ปิณฑิรา ตั้งศุภธวัช

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา
ตามหลักสูตรปริญญาปรัชญาดุษฎีบัณฑิต สาขาวิชาคณิตศาสตร์
ธันวาคม 2555

ปิณฑิรา ตั้งศุภธวัช. (2555). *ปัญหาค่าความเป็นบวกสำหรับลำดับที่สอดคล้องความสัมพันธ์เวียนเกิดเชิงเส้นที่มีสัมประสิทธิ์เป็นจำนวนเต็ม*. ปริญญานิพนธ์ ปร.ด. (คณิตศาสตร์). กรุงเทพฯ: บัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ. คณะกรรมการควบคุม: ศาสตราจารย์ ดร.ณรงค์ ปั้นนิ่ม, ศาสตราจารย์ ดร.วิเชียร เลาหโกศล, ศาสตราจารย์ ดร.Saad El-Zanati.

พิจารณาความสัมพันธ์เวียนเกิดเชิงเส้นอันดับ k ในรูป

$$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k} \qquad (n \geq k)$$

เมื่อ $a_1, a_2, \ldots, a_k\ (\neq 0)$ เป็นจำนวนเต็ม ความสัมพันธ์นี้ก่อให้เกิดลำดับของจำนวนจริง $(u_n)_{n=0}^{\infty}$ ได้เพียงลำดับเดียวเท่านั้น เมื่อ $u_1, u_2, \ldots, u_k$ เป็นจำนวนเต็มที่กำหนดให้

ปัญหาค่าความเป็นบวกกล่าวว่า "ลำดับ $(u_n)_{n=0}^{\infty}$ สามารถตัดสิน (decidable) ได้หรือไม่ว่าเป็นลำดับของจำนวนไม่ติดลบ" กล่าวคือ สามารถตัดสินได้หรือไม่ว่า $u_n \geq 0$ สำหรับทุก $n \geq 0$

ในปี ค.ศ. 2006 Vasa Halava, Tero Harju และ Mika Hirvensalo ได้ตอบปัญหาค่าความเป็นบวกของสมาชิกที่สอดคล้องความสัมพันธ์เวียนเกิดเชิงเส้นอันดับสอง นอกจากนี้ปัญหาค่าความเป็นบวกของสมาชิกที่สอดคล้องความสัมพันธ์เวียนเกิดเชิงเส้นอันดับสามได้ถูกตอบแล้วในปี 2009 วัตถุประสงค์หลักของวิทยานิพนธ์นี้คือ การแสดงว่าข้อสรุปดังกล่าวเป็นจริงสำหรับความสัมพันธ์เวียนเกิดเชิงเส้นอันดับ 4 ที่มีสัมประสิทธิ์เป็นจำนวนเต็ม ขั้นตอนการหาผลเฉลยดังกล่าว นำไปสู่การพิจารณาการจำแนกรากของสมการพีชคณิตอันดับ 4, การกระจายของฟังก์ชันโคไซน์ และพฤติกรรมผลรวมของค่าฟังก์ชันโคไซน์

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by

Pinthira Tangsupphathawat

has been approved by the Graduate School as partial fulfillment of the requirements for

the Doctor of Philosophy in Mathematics of Srinakharinwirot University.

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Lastly and most importantly, I dedicate this work to my parents, who were with me at the early stage of my life, and longed to see my academic accomplishment. I deeply miss them and it is rather unfortunate that they were not here to share with me this auspicious moment.

Pinthira Tangsupphathawat

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Pinthira Tangsupphathawat

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CHAPTER 1

INTRODUCTION

Consider a kth order linear recurrence relation of the form

$$u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3} + \cdots + a_ku_{n-k} \quad (n \geq k), \tag{1}$$

where $a_1, a_2, a_3, \ldots, a_k (\neq 0) \in \mathbb{Z}$. The recurrence (1) defines a unique sequence of integers provided the initial integers $u_0, u_1, u_2, \ldots, u_{k-1}$ are given.

The best-known example of a linear recurrence sequence was provided by Leonardo of Pisa in the 12th century: the so-called *Fibonacci sequence*

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, \ldots \rangle,$$

which satisfies the recurrence relation $u_{n+2} = u_{n+1} + u_n$ with initial conditions $u_0 = 0,\ u_1 = 1$. Leonardo of Pisa introduced this sequence as a means to model the growth of an idealised population of rabbits. Not only has the Fibonacci sequence been extensively studied since, but linear recurrence sequences now form a vast subject in their own right, with numerous applications in mathematics and other sciences. A deep and extensive treatise on the mathematical aspects of recurrence sequences is the recent monograph of Everest *et al.* [5].

In this dissertation, we are interested in the

Positivity Problem: Is it possible to decide whether the sequence $(u_n)_{n\geq 0}$ is nonnegative? Equivalently, is it decidable whether $u_n \geq 0$ for all $n \geq 0$?

The Positivity Problem is one of the three important closely related decision problems ([14]) which are yet to be settled, namely,

The Skolem Problem: does a given linear recurrence sequence have a zero?

The Positivity Problem: are all the terms of a given linear recurrence sequence positive? (Note that this problem comes in two natural flavors, according to whether strict or non-strict positivity is required.)

The Ultimate Positivity Problem: is the given linear recurrence sequence ultimately positive, i.e., are all the terms of the sequence positive except possibly for a finite number of exceptions ? (Note, likewise, that this problem admits two natural variants.)

The above three problems (and closely related variants) have applications in many different areas, such as theoretical biology (analysis of L-systems, population dynamics), software verification (termination of linear programs), probabilistic model checking (reachability in Markov chains, stochastic logics), quantum computing (threshold problems for quantum automata), as well as combinatorics, term rewriting, and the study of generating functions. For example, a particular term of a linear recurrence sequence usually has combinatorial significance only if it is non-negative. Likewise, a linear recurrence sequence modeling population growth is biologically meaningful only if it uniformly positive.

Related to the Skolem Problem is one of the most fundamental results about the zeros of linear recurrence sequences, the celebrated Skolem-Mahler-Lech Theorem:

Theorem 1.1 Let $(u_n)_{n=0}^{\infty}$ be a linear recurrence sequence over the reals. Its set of zeros $\{n : \ u_n = 0\}$ consists of a finite set F, together with a finite number of arithmetic progressions $A_1 \cup A_2 \cup \cdots \cup A_l$.

A perhaps surprising observation concerning the Positivity Problem is that its decidability would immediately entail the decidability of the Skolem Problem, since $u_n = 0$ if and only if $u_n^2 \leq 0$, and we know that the pointwise square of a linear recurrence sequence is again a linear recurrence sequence. In the worst case, however, this trick reduces an instance of the Skolem Problem for a linear recurrence sequence of order k to an instance of the Positivity Problem for a linear recurrence sequence of order $k^2 + 1$. An immediate consequence of the reduction is the NP-hardness of all versions of the Positivity Problem. Let us note, however, that no such reduction is known for the Ultimate Positivity Problem, and therefore that no non-trivial lower bounds are known for this problem.

In view of the above reduction, it might be natural to consider that the Positivity Problem has been open since the advent of the Skolem Problem. The earliest explicit reference that we have found in the literature, however, goes back to the 1970s: the problem is mentioned (in equivalent formulation) in papers of Salomaa [16] and Soittola [17]; in the latter, the author opined that " in our estimation, [Skolem and Positivity] will be very difficult problems."

Note that the variants and reductions which we discussed for the Skolem Problem apply almost verbatim to the Positivity and Ultimate Positivity Problems.

The Positivity Problem considered in this dissertation is to decide whether *all* elements of a sequence are nonnegative. In contrast, the Ultimate Positivity Problem, which asks whether the terms u_n are nonnegative for all sufficiently large n is not of interest here. Deciding the positivity of all sequence elements requires some further considerations to take care of the initial tail of the sequence.

As mentioned in [9], the Positivity Problem is a natural question and well-known in a number of situations, such as for matrices and in the Mortality Problem ([16], [17], [8], [15]). A closely related result, the Skolem-Mahler-Lech Theorem, provides information about when the terms of a linear recurrence sequence are zero ([2], [10], section 2.1.1 of [5]). Another related problem, known as the Orbit Problem, is, in an equivalent form, a question about simultaneous solvability of a system of equations in linear recurrence sequences (see section 14.2.3 of [5]).

The Positivity Problem for sequences satisfying a second order linear recurrence has already been shown to be decidable by Halava, Harju and Hirvensalo [9], in 2006; see also [1] and [3]. This means that there is an algorithmic procedure to decide whether elements in a sequence satisfying a second order linear recurrence with integer coefficients are nonnegative.

The Positivity Problem for sequences satisfying a third order linear recurrence was recently shown to be decidable in [18] by the proposer and her mentor. The underlying methodology there is based on the fact that the roots of a third degree algebraic equation with integer coefficients can be explicitly computed, and in one instance on a result in Diophantine approximation. Similar approximation results have been used in this context before e.g. in [3] and [6].

The main objective of this dissertation is to prove that the Positivity Problem for fourth order recurrence sequences is decidable. To attack this problem, we are guided by the analysis in the third order case. There the procedure involves the following steps, and in turn necessitates us to handle three particular subproblems.

Step 1: The starting step is to explicitly classify the roots of a third degree algebraic equation with integer coefficients. This leads us to

Subproblem I: To explicitly classify the roots of a fourth degree algebraic equation with integer coefficients.

Step 2: Having determined all possible shapes of the sequence elements explicitly through Step 1, the analysis is to determine signs of sequence and/or related elements. In some cases, one is confronted with signs of various cosine values, whose determination involves the use of Dirichlet's theorem. The corresponding subproblem for the fourth order cases is:

Subproblem II: To investigate the distribution of the values of $\cos(n\theta+\beta)$ as n varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, where $\theta,\beta\in[-\pi,\pi)$ and $\theta\notin\{-\pi,0\}$. This proof appears in [19].

Step 3: Through our trial and error pilot investigation of the fourth order case, another difficulty, not confronted in the lower order situations, arises from a combination of two cosine functions. This leads to

Subproblem III: To investigate the distribution of the values of $f(n\theta+\beta)$ as n varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, where $f:\mathbb{R}\to\mathbb{R}$ is a nonzero purely periodic function with least period P and $\theta(\neq 0)$ and β in the interval $[0,P)$. This proof appears in [4].

CHAPTER 2

LITERATURE REVIEWS

There is a number of preliminary definitions and results related to our work, most notable are the determination of the roots of polynomials of small degrees, the concept of recurrence sequences, the theorem of Dirichlet dealing with the uniform distribution of multiples of an irrational real number, the previous works of Halava-Harju-Hirvensalo in the second order case and of Laohakosol-Tangsupphathawat in the third order case. The remaining sections briefly describe these topics. In addition, a recent measure-theoretic work of Bell-Gerhold in 2006 enebles us to give a second and shorter proof of our main theorem through reducing the number of considered cases by roughly one half.

1 Roots of polynomial equations of degrees $2, 3$ and 4

General references for results in this section are [7, Chapter 5] and [23]. Let K be an algebraically closed field.

1.1 Quadratic equations

The discriminant of a quadratic polynomial

$$q(x) = ax^2 + bx + c \in K[x] \quad (a \neq 0),$$

is $D = D(q) = b^2 - 4ac$.

If $q(x)$ has roots r and s, then $D = a^2(r-s)^2$. For example, let $K = \mathbb{C}$ and $q \in \mathbb{R}[x]$. Then either $r, s \in \mathbb{R}$ and $D \geq 0$, or $r, s \in \mathbb{C} \setminus \mathbb{R}, s = \bar{r},\ r - s$ is a real multiple of $i := \sqrt{-1}$, and $D < 0$. In this case the sign of the discriminant indicates whether q are real.

When K does not have characteristic 2, the general quadratic equation $q(x) = 0 \ \ (a \neq 0)$ has solutions

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1.2 Equations of degree 3

If the field K does not have characteristic 3, a polynomial $f(x) = ax^3 + bx^2 + cx + d \in K[x]$ of degree 3 can be put in the form $a(x^3 + px + q)$ by substituting $x - b/3a$ for x, and we see that p and q are rational functions of a, b, c, d. The discriminant of $x^3 + px + q$ is

$$D = D(x^3 + px + q) = -4p^3 - 27q^2.$$

If the roots of $x^3 + px + q$ are r, s, t, then $D = (r - s)^2(r - t)^2(s - t)^2$. For example, let $K = \mathbb{C}$ and $p, q \in \mathbb{R}$. Then either

- $r, s, t \in \mathbb{R}$ and $D \geq 0$, or
- say, $t \in \mathbb{R}$ and $r, s \in \mathbb{C} \setminus \mathbb{R}$, $s = \bar{r}$ and $D < 0$.

Hence, the sign of the discriminant indicates whether all zeros of $f(x)$ are real.

The third degree equation $x^3 + px + q = 0$ can be solved by Cardano's method when K does not have characteristic 2 or 3. With $x = u + v$, the equation becomes

$$(u + v)^3 + p(u + v) + q = u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

If arrange so that $3uv = -p$, then u^3 and v^3 satisfy $u^3v^3 = -p^3/27$ and $u^3 + v^3 = -q$; hence $u' = u^3$ and $v' = v^3$ are the roots

$$u' = \frac{-q + \sqrt{q^2 + 4p^3/27}}{2}, \quad v' = \frac{-q - \sqrt{q^2 + 4p^3/27}}{2}$$

of the resolvant polynomial $x^2 + qx - p^3/27$. The roots x_1, x_2, x_3 of $x^3 + px + q$ are

$$x = u + v = \sqrt[3]{\frac{-q + \sqrt{q^2 + 4p^3/27}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 + 4p^3/27}}{2}}$$

in which u and v are cube roots of u' and v' such that $uv = -p/3$.

The roots r_1, r_2, r_3 of the original polynomial $ax^3 + bx^2 + cx + d$ are

$$r_i = x_i - b/3a \quad (i = 1, 2, 3).$$

1.3 Equations of degree 4

The roots r_1, r_2, r_3, r_4 of the fourth degree equation

$$0 = f(x) = ax^4 + bx^3 + cx^2 + dx + e \in K[x] \tag{2}$$

can be found by the following method when K does not have characteristic 2. Substituting $x - b/4a$ for x puts $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ in the form

$$g(x) = \frac{f(x - b/4a)}{a} = x^4 + px^2 + qx + r.$$

We see that p, q, r are rational functions of a, b, c, d, e.

Let x_1, x_2, x_3, x_4 be the roots of the equation

$$x^4 + px^2 + qx + r = 0, \tag{3}$$

then $x_i = r_i + b/4a \ \ (i = 1, 2, 3, 4)$ and $x_1 + x_2 + x_3 + x_4 = 0$. Let

$$\begin{aligned} u &= (x_1 + x_2)(x_3 + x_4) = -(x_1 + x_2)^2, \\ v &= (x_1 + x_3)(x_2 + x_4) = -(x_1 + x_3)^2, \\ w &= (x_1 + x_4)(x_2 + x_3) = -(x_1 + x_4)^2; \end{aligned}$$

equivalently,

$$\begin{aligned} u &= (r_1 + r_2 + b/2a)(r_3 + r_4 + b/2a) = -(r_1 + r_2 + b/2a)^2, \\ v &= (r_1 + r_3 + b/2a)(r_2 + r_4 + b/2a) = -(r_1 + r_3 + b/2a)^2, \\ w &= (r_1 + r_4 + b/2a)(r_2 + r_3 + b/2a) = -(r_1 + r_4 + b/2a)^2. \end{aligned}$$

To solve (2) and (3), we construct a polynomial of degree 3 whose roots are u, v, w. Its coefficients are found by computing $u + v + w, uv + uw + vw, uvw$ from the elementary symmetric functions

$$\sum_{i<j} x_i x_j = p, \quad \sum_{i<j<k} x_i x_j x_k = -q, x_1 x_2 x_3 x_4 = r.$$

First,

$$u + v + w = 2 \sum_{i<j} x_i x_j = 2p.$$

Next,

$$uv + 6uw + vw = \sum_{i<j} x_i^2 x_j^2 + 3 \sum_{j<k;j,k\neq i} x_i^2 x_j x_k + 6x_1x_2x_3x_4.$$

Since

$$0 = \left(\sum_i x_i\right)\left(\sum_{i<j<k} x_i x_j x_k\right) = \sum_{j<k;j,k\neq i} x_i^2 x_j x_k + 4x_1x_2x_3x_4$$

$$p^2 = \left(\sum_{i<j} x_i x_j\right)^2 = \sum_{i<j} x_i^2 x_j^2 + 2 \sum_{j<k;j,k\neq i} x_i^2 x_j x_k + 6x_1x_2x_3x_4,$$

we have

$$\sum_{j<k;j,k\neq i} x_i^2 x_j x_k = -4r, \quad uv + uw + vw = p^2 - 4r.$$

Finally,

$$uvw = \sum_{i,j,k} x_i^3 x_j^2 x_k + 2 \sum_{i;j<k<\ell} x_i^3 x_j x_k x_\ell + 2 \sum_{i<j<k} x_i^2 x_j^2 x_k^2 + 4 \sum_{i<j;k<\ell} x_i^2 x_j^2 x_k x_\ell,$$

where it is understood that i, j, k, ℓ are all distinct in the summation. Now

$$pr = \left(\sum_{i<j} x_i x_j\right) x_1x_2x_3x_4 = \sum_{i<j;k<\ell} x_i^2 x_j^2 x_k x_\ell$$

$$q^2 = \left(\sum_{i<j<k} x_i x_j x_k\right)^2 = \sum_{i;j<k<\ell} x_i^3 x_j x_k x_\ell + 2 \sum_{i<j;k<\ell} x_i^2 x_j^2 x_k x_\ell$$

$$0 = \left(\sum_i x_i\right)^2 x_1x_2x_3x_4 = \sum_{i;j<k<\ell} x_i^3 x_j x_k x_\ell + 2 \sum_{i<j;k<\ell} x_i^2 x_j^2 x_k x_\ell$$

$$\begin{aligned} 0 &= \left(\sum_i x_i\right)\left(\sum_{i<j} x_i x_j\right)\left(\sum_{i<j<k} x_i x_j x_k\right) \\ &= \sum_{i,j,k} x_i^3 x_j^2 x_k + 3 \sum_{i;j<k<\ell} x_i^3 x_j x_k x_\ell + 3 \sum_{i<j<k} x_i^2 x_j^2 x_k^2 + 8 \sum_{i<j;k<\ell} x_i^2 x_j^2 x_k x_\ell \end{aligned}$$

and it readily follows that

$$uvw = -q^2.$$

Hence, u, v, w are the solutions of the equation

$$x^3 - 2px^2 + (p^2 - 4r)x + q^2 = 0. \tag{4}$$

The polynomial $r(x) = x^3 - 2px^2 + (p^2 - 4r)x + q^2 \in K[x]$ whose roots are u, v, w is the resolvant polynomial of $f(x)$. Since

$$u - v = (x_1 - x_4)(x_3 - x_2)$$

$$u - w = (x_1 - x_3)(x_4 - x_2)$$

$$v - w = (x_1 - x_2)(x_4 - x_3),$$

$r(x)$ and $g(x)$ have the same discriminant. The discriminant of f is

$$D(f) = a^6 D(f/a) = a^6 D(g) = a^6 D(r).$$

To solve (3), first solve equation (4) to determine u, v, w provided K does not have characteristic 3. Then $x_1 + x_2 = u'$ is square root of $-u$, $x_1 + x_3 = v'$ is a square root of $-v$, and $x_1 + x_4 = w'$ is a square root of $-w$. The square root must be chosen so that

$$u'v'w' = (x_1 + x_+2)(x_1 + x_3)(x_1 + x_4) = x_1^2 \sum_i x_i + \sum_{i<j<k} x_i x_j x_k = -q.$$

Then

$$u' + v' + w' = 3x_1 + x_2 + x_3 + x_4 = 2x_1, \; u' - v' - w' = x_2 - x_1 - x_3 - x_4 = 2x_2$$

and similarly

$$-u' + v' - w' = 2x_3, \quad -u' - v' + w' = 2x_4.$$

Thus, the solutions of (3) are

$$x_1 = \frac{1}{2}(u' + v' + w')$$
$$x_2 = \frac{1}{2}(u' - v' - w')$$
$$x_3 = \frac{1}{2}(-u' + v' - w')$$
$$x_4 = \frac{1}{2}(-u' - v' + w'),$$

where $u'v'w' = -q$. The solutions of (2) are

$$r_1 = -\frac{b}{4a} + \frac{1}{2}(u' + v' + w')$$
$$r_2 = -\frac{b}{4a} + \frac{1}{2}(u' - v' - w')$$
$$r_3 = -\frac{b}{4a} + \frac{1}{2}(-u' + v' - w')$$
$$r_4 = -\frac{b}{4a} + \frac{1}{2}(-u' - v' + w').$$

2 Recurrence sequences

A recurrence relation is perhaps simplest to describ in the language of counting, [22, Chapter 4]. It is a formula that counts the number of ways to do a procedure involving n objects in terms of the number of ways to do it with fewer objects. That is, if u_k is the number of ways to do the procedure with k objects, for $k = 0, 1, 2, \ldots$, then a recurrence relation is an equation that expresses u_n as some function of preceding u_k's, $k < n$. A sequence $(u_n)_{n \geq 0}$ whose elements satisfy a recurrence relation is called a recurrence sequence. In this dissertation, we are interested in recurrence relations of the form

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_r u_{n-r} \qquad (n > r) \tag{5}$$

where $r \in \mathbb{N}$ is a fixed natural number and the c_i's are given complex numbers with $c_1 \neq 0$. We also need to be given the first r values $u_0, u_1, \ldots, u_{r-1}$ in order to compute the next u_n correctly. The information about the starting values needed to compute with a recurrence relation is called the initial conditions. Associated with the relation (5) is the so-called characteristic equation

$$x^r - c_1 x^{r-1} - \cdots - c_r = 0, \tag{6}$$

where the left-hand polynomial is referred to as the characteristic polynomial of the recurrence relation (5). We have, [13, Chapter 4], [5]:

Theorem 2.1 Let $c_1 (\neq 0), c_2, \ldots, c_r, u_0, u_1, \ldots, u_{r-1}$ be given complex numbers. If the characteristic polynomial

$$Q(z) = z^r - c_1 z^{r-1} - \cdots - c_r$$

has k distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively, where

$$m_1 + m_2 + \cdots + m_k = r,$$

then

$$u_n = \sum_{j=1}^{r} \left(a_{0,j} + a_{1,j} n + \cdots + a_{m_j - 1, j} n^{m_j - 1} \right) \alpha_j^n \qquad (n \geq 0),$$

where the a_{ij} are computed by solving the system of equations arising from substituting the r initial values.

3 Dirichlet's theorem

For an irrational real number ξ, our work leads us to deal with the distribution of the integral and fractional parts of multiples of ξ, and the following classical results are well-known, [12], [11].

Theorem 2.2 [12, Theorem 6.3] If ξ is an irrational real number, then the sequence $\{n\xi\}_{n\geq 1}$ is uniformly distributed modulo 1, in particular, this sequence is dense in the interval $[0,1]$.

Corollary 2.3 [12, Corollary 6.4] The real number ξ is irrational if and only if the points $(\xi),(2\xi),(3\xi),\ldots$ are everywhere dense in the unit interval $[0,1]$, where (x) denotes the fractional part of the real number x.

4 The work of Halava, Harju and Hirvensalo [9]

Consider a second order linear recurrence of the form

$$u_n = au_{n-1} + bu_{n-2} \quad (n \geq 3), \tag{7}$$

where $a, b(\neq 0)$ are given integers.

Lemma 2.4 Let $p(x) = x^2 - ax - b$ be the characteristic polynomial of (7) with discriminant $D = a^2 + 4b$.

I. If $D > 0$, then

$$u_n = A\lambda_1^n + B\lambda_2^n,$$

where $\lambda_1 \neq \lambda_2$ are the real roots of $p(x)$ and

$$A = \frac{u_1 - u_0\lambda_2}{\sqrt{D}}, \quad B = \frac{u_0\lambda_1 - u_1}{\sqrt{D}}.$$

II. If $D = 0$, then

$$u_n = (A + Bn)\lambda^n,$$

where $\lambda = a/2$ is the double root of $p(x)$ and

$$A = u_0, \quad B = \frac{2u_1 - u_0a}{a}.$$

III. If $D < 0$, then

$$u_n = A\lambda^n + \overline{A\lambda_2^n},$$

where λ and $\bar{\lambda}$ are the complex roots of $p(x)$, and A is as in Case I.

Lemma 2.5 If

$$u_n = A\lambda^n + \overline{A\lambda_2^n} \quad (n \geq 0)$$

and $\lambda \notin \mathbb{R}$, then $(u_n)_{n\geq 0}$ has negative elements.

Theorem 2.6 The Positivity Problem is decidable for second order recurrence sequences.

5 The work of Laohakosol and Tangsupphathawat [18]

Consider a third order linear recurrence of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} \quad (n \geq 3), \tag{8}$$

where $a_1, a_2, a_3 (\neq 0)$ are given integers. Recall that the characteristic polynomial associated with the recurrence (8) is

$$p(x) = x^3 - a_1 x^2 - a_2 x - a_3 \in \mathbb{Z}[x].$$

Following the derivation on pages 169-170 of [7], we start by reviewing some facts. Write

$$p\left(x + \frac{a_1}{3}\right) = x^3 + \alpha x + \beta, \quad \alpha := \frac{-a_1^2 - 3a_2}{3}, \;\; \beta := \frac{-2a_1^3 - 9a_2 a_1 - 27a_3}{27}.$$

Let x_1, x_2, and x_3 be all the three roots of $p\left(x + a_1/3\right)$. The discriminant of $p\left(x + a_1/3\right)$ is

$$D = -4\alpha^3 - 27\beta^2 = (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2.$$

Proposition 2.7 Let $(u_n)_{n\geq 0}$ be a sequence of integers satisfying (8) with integer coefficients $a_1, a_2, a_3 (\neq 0)$, and initial integral values u_0, u_1, u_2. Let $p(x)$ be the characteristic polynomial of (8) whose roots are λ_i $(i = 1, 2, 3)$, and whose discriminant is $D = -4\alpha^3 - 27\beta^2$, where $\alpha = (-a_1^2 - 3a_2)/3$ and $\beta = (-2a_1^3 - 9a_2 a_1 - 27a_3)/27$.

I. If $D > 0$, then $\lambda_1, \lambda_2, \lambda_3$ are distinct nonzero real numbers and

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2\lambda_3 u_0 - (\lambda_3 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \in \mathbb{R},\ B = \frac{-\lambda_1\lambda_3 u_0 + (\lambda_3 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R},$$
$$C = \frac{\lambda_1\lambda_2 u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R}\,.$$

II. (II.1) If $D = 0$ and $\alpha = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda \in \mathbb{R} \setminus \{0\}$ and

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0),$$

where

$$A = u_0 \in \mathbb{R},\ B = \frac{-3u_0}{2} + \frac{2u_1}{\lambda} - \frac{u_2}{2\lambda^2} \in \mathbb{R},\ C = \frac{u_0}{2} - \frac{u_1}{\lambda} + \frac{u_2}{2\lambda^2} \in \mathbb{R}\,.$$

(II.2) If $D = 0$ and $\alpha \neq 0$, then $p(x)$ has two distinct real roots $\lambda_1 (\neq 0)$ of multiplicity 1, $\lambda_2 = \lambda_3 (\neq 0)$ of multiplicity 2, and

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2^2 u_0 - 2\lambda_2 u_1 + u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R},\ B = \frac{-\lambda_1(2\lambda_2 - \lambda_1)u_0 + 2\lambda_2 u_1 - u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R},$$
$$C = \frac{\lambda_1\lambda_2 u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{\lambda_2(\lambda_2 - \lambda_1)} \in \mathbb{R}\,.$$

III. If $D < 0$, then $p(x)$ has one real root $\lambda_1 (\neq 0)$, two complex conjugate roots $\lambda_2, \lambda_3 = \bar{\lambda}_2 \in \mathbb{C} \setminus \mathbb{R}$ and

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2\bar{\lambda}_2 u_0 - (\bar{\lambda}_2 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{C},\ B = \frac{-\lambda_1\bar{\lambda}_2 u_0 + (\bar{\lambda}_2 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{C}\,.$$

Lemma 2.8 Let a, b be positive real numbers belonging to the interval $(0,1)$ and let $\mathcal{B}$ and $\mathcal{C}$ be two positive real numbers. Consider the following three exponential polynomials with argument $n \in \mathbb{N} \cup \{0\}$,

$$P(n) = a^n (\mathcal{B} - \mathcal{C}b^n), \quad Q(n) = a^n (\mathcal{B} + \mathcal{C}b^n), \quad R(n) = a^n (-\mathcal{B} + \mathcal{C}b^n).$$

The following conclusions hold:

(Type F-Z) there are explicitly computable integers $N_0 \in \mathbb{N} \cup \{0\}$ and $N_1 \in \mathbb{N}$ such that

$$\sup\{P(n) \; ; \; n \geq 0\} = \max\{P(n) \; ; \; N_0 \leq n \leq N_1\};$$

(Type F+Z) $\sup\{Q(n) \; ; \; n \geq 0\} = Q(0)$;

(Type -F+Z) there is an explicitly computable integer $N_2 \in \mathbb{N} \cup \{0\}$ such that

$$\sup\{R(n) \; ; \; n \geq 0\} = \max\{0, R(n) \; ; \; 0 \leq n \leq N_2\}.$$

Lemma 2.9 Let $\varphi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.

(*a*) If θ is a rational multiple of π, then $\cos(\varphi + n\theta)$ is periodic and takes only finitely many explicitly computable values as n varies over $\mathbb{N} \cup \{0\}$.

(*b*) If θ is not a rational multiple of π, then as n varies over $\mathbb{N} \cup \{0\}$ the range of values of $\cos(\varphi + n\theta)$ is dense in $[-1, 1]$.

Theorem 2.10 The Positivity Problem is decidable for each sequence of integers satisfying a third order linear recurrence with integer coefficients.

6 The work of Bell and Gerhold [1]

A sequence $(u_n)_{n \geq 0}$ is called a (linear) recurrence sequence of order $k \in \mathbb{N}$ if it satisfies a recurrence relation of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k} \quad (n \geq k), \tag{9}$$

where $a_1, \ldots, a_k$ $(\neq 0)$ are given real numbers. The characteristic polynomial associated with (9) is

$$\mathrm{Char}(z) := z^k - a_1 z^{k-1} - \cdots - a_{k-1} z - a_k.$$

Let all the distinct (nonzero) real roots of $\mathrm{Char}(z)$ be λ_i with multiplicities $a_i + 1$ ($i = 1, \ldots, r$), and let all its distinct (nonzero) complex conjugate pairs of roots be γ_j $(= |\gamma_j|e^{i\theta_j})$, $\overline{\gamma_j}$ with multiplicities $b_j + 1$ ($j = 1, \ldots, s$). Thus,

$$a_1 + \cdots + a_r + 2b_1 + \cdots + 2b_s + r + 2s = k.$$

Each sequence element satisfying (9) can be written as

$$u_n = \sum_{i=1}^{r} P_i(n)\lambda_i^n + 2\sum_{j=1}^{s} Q_j(n)|\gamma_j|^n \qquad (n \geq 0), \tag{10}$$

where

$$P_i(n) := \sum_{a=0}^{a_i} A_{i,a}n^a \in \mathbb{R}[n],\ A_{i,a_i} \neq 0,\ a_i \in \mathbb{N}\cup\{0\} \ \ (i = 1, \ldots, r),$$
$$Q_j(n) := \sum_{b=0}^{b_j} |B_{j,b}| \cos(\varphi_{j,b} + n\theta_j)n^b \in \mathbb{R}[n],$$
$$B_{j,1} = |B_{j,1}|e^{i\varphi_j}, \ldots, B_{j,b_j} = |B_{j,b_j}|e^{i\varphi_j} (\neq 0) \in \mathbb{C},\ b_j \in \mathbb{N}\cup\{0\} \ \ (j = 1, \ldots, s).$$

The roots of $\mathrm{Char}(z)$ having the largest absolute value are called *dominating roots*. Such roots play a crucial role in the positivity of the sequence (u_n) as witnessed in the following result of Bell-Gerhold, [1, Theorem 2], which helps in reducing the number of possible cases considerably.

Theorem 2.11 Let (u_n) be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N} : u_n > 0\}$ and $\{n \in \mathbb{N} : u_n < 0\}$ have positive density, and thus both sets contain infinitely many elements.

Lemma 2.12 Let $\theta_1, \theta_2, \ldots, \theta_d$ be rational numbers in $(0, 1)$, and let a_i, β_i be real numbers such that the purely periodic sequence

$$u_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i)$$

is not identically zero. Then u_n has a positive and a negative value.

CHAPTER 3

FOURTH ORDER POSITIVITY PROBLEM

Consider a fourth order linear recurrence of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} + a_4 u_{n-4} \quad (n \geq 4), \tag{11}$$

where $a_1, a_2, a_3, a_4 (\neq 0) \in \mathbb{Z}$.

We show here that the Positivity Problem is decidable for each sequence satisfying a fourth order linear recurrence with integer coefficients, viz.,

The Main Theorem: The Positivity Problem is decidable for each sequence of integers satisfying a fourth order linear recurrence with integer coefficients.

We give two proofs of our main theorem. In order to facilitate the presentation, we proceed by first settling the three subproblems mentioned in Chapter 1. The first proof of our main theorem is rather long but elementary in nature. It makes essential use of the root classification presented in Subproblem I. This proof appears in [21]. The second proof though much shorter is non-elementary in the sense that some measure-theoretic results of Bell-Gerhold are employed in order to reduce the number of cases to be considered by one half. It is worth mentioning that the root classification used in the first proof is not required in the second proof. This proof appears in [20].

1 Results related to Subproblem I

Following the elaboration in Section 1.3 of Chapter 2, consider a fourth order linear recurrence of the form (11) (with $k = 4$). Its characteristic polynomial is

$$p(x) := x^4 - a_1 x^3 - a_2 x^2 - a_3 x - a_4 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4) \in \mathbb{Z}[x] \tag{12}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the fourth roots of $p(x)$. Write

$$p\left(x + a_1/4\right) = x^4 + \alpha x^2 + \beta x + \gamma = (x - x_1)(x - x_2)(x - x_3)(x - x_4) \in \mathbb{Q}[x],$$

where x_1, x_2, x_3, x_4 are all fourth roots of $p(x + a_1/4) \in \mathbb{Q}[x]$ and

$$\alpha = -\frac{3a_1^2 + 8a_2}{8},\ \beta = -\frac{a_1^3 + 4a_1a_2 + 8a_3}{8},\ \gamma = -\frac{3a_1^4 + 16a_1^2a_2 + 64a_1a_3 + 256a_4}{256}.$$

Then, $x_i = \lambda_i - a_1/4 \ \ (i = 1, 2, 3, 4)$ and $x_1 + x_2 + x_3 + x_4 = 0$. Let

$$u = (x_1 + x_2)(x_3 + x_4) = -(x_1 + x_2)^2, \tag{13}$$

$$v = (x_1 + x_3)(x_2 + x_4) = -(x_1 + x_3)^2, \tag{14}$$

$$w = (x_1 + x_4)(x_2 + x_3) = -(x_1 + x_4)^2. \tag{15}$$

We construct a polynomial of degree 3 whose roots are u, v, and w. Its coefficients are found from the elementary symetric functions

$$\sum_{i<j} x_ix_j = \alpha,\ \sum_{i<j<k} x_ix_jx_k = -\beta,\ x_1x_2x_3x_4 = \gamma$$

by computing

$$u + v + w = 2\alpha,\ uv + uw + vw = \alpha^2 - 4\gamma,\ uvw = -\beta^2.$$

This polynomial is

$$q(x) := x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x - u)(x - v)(x - w) \in \mathbb{Q}[x]. \tag{16}$$

Since

$$u - v = (x_1 - x_4)(x_3 - x_2) = (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2),$$
$$u - w = (x_1 - x_3)(x_4 - x_2) = (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2),$$
$$v - w = (x_1 - x_2)(x_4 - x_3) = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3),$$

the three polynomials $q(x)$, $p(x + a_1/4)$ and $p(x)$ have the same discriminant

$$\begin{aligned}\mathcal{D} &= \mathrm{disc}(q) = \mathrm{disc}\left(p(x + a_1/4)\right) = \mathrm{disc}(p)\\ &= 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3 \quad ([23, p.\ 192])\\ &= (u - v)^2(u - w)^2(v - w)^2.\end{aligned}$$

To find the roots x_1, x_2, x_3, x_4 of the equation $p(x+a_1/4) = 0$, we first solve equation (16) to determine the roots u, v, w of $q(x) = 0$. From (13), (14), (15), setting

$$x_1 + x_2 := u' = \text{square root of} - u \tag{17}$$

$$x_1 + x_3 := v' = \text{square root of} - v \tag{18}$$

$$x_1 + x_4 := w' = \text{square root of} - w. \tag{19}$$

where these square roots must be chosen so that

$$u'v'w' = (x_1 + x_2)(x_1 + x_3)(x_1 + x_4) = x_1^2 \sum_i x_i + \sum_{i<j<k} x_i x_j x_k = -\beta, \tag{20}$$

we get

$$u' + v' + w' = 3x_1 + x_2 + x_3 + x_4 = 2x_1, \quad -u' + v' - w' = 2x_3,$$
$$u' - v' - w' = x_2 - x_1 - x_3 - x_4 = 2x_2, \quad -u' - v' + w' = 2x_4.$$

The solutions of $p(x + a_1/4) = 0$ are given by

$$x_1 = \frac{1}{2}(u' + v' + w') \tag{21}$$
$$x_2 = \frac{1}{2}(u' - v' - w') \tag{22}$$
$$x_3 = \frac{1}{2}(-u' + v' - w') \tag{23}$$
$$x_4 = \frac{1}{2}(-u' - v' + w'). \tag{24}$$

To determine the nature of these roots, we subdivide our consideration into three cases depending on the signs of $\mathcal{D}$.

Case I: $\mathcal{D} > 0$

As seen in Section 5 of Chapter 2, the three roots u, v, w of

$$q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x - u)(x - v)(x - w) \in \mathbb{Q}[x]$$

are distinct real numbers.
We subdivide into further subcases depending on whether $\beta = 0$.

I.1 $\beta = 0$.

Then $q(x)$ has a zero root, say $u = 0$, implying by (17) that $u' = 0$, and so

$$q(x) = x(x^2 - 2\alpha x + (\alpha^2 - 4\gamma)) = x(x^2 - (v + w)x + vw) = x(x - v)(x - w).$$

I.1.1 $\alpha^2 - 4\gamma \geq 0$. Then v, w have the same sign, distinct and nonzero.

I.1.1.1 $\alpha \leq 0$. Then v, w are two distinct negative real numbers which by (18), (19) implies that v', w' are nonzero real numbers with $v' \neq \pm\, w'$. By (21),(22),(23) and (24), we have

$$x_1 = \frac{1}{2}(v' + w'),\ x_2 = \frac{1}{2}(-v' - w'),\ x_3 = \frac{1}{2}(v' - w'),\ x_4 = \frac{1}{2}(-v' + w).$$

Hence, $p\left(x + a_1/4\right)$ has four distinct nonzero real roots.

I.1.1.2 $\alpha > 0$. Then v, w are two distinct positive real numbers, which by (18), (19) implies that $v' = \Box i,\ w' = \heartsuit i$ are purely imaginary with $\Box \neq \pm \heartsuit$ being two nonzero real numbers. By (21),(22),(23) and (24), we have

$$\begin{aligned} x_1 &= \frac{1}{2}(\Box + \heartsuit)i,\ x_2 = -\frac{1}{2}(\Box + \heartsuit)i = \bar{x}_1, \\ x_3 &= \frac{1}{2}(\Box - \heartsuit)i,\ x_4 = -\frac{1}{2}(\Box - \heartsuit)i = -\bar{x}_3. \end{aligned}$$

Thus, the roots of $p(x + a_1/4)$ consist of two nonzero distinct purely imaginary conjugate pairs.

I.1.2 $\alpha^2 - 4\gamma < 0$. Then v and w are two nonzero real numbers of opposite sign, say $v \in \mathbb{R}^-$ and $w \in \mathbb{R}^+$, which by (18), (19) implies that v' is a nonzero real number and $w' = \heartsuit i$ is nonzero purely imaginary. By (21),(22),(23) and (24), we have

$$\begin{aligned} x_1 &= \frac{1}{2}(v' + \heartsuit i),\ x_2 = \frac{1}{2}(-v' - \heartsuit i) = -x_1, \\ x_3 &= \frac{1}{2}(v' - \heartsuit i) = \bar{x}_1,\ x_4 = \frac{1}{2}(-v' + \heartsuit i) = -\bar{x}_1. \end{aligned}$$

Thus, the roots of $p(x + a_1/4)$ consists of two distinct, nonzero, non-real complex conjugate pairs.

I.2 $\beta \neq 0$.

Then $q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x-u)(x-v)(x-w) \in \mathbb{Q}[x]$ has three distinct roots $u, v, w \in \mathbb{R} \setminus \{0\}$.

I.2.1 $\alpha^2 - 4\gamma \geq 0$.

I.2.1.1 $\alpha \leq 0$. Since the coefficients of $q(x)$ are all nonnegative, by Descartes' rule of signs, the three roots $u, v, w \in \mathbb{R}^-$, which by (17), (18), (19) implies that

u', v', w' are three nonzero real numbers with

$$x_1 + x_2 = u' \neq \pm\, v' = \pm(x_1 + x_3),$$
$$x_1 + x_2 = u' \neq \pm\, w' = \pm(x_1 + x_4),$$
$$x_1 + x_3 = v' \neq \pm\, w' = \pm(x_1 + x_4).$$

By (21),(22),(23) and (24), we have

$$x_1 = \frac{1}{2}(u' + v' + w'),\ x_2 = \frac{1}{2}(u' - v' - w'),$$
$$x_3 = \frac{1}{2}(-u' + v' - w'),\ x_4 = \frac{1}{2}(-u' - v' + w').$$

Hence, $p(x + a_1/4)$ has four distinct real roots x_1, x_2, x_3, x_4.

I.2.1.2 $\alpha > 0$. Since $\beta^2 > 0$, we know that $uvw > 0$ and so two roots of $q(x)$, say u, v, are negative real numbers and one root, say w, is a positive real number, which by (17), (18), (19) imply that $u', v' \in \mathbb{R}$, $u' \neq \pm\, v'$ and $w' = \heartsuit i$, $\heartsuit \in \mathbb{R}\setminus\{0\}$. Thus,

$$x_1 = \frac{1}{2}(u' + v' + \heartsuit i),\ x_2 = \frac{1}{2}(u' - v' - \heartsuit i),$$
$$x_3 = \frac{1}{2}(-u' + v' - \heartsuit i),\ x_4 = \frac{1}{2}(-u' - v' + \heartsuit i).$$

Since $p(x + a_1/4) \in \mathbb{Q}[x]$, its four complex roots x_1, x_2, x_3, x_4 must occur in two conjugate pairs. From their shapes, there are two possibilities, viz., ($\bar{x}_1 = x_2$ and $\bar{x}_4 = x_3$) or ($\bar{x}_1 = x_3$ and $\bar{x}_4 = x_2$). If $\bar{x}_1 = x_2$, then $v' = 0,\ u' \neq 0$ and so

$$x_1 = \frac{1}{2}(u' + i\heartsuit) \in \mathbb{C} \setminus \mathbb{R},\ x_2 = \bar{x}_1,\ x_3 = -x_1,\ x_4 = -\bar{x}_1.$$

If $\bar{x}_1 = x_3$, then $u' = 0,\ v' \neq 0$ and we deduce that the solutions are of the same form as in the last possibility, i.e., two distinct complex conjugate pairs.

I.2.2 $\alpha^2 - 4\gamma < 0$. Since $\beta^2 > 0$, we know that $uvw > 0$ and so two roots of $q(x)$, say u, v are negative real numbers, and one root, say w is a positive real number, which by (17), (18), (19) imply that $u', v' \in \mathbb{R},\ u' \neq \pm\, v'$ and $w' = \heartsuit i,\ \heartsuit \in \mathbb{R}\setminus\{0\}$. The roots of $p(x + a_1/4)$ are of the same shape as those in the subcase [I.2.1.2], i.e., two distinct complex conjugate pairs.

Case II: $\mathcal{D} = 0$

As seen in Section 5 of Chapter 2, the three roots u, v, w are real numbers and since

$$\begin{aligned}\mathcal{D} &= 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3 \quad ([23, p.\ 192]) \\ &= (u-v)^2(u-w)^2(v-w)^2,\end{aligned}$$

we must have

$$u = v \quad \text{or} \quad u = w \quad \text{or} \quad v = w. \tag{25}$$

Recall that

$$q(x) = x^3 - 2\alpha x^2 + (\alpha^2 - 4\gamma)x + \beta^2 = (x-u)(x-v)(x-w) \in \mathbb{Q}[x].$$

Let $\alpha^* := -(\alpha^2 + 12\gamma)/3$.

II.1 If $\alpha^* = 0$, then $\gamma = -\alpha^2/12$ and putting this into the expression for $\mathcal{D}$, we get $\beta^2 = -8\alpha^3/27$. Substituting these values into the expression for $q(x)$, we find that $q(x) = (x - 2\alpha/3)^3$ showing that $u = v = w = 2\alpha/3$.

II.1.1 If $\beta = 0$, then $u = v = w = 0$. By (17),(18) and (19), we get $u' = v' = w' = 0$. Thus, (21),(22),(23) and (24) yield $x_1 = x_2 = x_3 = x_4 = 0$.

II.1.2 If $\beta \neq 0$, then $\beta^2 = -uvw = -u^3 > 0$, so that $u = v = w \in \mathbb{R}^-$. By (17),(18) and (19), we get $u' = v' = w' \in \mathbb{R}\backslash\{0\}$. Thus, (21),(22),(23) and (24) yield $x_1 = \frac{3u'}{2}$, $x_2 = \frac{-u'}{2} = x_3 = x_4$.

II.2 If $\alpha^* \neq 0$, we have u, v and w are not all the same.

II.2.1 If $\beta = 0$, then $uvw = 0$; without loss of generality, assume $u = 0$. By (25), we deduce $u = v = 0$. From the shape of $q(x)$, we must have $\alpha^2 - 4\gamma = 0$ and $w = 2\alpha$. By (17),(18) and (19), we get

$$u' = v' = 0,\ w' = \text{square root of} - 2\alpha,$$

so that (21),(22),(23) and (24) yield $x_1 = w'/2 = x_4$, $x_2 = -w'/2 = x_3$. Clearly, $\alpha \neq 0$, for otherwise $u = v = w = 0$. Let $r = \frac{1}{2}\sqrt{|-2\alpha|} \in \mathbb{R}\backslash\{0\}$. Taking into account that α can be either positive or negative, we find that there are two possible set of solutions

$$\{x_1, x_2, x_3, x_4\} = \{r,\ r,\ -r,\ -r\} \quad \text{or} \quad \{ir,\ ir,\ -ir,\ -ir\}.$$

II.2.2 If $\beta \neq 0$, since $\beta \in \mathbb{Q}$, we deduce that $-uvw > 0$ so that two elements, say $u, v \in \mathbb{R}\setminus\{0\}$, are of the same sign (and so by (25), $u = v$), while $w \in \mathbb{R}\setminus\{0\}$ is negative. If $u = v \in \mathbb{R}^-$, $w \in \mathbb{R}^-$, then (17),(18), (19) show that $u' = v', w' \in \mathbb{R}\setminus\{0\}$. Thus, (21),(22),(23),(24) yield four real roots

$$x_1 = \frac{1}{2}(2u' + w'), \quad x_2 = -\frac{w'}{2} = x_3, \quad x_4 = \frac{1}{2}(-2u' + w').$$

Apart from $x_2 = x_3$, all other roots are distinct for if $x_1 = x_2$, then $u' = -w'$ implying that $(v =)u = w$, contradicting the fact that all three roots are not the same; if $x_1 = x_4$, then $u' = 0$, implying that $u = v = 0$ which contradicts $\beta \neq 0$; if $x_2 = x_4$, then $u' = w'$ implying that $(v =)u = w$, again a contradiction. If $u = v \in \mathbb{R}^+$, $w \in \mathbb{R}^-$, then (17),(18), (19) show that $w' = 2W_1 \in \mathbb{R}\setminus\{0\}$, and we have $u' = \pm v' = ir_2$ ($r_2 \in \mathbb{R}\setminus\{0\}$). If $u' = v'$, then (21),(22),(23),(24) yield

$$\{x_1,\ x_2,\ x_3,\ x_4\} = \{W_1 + ir_2,\ -W_1,\ -W_1,\ W_1 - ir_2\},$$

while if $u' = -v'$, we get

$$\{x_1,\ x_2,\ x_3,\ x_4\} = \{W_1,\ -W_1 + ir_2,\ -W_1 - ir_2,\ W_1\}.$$

Clearly, $x_1 \neq x_2$, $x_1 \neq x_4$, $x_2 \neq x_4$ in both situations.

Case III: $\mathcal{D} < 0$

As seen in Section 5 of Chapter 2, $q(x)$ has one real root and two complex conjugate roots, say $u \in \mathbb{R}$, $v = \overline{w} \in \mathbb{C}\setminus\mathbb{R}$.

III.1 $\beta = 0$ (so that $uvw = 0$).

If $u \neq 0$, then $v = w = 0$, contradicting $v = \overline{w} \in \mathbb{C}\setminus\mathbb{R}$ and so we must have $u = 0$. Thus, (17) yields $u' = 0$. Since $v = \overline{w}$, (18) and (19) show that either $v' = \overline{w'} \in \mathbb{C}\setminus\mathbb{R}$ or $v' = -\overline{w'} \in \mathbb{C}\setminus\mathbb{R}$. As (21),(22),(23) and (24) are symmetric in v' and w', we can assume that $v' = \overline{w'} = r_3 + ir_4$ ($r_3, r_4(\neq 0) \in \mathbb{R}$), as the other alternative yields roots of the same shape. Thus, (21),(22),(23) and (24) yield $x_1 = (v' + w')/2 = r_3 = -x_2$, $x_3 = (v' - w')/2 = ir_4 = -x_4$. Here, $r_3 \neq 0$, for otherwise $x_1 = x_2 = 0$ implying $D = 0$ which is a contradiction.

III.2 $\beta \neq 0$ (so that $u \in \mathbb{R}\setminus\{0\}$).

• If $u < 0$, then $u' = r \in \mathbb{R}\setminus\{0\}$. Similar to the last case, we can assume that $v' = \overline{w'} = r_3 + ir_4$ $(r_3, r_4(\neq 0) \in \mathbb{R})$, as the possibility $v' = -\overline{w'}$ yields roots of the same shape. Thus, (21),(22),(23) and (24) yield

$$x_1 = (r + 2r_3)/2,\ x_2 = (r - 2r_3)/2,\ x_3 = (-r + i2r_4)/2,\ x_4 = (-r - i2r_4)/2.$$

Here again $r_3 \neq 0$ (so that $x_1 \neq x_2$), for otherwise $x_1 = x_2$ implying $\mathcal{D} = 0$, a contradiction

• If $u > 0$, then $u' = ir$ $(r \in \mathbb{R}\setminus\{0\})$. Again by choosing the sign of u' appropriately, we can assume that $v' = \overline{w'} = r_3 + ir_4$ $(r_3, r_4(\neq 0) \in \mathbb{R})$. Thus, (21),(22),(23) and (24) yield

$$x_1 = (2r_3 + ir)/2,\ x_2 = (-2r_3 + ir)/2,$$
$$x_3 = i(-r + 2r_4)/2,\ x_4 = i(-r - 2r_4)/2.$$

Since $p(x + a_1/4) \in \mathbb{Q}[x]$, its four complex roots x_1, x_2, x_3, x_4 must occur in two conjugate pairs. From their shapes, there are two possibilities, viz., $\overline{x_1} = x_3$ or $\overline{x_1} = x_4$. In either case, we deduce that $r_4 = 0$, a contradiction. There is then no solution in this case.

From the above information about the nature of the four zero of $p(x+a_1/4)$, we deduce the following result about the four zero of $p(x)$.

Theorem 3.1 Let

$$\begin{aligned}
p(x) :=& x^4 - a_1x^3 - a_2x^2 - a_3x - a_4 \\
=& (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4) \in \mathbb{Z}[x], \quad a_4 \neq 0,
\end{aligned}$$

$$p\left(x + a_1/4\right) := x^4 + \alpha x^2 + \beta x + \gamma = (x-x_1)(x-x_2)(x-x_3)(x-x_4) \in \mathbb{Q}[x],$$

$$\alpha = -\frac{3a_1^2 + 8a_2}{8} \in \mathbb{Q}, \qquad \alpha^* := -\frac{\alpha^2 + 12\gamma}{3} \in \mathbb{Q},$$

$$\beta = -\frac{a_1^3 + 4a_1a_2 + 8a_3}{8} \in \mathbb{Q}, \quad \gamma = -\frac{3a_1^4 + 16a_1^2a_2 + 64a_1a_3 + 256a_4}{256} \in \mathbb{Q},$$

$$\begin{aligned}
\mathcal{D} &= \operatorname{disc}(p(x)) = \operatorname{disc}(p(x + a_1/4)) \\
&= 16\alpha^4\gamma - 4\alpha^3\beta^2 - 128\alpha^2\gamma^2 + 144\alpha\beta^2\gamma - 27\beta^4 + 256\gamma^3 \in \mathbb{Q}\,.
\end{aligned}$$

Then the nature of the four zeros of $p(x)$ can be classified as displayed in Figure 1.

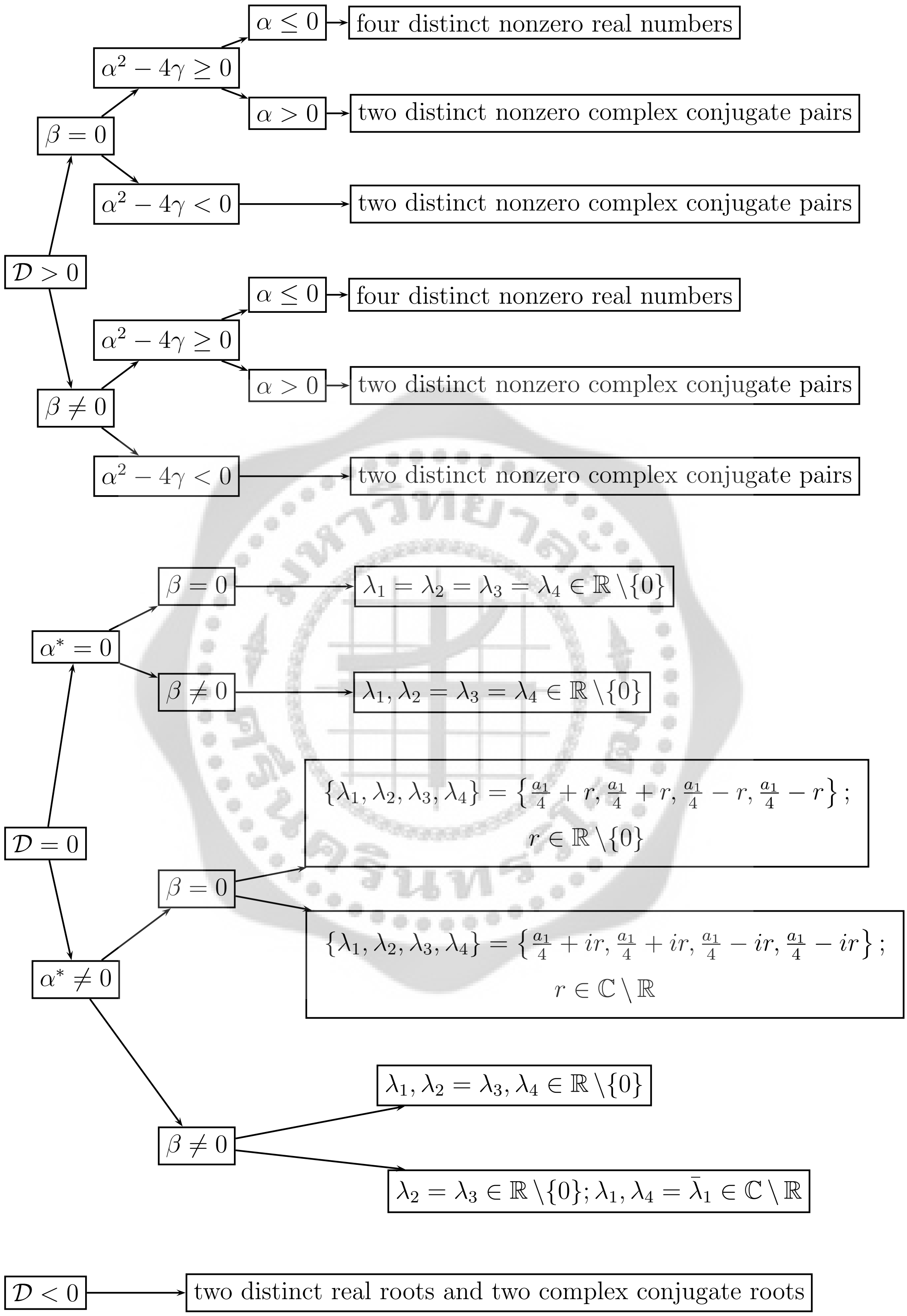

Figure 1: The nature of the four zeros of $p(x)$.

Using the information about the four zeros of $p(x)$, we now summarize the shapes of the sequence elements.

Theorem 3.2 Let $\{u_n\}_{n\geq 0}$ be a sequence of elements satisfying a fourth order linear recurrence of the form (1), let

$$p(x) := x^4 - a_1x^3 - a_2x^2 - a_3x - a_4 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4) \in \mathbb{Z}[x],\ a_4 \neq 0,$$

be its characteristic polynomial having $\lambda_1, \ldots, \lambda_4$ as all its (non-vanishing) zeros and let $\mathcal{D}$ be the discriminant of $p(x)$. Then the general term of the sequence takes one of the following forms displayed in Figure 2.

$\mathcal{D} > 0$

- $\mathbf{C(r_1r_2r_3r_4)}$: $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n$
- $\mathbf{C(z_1\bar{z}_1z_2\bar{z}_2)}$: $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n$

$\mathcal{D} = 0$

- $\mathbf{C(r_1^4)}$: $u_n = (A + Bn + Cn^2 + Dn^3)\lambda_1^n$
- $\mathbf{C(r_1r_2^3)}$: $u_n = A\lambda_1^n + (B + Cn + Dn^2)\lambda_2^n$
- $\mathbf{C(r_1^2r_2^2)}$: $u_n = (A + Bn)\lambda_1^n + (C + Dn)\lambda_2^n$
- $\mathbf{C(z^2\bar{z}^2)}$: $u_n = (A + Bn)\lambda^n + (\bar{A} + \bar{B}n)\bar{\lambda}^n$
- $\mathbf{C(r_1r_2r_3^2)}$: $u_n = A\lambda_1^n + B\lambda_2^n + (C + Dn)\lambda_3^n$
- $\mathbf{C(r_1^2z\bar{z})}$: $u_n = (A + Bn)\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$

$\mathcal{D} < 0$

- $\mathbf{C(r_1r_2z\bar{z})}$: $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$

Figure 2: The general term of the sequence.

The coefficients of the sequence elements are determined by solving the system arising from substituting the four positive initial values, and we have the following results.

In the case $\mathbf{C}(r_1r_2r_3r_4)$, we have $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n \quad (n \geq 0)$, where

$$
\begin{aligned}
\Lambda(r_1r_2r_3r_4) &= (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3) \in \mathbb{R}\setminus\{0\},\\
A\,\Lambda(r_1r_2r_3r_4) &= \lambda_2\lambda_3\lambda_4\left\{\lambda_2\lambda_3(\lambda_3-\lambda_2)-\lambda_2\lambda_4(\lambda_4-\lambda_2)+\lambda_3\lambda_4(\lambda_4-\lambda_3)\right\}u_0\\
&\quad-\left\{\lambda_2^2\lambda_3^2(\lambda_3-\lambda_2)-\lambda_2^2\lambda_4^2(\lambda_4-\lambda_2)+\lambda_3^2\lambda_4^2(\lambda_4-\lambda_3)\right\}u_1\\
&\quad+\left\{\lambda_2\lambda_3(\lambda_3^2-\lambda_2^2)-\lambda_2\lambda_4(\lambda_4^2-\lambda_2^2)+\lambda_3\lambda_4(\lambda_4^2-\lambda_3^2)\right\}u_2\\
&\quad-\left\{\lambda_2\lambda_3(\lambda_3-\lambda_2)-\lambda_2\lambda_4(\lambda_4-\lambda_2)+\lambda_3\lambda_4(\lambda_4-\lambda_3)\right\}u_3 \in \mathbb{R},\\
B\,\Lambda(r_1r_2r_3r_4) &= -\lambda_1\lambda_3\lambda_4\left\{\lambda_1\lambda_3(\lambda_3-\lambda_1)-\lambda_1\lambda_4(\lambda_4-\lambda_1)+\lambda_3\lambda_4(\lambda_4-\lambda_3)\right\}u_0\\
&\quad+\left\{\lambda_1^2\lambda_3^2(\lambda_3-\lambda_1)-\lambda_1^2\lambda_4^2(\lambda_4-\lambda_1)+\lambda_3^2\lambda_4^2(\lambda_4-\lambda_3)\right\}u_1\\
&\quad-\left\{\lambda_1\lambda_3(\lambda_3^2-\lambda_1^2)-\lambda_1\lambda_4(\lambda_4^2-\lambda_1^2)+\lambda_3\lambda_4(\lambda_4^2-\lambda_3^2)\right\}u_2\\
&\quad+\left\{\lambda_1\lambda_3(\lambda_3-\lambda_1)-\lambda_1\lambda_4(\lambda_4-\lambda_1)+\lambda_3\lambda_4(\lambda_4-\lambda_3)\right\}u_3 \in \mathbb{R},\\
C\,\Lambda(r_1r_2r_3r_4) &= \lambda_1\lambda_2\lambda_4\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1)-\lambda_1\lambda_4(\lambda_4-\lambda_1)+\lambda_2\lambda_4(\lambda_4-\lambda_2)\right\}u_0\\
&\quad-\left\{\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1)-\lambda_1^2\lambda_4^2(\lambda_4-\lambda_1)+\lambda_2^2\lambda_4^2(\lambda_4-\lambda_2)\right\}u_1\\
&\quad+\left\{\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)-\lambda_1\lambda_4(\lambda_4^2-\lambda_1^2)+\lambda_2\lambda_4(\lambda_4^2-\lambda_2^2)\right\}u_2\\
&\quad-\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1)-\lambda_1\lambda_4(\lambda_4-\lambda_1)+\lambda_2\lambda_4(\lambda_4-\lambda_2)\right\}u_3 \in \mathbb{R},\\
D\,\Lambda(r_1r_2r_3r_4) &= -\lambda_1\lambda_2\lambda_3\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1)-\lambda_1\lambda_3(\lambda_3-\lambda_1)+\lambda_2\lambda_3(\lambda_3-\lambda_2)\right\}u_0\\
&\quad+\left\{\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1)-\lambda_1^2\lambda_3^2(\lambda_3-\lambda_1)+\lambda_2^2\lambda_3^2(\lambda_3-\lambda_2)\right\}u_1\\
&\quad-\left\{\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)-\lambda_1\lambda_3(\lambda_3^2-\lambda_1^2)+\lambda_2\lambda_3(\lambda_3^2-\lambda_2^2)\right\}u_2\\
&\quad+\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1)-\lambda_1\lambda_3(\lambda_3-\lambda_1)+\lambda_2\lambda_3(\lambda_3-\lambda_2)\right\}u_3 \in \mathbb{R}\,.
\end{aligned}
$$

In the case $\mathbf{C}(z_1\bar{z}_1z_2\bar{z}_2)$, we have $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0)$, where

$$
\begin{aligned}
\Lambda(z_1\bar{z}_1z_2\bar{z}_2) &= (\bar{\lambda}_1-\lambda_1)(\lambda_2-\lambda_1)(\bar{\lambda}_2-\lambda_1)(\lambda_2-\bar{\lambda}_1)(\bar{\lambda}_2-\bar{\lambda}_1)(\bar{\lambda}_2-\lambda_2) \in \mathbb{R}\setminus\{0\},\\
A\,\Lambda(z_1\bar{z}_1z_2\bar{z}_2) &= \bar{\lambda}_1\lambda_2\bar{\lambda}_2\left\{\bar{\lambda}_1\lambda_2(\lambda_2-\bar{\lambda}_1)-\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2-\bar{\lambda}_1)+\lambda_2\bar{\lambda}_2(\bar{\lambda}_2-\lambda_2)\right\}u_0\\
&\quad-\left\{\bar{\lambda}_1^2\lambda_2^2(\lambda_2-\bar{\lambda}_1)-\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_2-\bar{\lambda}_1)+\lambda_2^2\bar{\lambda}_2^2(\bar{\lambda}_2-\lambda_2)\right\}u_1\\
&\quad+\left\{\bar{\lambda}_1\lambda_2(\lambda_2^2-\bar{\lambda}_1^2)-\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2^2-\bar{\lambda}_1^2)+\lambda_2\bar{\lambda}_2(\bar{\lambda}_2^2-\lambda_2^2)\right\}u_2\\
&\quad-\left\{\bar{\lambda}_1\lambda_2(\lambda_2-\bar{\lambda}_1)-\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2-\bar{\lambda}_1)+\lambda_2\bar{\lambda}_2(\bar{\lambda}_2-\lambda_2)\right\}u_3 \in \mathbb{C},
\end{aligned}
$$

$$\begin{aligned}
B\,\Lambda(z_1\bar{z}_1z_2\bar{z}_2) = \lambda_1\bar{\lambda}_1\bar{\lambda}_2\left\{\lambda_1\bar{\lambda}_1(\bar{\lambda}_1-\lambda_1)-\lambda_1\bar{\lambda}_2(\bar{\lambda}_2-\lambda_1)+\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2-\bar{\lambda}_1)\right\}u_0 \\
-\left\{\lambda_1^2\bar{\lambda}_1^2(\bar{\lambda}_1-\lambda_1)-\lambda_1^2\bar{\lambda}_2^2(\bar{\lambda}_2-\lambda_1)+\bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_2-\bar{\lambda}_1)\right\}u_1 \\
+\left\{\lambda_1\bar{\lambda}_1(\bar{\lambda}_1^2-\lambda_1^2)-\lambda_1\bar{\lambda}_2(\bar{\lambda}_2^2-\lambda_1^2)+\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2^2-\bar{\lambda}_1^2)\right\}u_2 \\
-\left\{\lambda_1\bar{\lambda}_1(\bar{\lambda}_1-\lambda_1)-\lambda_1\bar{\lambda}_2(\bar{\lambda}_2-\lambda_1)+\bar{\lambda}_1\bar{\lambda}_2(\bar{\lambda}_2-\bar{\lambda}_1)\right\}u_3 \in \mathbb{C}\,.
\end{aligned}$$

In the case $\mathbf{C}(r_1^4)$, we have $u_n = (A+Bn+Cn^2+Dn^3)\lambda_1^n \quad (n \geq 0)$, where

$$\begin{aligned}
\Lambda(r_1^4) &= 12\lambda_1^6 \in \mathbb{R}\setminus\{0\}, \\
A\,\Lambda(r_1^4) &= 12\lambda_1^6u_0 \in \mathbb{R}, \quad B\,\Lambda(r_1^4) = -22\lambda_1^6u_0+36\lambda_1^5u_1-18\lambda_1^4u_2+4\lambda_1^3u_3 \in \mathbb{R}, \\
C\,\Lambda(r_1^4) &= 12\lambda_1^6u_0-30\lambda_1^5u_1+24\lambda_1^4u_2-6\lambda_1^3u_3 \in \mathbb{R}, \\
D\,\Lambda(r_1^4) &= -2\lambda_1^6u_0+6\lambda_1^5u_1-6\lambda_1^4u_2+2\lambda_1^3u_3 \in \mathbb{R}\,.
\end{aligned}$$

In the case $\mathbf{C}(r_1r_2^3)$, we have $u_n = A\lambda_1^n+(B+Cn+Dn^2)\lambda_2^n \quad (n \geq 0)$, where

$$\begin{aligned}
\Lambda(r_1r_2^3) &= 2\lambda_2^3(\lambda_2^3-3\lambda_1\lambda_2^2+3\lambda_1^2\lambda_2-2\lambda_1^3) \in \mathbb{R}\setminus\{0\}, \\
A\,\Lambda(r_1r_2^3) &= 2\lambda_2^6u_0-6\lambda_2^5u_1+6\lambda_2^4u_2-2\lambda_2^3u_3 \in \mathbb{R}, \\
B\,\Lambda(r_1r_2^3) &= -2\lambda_1\lambda_2^3(3\lambda_2^2+\lambda_1^2-3\lambda_1\lambda_2)u_0+6\lambda_2^5u_1-6\lambda_2^4u_2+2\lambda_2^3u_3 \in \mathbb{R}, \\
C\Lambda(r_1r_2^3) &= \lambda_1\lambda_2^3(5\lambda_2-3\lambda_1)(\lambda_2-\lambda_1)u_0-\lambda_2^2(5\lambda_2^3-9\lambda_1^2\lambda_2+4\lambda_1^3)u_1 \\
&\quad +\lambda_2(8\lambda_2^3-9\lambda_1\lambda_2^2+\lambda_1^3)u_2-\lambda_2(3\lambda_2-\lambda_1)(\lambda_2-\lambda_1)u_3 \in \mathbb{R}, \\
D\,\Lambda(r_1r_2^3) &= -\lambda_1\lambda_2^3(\lambda_2-\lambda_1)^2u_0+\lambda_2^2(\lambda_2^3-3\lambda_1^2\lambda_2+2\lambda_1^3)u_1 \\
&\quad -\lambda_2(2\lambda_2^3-3\lambda_1\lambda_2^2+\lambda_1^3)u_2+\lambda_2(\lambda_2-\lambda_1)^2u_3 \in \mathbb{R}\,.
\end{aligned}$$

In the case $\mathbf{C}(r_1^2r_2^2)$, we have $u_n = (A+Bn)\lambda_1^n+(C+Dn)\lambda_2^n \quad (n \geq 0)$, where

$$\begin{aligned}
\Lambda(r_1^2r_2^2) =& \lambda_1\lambda_2(\lambda_2-\lambda_1)^4 \in \mathbb{R}\setminus\{0\}, \\
A\,\Lambda(r_1^2r_2^2) =& \lambda_1\lambda_2^3(\lambda_2-3\lambda_1)(\lambda_2-\lambda_1)u_0+6\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1)u_1 \\
&-3\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)u_2+2\lambda_1\lambda_2(\lambda_2-\lambda_1)u_3 \in \mathbb{R}, \\
B\,\Lambda(r_1^2r_2^2) =& -\lambda_1\lambda_2^3(\lambda_2-\lambda_1)^2u_0+\lambda_2^2(2\lambda_1^3-3\lambda_1^2\lambda_2+\lambda_2^3)u_1 \\
&-\lambda_2(\lambda_1^3-3\lambda_1\lambda_2^2+2\lambda_2^3)u_2+\lambda_2(\lambda_2-\lambda_1)^2u_3 \in \mathbb{R}, \\
C\,\Lambda(r_1^2r_2^2) =& \lambda_1^3\lambda_2(3\lambda_2-\lambda_1)(\lambda_2-\lambda_1)u_0-6\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1)u_1 \\
&+3\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)u_2-2\lambda_1\lambda_2(\lambda_2-\lambda_1)u_3 \in \mathbb{R},
\end{aligned}$$

$$\begin{aligned} D\,\Lambda(r_1^2r_2^2) = &- \lambda_1^3\lambda_2(\lambda_2-\lambda_1)^2u_0 + \lambda_1^2(\lambda_1^3 - 3\lambda_1\lambda_2^2 + 2\lambda_2^3)u_1 \\ &- \lambda_1(2\lambda_1^3 - 3\lambda_1^2\lambda_2 + \lambda_2^3)u_2 + \lambda_1(\lambda_2-\lambda_1)^2u_3 \in \mathbb{R}\,. \end{aligned}$$

In the case $\mathbf{C}(z^2\bar{z}^2)$, we have $u_n = (A+Bn)\lambda_1^n + (\bar{A}+\bar{B}n)\bar{\lambda}_1^n \quad (n \geq 0)$, where

$$\begin{aligned} \Lambda(z^2\bar{z}^2) =&\lambda_1\bar{\lambda}_1(\bar{\lambda}_1-\lambda_1)^4 \in \mathbb{R}\setminus\{0\}, \\ A\,\Lambda(z^2\bar{z}^2) =&\lambda_1\bar{\lambda}_1^3(\bar{\lambda}_1-3\lambda_1)(\bar{\lambda}_1-\lambda_1)u_0 + 6\lambda_1^2\bar{\lambda}_1^2(\bar{\lambda}_1-\lambda_1)u_1 \\ &- 3\lambda_1\bar{\lambda}_1(\bar{\lambda}_1^2-\lambda_1^2)u_2 + 2\lambda_1\bar{\lambda}_1(\bar{\lambda}_1-\lambda_1)u_3 \in \mathbb{C}, \\ B\,\Lambda(z^2\bar{z}^2) =&- \lambda_1\bar{\lambda}_1^3(\bar{\lambda}_1-\lambda_1)^2u_0 + \bar{\lambda}_1^2(2\lambda_1^3 - 3\lambda_1^2\bar{\lambda}_1 + \bar{\lambda}_1^3)u_1 \\ &- \bar{\lambda}_1(\lambda_1^3 - 3\lambda_1\bar{\lambda}_1^2 + 2\bar{\lambda}_1^3)u_2 + \bar{\lambda}_1(\bar{\lambda}_1-\lambda_1)^2u_3 \in \mathbb{C}\,. \end{aligned}$$

In the case $\mathbf{C}(r_1r_2r_3^2)$, we have $u_n = A\lambda_1^n + B\lambda_2^n + (C+Dn)\lambda_3^n \quad (n \geq 0)$, where

$$\begin{aligned} \Lambda(r_1r_2r_3^2) =\ & \lambda_2\lambda_3^3(\lambda_3-\lambda_2)^2 - \lambda_1\lambda_3^3(\lambda_3-\lambda_1)^2 + \lambda_1\lambda_2^2\lambda_3^2(3\lambda_3-2\lambda_2) \\ &- \lambda_1^2\lambda_2\lambda_3^2(3\lambda_3-2\lambda_1) + \lambda_1^2\lambda_2^2\lambda_3(\lambda_2-\lambda_1) \in \mathbb{R}\setminus\{0\}, \\ A\,\Lambda(r_1r_2r_3^2) =\ & \lambda_2\lambda_3^3(\lambda_3-\lambda_2)^2u_0 - \lambda_3^2(\lambda_3^3 + 2\lambda_2^3 - 3\lambda_2^2\lambda_3)u_1 \\ &+ \lambda_3(2\lambda_3^3 + \lambda_2^3 - 3\lambda_2\lambda_3^2)u_2 - \lambda_3(\lambda_3-\lambda_2)^2u_3 \in \mathbb{R}, \\ B\,\Lambda(r_1r_2r_3^2) =\ & -\lambda_1\lambda_3^3(\lambda_3-\lambda_1)^2u_0 + \lambda_3^2(\lambda_3^3 + 2\lambda_1^3 - 3\lambda_1^2\lambda_3)u_1 \\ &- \lambda_3(2\lambda_3^3 + \lambda_1^3 - 3\lambda_1\lambda_3^2)u_2 + \lambda_3(\lambda_3-\lambda_1)^2u_3 \in \mathbb{R}, \\ C\,\Lambda(r_1r_2r_3^2) =\ & \lambda_1\lambda_2\lambda_3\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\lambda_3(3\lambda_3-2\lambda_1) + \lambda_2\lambda_3(3\lambda_3-2\lambda_2)\right\}u_0 \\ &- \left\{-\lambda_1^2\lambda_3^2(3\lambda_3-2\lambda_1) + \lambda_2^2\lambda_3^2(3\lambda_3-2\lambda_2)\right\}u_1 \\ &+ \left\{-\lambda_1\lambda_3(3\lambda_3^2-\lambda_1^2) + \lambda_2\lambda_3(3\lambda_3^2-\lambda_2^2)\right\}u_2 \\ &- \left\{-\lambda_1\lambda_3(2\lambda_3-\lambda_1) + \lambda_2\lambda_3(2\lambda_3-\lambda_2)\right\}u_3 \in \mathbb{R}, \\ D\,\Lambda(r_1r_2r_3^2) =\ & -\lambda_1\lambda_2\lambda_3\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\lambda_3(\lambda_3-\lambda_1) + \lambda_2\lambda_3(\lambda_3-\lambda_2)\right\}u_0 \\ &+ \left\{\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1) - \lambda_1^2\lambda_3^2(\lambda_3-\lambda_1) + \lambda_2^2\lambda_3^2(\lambda_3-\lambda_2)\right\}u_1 \\ &- \left\{\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2) - \lambda_1\lambda_3(\lambda_3^2-\lambda_1^2) + \lambda_2\lambda_3(\lambda_3^2-\lambda_2^2)\right\}u_2 \\ &+ \left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\lambda_3(\lambda_3-\lambda_1) + \lambda_2\lambda_3(\lambda_3-\lambda_2)\right\}u_3 \in \mathbb{R}\,. \end{aligned}$$

In the case $\mathbf{C}(r_1^2 z\bar{z})$, we have $u_n = (A+Bn)\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n \quad (n \geq 0)$, where

$$\begin{aligned}
\Lambda(r_1^2 z\bar{z}) &= \lambda_1^3\bar{\lambda}_2(\bar{\lambda}_2-\lambda_1)^2 - \lambda_1^3\lambda_2(\lambda_2-\lambda_1)^2 + \lambda_1\lambda_2^2\bar{\lambda}_2^2(\bar{\lambda}_2-\lambda_2) \\
&\quad - \lambda_1^2\lambda_2\bar{\lambda}_2^2(2\bar{\lambda}_2-3\lambda_1) + \lambda_1^2\lambda_2^2\bar{\lambda}_2(2\lambda_2-3\lambda_1) \in \mathbb{C}\setminus\{0\}, \\
A\,\Lambda(r_1^2 z\bar{z}) &= \lambda_1\lambda_2\bar{\lambda}_2\left\{\lambda_1\lambda_2(2\lambda_2-3\lambda_1) - \lambda_1\bar{\lambda}_2(2\bar{\lambda}_2-3\lambda_1) + \lambda_2\bar{\lambda}_2(\bar{\lambda}_2-\lambda_2)\right\}u_0 \\
&\quad - \left\{\lambda_1^2\lambda_2^2(2\lambda_2-3\lambda_1) - \lambda_1^2\bar{\lambda}_2^2(2\bar{\lambda}_2-3\lambda_1)\right\}u_1 \\
&\quad + \left\{\lambda_1\lambda_2(\lambda_2^2-3\lambda_1^2) - \lambda_1\bar{\lambda}_2(\bar{\lambda}_2^2-3\lambda_1^2)\right\}u_2 \\
&\quad - \left\{\lambda_1\lambda_2(\lambda_2-2\lambda_1) - \lambda_1\bar{\lambda}_2(\bar{\lambda}_2-2\lambda_1)\right\}u_3 \in \mathbb{C}, \\
B\,\Lambda(r_1^2 z\bar{z}) &= -\lambda_1\lambda_2\bar{\lambda}_2\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\bar{\lambda}_2(\bar{\lambda}_2-\lambda_1) + \lambda_2\bar{\lambda}_2(\bar{\lambda}_2-\lambda_2)\right\}u_0 \\
&\quad + \left\{\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1) - \lambda_1^2\bar{\lambda}_2^2(\bar{\lambda}_2-\lambda_1) + \lambda_2^2\bar{\lambda}_2^2(\bar{\lambda}_2-\lambda_2)\right\}u_1 \\
&\quad - \left\{\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2) - \lambda_1\bar{\lambda}_2(\bar{\lambda}_2^2-\lambda_1^2) + \lambda_2\bar{\lambda}_2(\bar{\lambda}_2^2-\lambda_2^2)\right\}u_2 \\
&\quad + \left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\bar{\lambda}_2(\bar{\lambda}_2-\lambda_1) + \lambda_2\bar{\lambda}_2(\bar{\lambda}_2-\lambda_2\right\}u_3 \in \mathbb{C}, \\
C\,\Lambda(r_1^2 z\bar{z}) &= \lambda_1^3\bar{\lambda}_2(\lambda_1-\bar{\lambda}_2)^2u_0 - \lambda_1^2(\lambda_1^3-3\lambda_1\bar{\lambda}_2^2+2\bar{\lambda}_2^3)u_1 \\
&\quad + \lambda_1(2\lambda_1^3-3\lambda_1^2\bar{\lambda}_2+\bar{\lambda}_2^3)u_2 - \lambda_1(\lambda_1-\bar{\lambda}_2)^2u_3 \in \mathbb{C}\,.
\end{aligned}$$

In the case $\mathbf{C}(r_1 r_2 z\bar{z})$, we have $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0)$, where

$$\begin{aligned}
\Lambda(r_1r_2z\bar{z}) &= (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\bar{\lambda}_3-\lambda_1)(\lambda_3-\lambda_2)(\bar{\lambda}_3-\lambda_2)(\bar{\lambda}_3-\lambda_3) \in \mathbb{C}\setminus\mathbb{R}, \\
A\,\Lambda(r_1r_2z\bar{z}) &= \lambda_2\lambda_3\bar{\lambda}_3\left\{\lambda_2\lambda_3(\lambda_3-\lambda_2) - \lambda_2\bar{\lambda}_3(\bar{\lambda}_3-\lambda_2) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3-\lambda_3)\right\}u_0 \\
&\quad - \left\{\lambda_2^2\lambda_3^2(\lambda_3-\lambda_2) - \lambda_2^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_2) + \lambda_3^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_3)\right\}u_1 \\
&\quad + \left\{\lambda_2\lambda_3(\lambda_3^2-\lambda_2^2) - \lambda_2\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_2^2) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_3^2)\right\}u_2 \\
&\quad - \left\{\lambda_2\lambda_3(\lambda_3-\lambda_2) - \lambda_2\bar{\lambda}_3(\bar{\lambda}_3-\lambda_2) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3-\lambda_3)\right\}u_3 \in \mathbb{C}, \\
B\,\Lambda(r_1r_2z\bar{z}) &= -\lambda_1\lambda_3\bar{\lambda}_3\left\{\lambda_1\lambda_3(\lambda_3-\lambda_1) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3-\lambda_1) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3-\lambda_3)\right\}u_0 \\
&\quad + \left\{\lambda_1^2\lambda_3^2(\lambda_3-\lambda_1) - \lambda_1^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_1) + \lambda_3^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_3)\right\}u_1 \\
&\quad - \left\{\lambda_1\lambda_3(\lambda_3^2-\lambda_1^2) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_1^2) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_3^2)\right\}u_2 \\
&\quad + \left\{\lambda_1\lambda_3(\lambda_3-\lambda_1) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3-\lambda_1) + \lambda_3\bar{\lambda}_3(\bar{\lambda}_3-\lambda_3)\right\}u_3 \in \mathbb{C}, \\
C\,\Lambda(r_1r_2z\bar{z}) &= \lambda_1\lambda_2\bar{\lambda}_3\left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3-\lambda_1) + \lambda_2\bar{\lambda}_3(\bar{\lambda}_3-\lambda_2)\right\}u_0 \\
&\quad - \left\{\lambda_1^2\lambda_2^2(\lambda_2-\lambda_1) - \lambda_1^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_1) + \lambda_2^2\bar{\lambda}_3^2(\bar{\lambda}_3-\lambda_2)\right\}u_1 \\
&\quad + \left\{\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_1^2) + \lambda_2\bar{\lambda}_3(\bar{\lambda}_3^2-\lambda_2^2)\right\}u_2 \\
&\quad - \left\{\lambda_1\lambda_2(\lambda_2-\lambda_1) - \lambda_1\bar{\lambda}_3(\bar{\lambda}_3-\lambda_1) + \lambda_2\bar{\lambda}_3(\bar{\lambda}_3-\lambda_2)\right\}u_3 \in \mathbb{C}\,.
\end{aligned}$$

2 Results related to Subproblem II

Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$. The distribution of the values of $\cos(n\theta + \beta)$ as n varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, has been of much interest recently. To mention one instance, in the course of establishing the positivity of elements in a binary sequence, it is shown by Halava, Harju and Hirvensalo [Lemma 2.5], using elementary means that the inequality $\cos(n\theta + \beta) \geq 0$ cannot hold for all $n \in \mathbb{N}_0$. We endeavor here to carry out two particular tasks. In the next subsection, we improve upon the above-mentioned result of Halava-Harju-Hirvensalo by showing that both positive and negative values occur infinitely often. When θ is not a rational multiple of π, through the use of a theorem of Dirichlet in Diophantine approximation, we show even more that the value set of such cosine function is dense in the closed interval $[-1, 1]$.

Our second task arises from a result of Bell and Gerhold [Lemma 2.12], dealing with a linear combination of cosine functions which states that:
let $\theta_1, \ldots, \theta_d$ be rational numbers in the open interval $(0, 1)$, and let α_i, β_i be real numbers such that the purely periodic sequence

$$u_n = \sum_{i=1}^{d} \alpha_i \cos(2\pi\theta_i n + \beta_i)$$

is not identically zero. Then the sequence (u_n) takes both positive and negative values.

We shall obtain more information about the periodicity of the sequence (u_n). We show that the least period of the sequence (u_n) is exactly equal to T, the least common multiple of the denominators $t_1, \ldots, t_d$.

2.1 Positive and Negative Values

We begin with a result on negative values. For an angle γ, it is convenient to use the phrase γ mod $(-\pi, \pi]$, when we mean an angle $\gamma_0 \in (-\pi, \pi]$ for which $\gamma \equiv \gamma_0$ mod 2π. Clearly, the method employed here can also be adapted to examine the values taken by other trigonometric functions.

Proposition 3.3 Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$. There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) < 0$.

Proof. Assume that there are only finitely many $n \in \mathbb{N}_0$, say $n_1, \ldots, n_L$ such that

$$\cos(n\theta + \beta) < 0 \quad (n = n_1, \ldots, n_L).$$

Thus,

$$\cos\left(((n_L + 1)\theta + \beta) + k\theta\right) = \cos\left((n_L + 1 + k)\theta + \beta\right) \geq 0$$

for all $k \in \mathbb{N}_0$, which contradicts the result in Lemma 2.5 when φ is replaced by $(n_L + 1)\theta + \beta \bmod (-\pi, \pi]$. $\square$

Proposition 3.4 Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$.

(i) The relation $\cos(n\theta + \beta) \leq 0$ cannot hold for all $n \in \mathbb{N}_0$.

(ii) There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) > 0$.

Proof. (i) Assume that $\cos(n\theta + \beta) \leq 0$ for all $n \in \mathbb{N}_0$, which implies that for each $n \in \mathbb{N}_0$,

$$n\theta + \beta \in \left[\frac{-3\pi}{2} + k2\pi, \frac{-\pi}{2} + k2\pi\right],$$

for some $k \in \mathbb{Z}$. Then $\beta \in [-\pi, -\pi/2]$ or $\beta \in [\pi/2, \pi)$.

If $\beta \in [-\pi, -\pi/2]$, since $|\theta| < \pi$, we have

$$\theta + \beta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right].$$

Inductively, we obtain $n\theta + \beta \in [-3\pi/2, -\pi/2]$ for all $n \in \mathbb{N}_0$. Since $\theta \neq 0$, this implies $|n\theta + \beta| \to \infty \quad (n \to \infty)$, which is a contradiction.

If $\beta \in [\pi/2, \pi)$, since $|\theta| < \pi$, we have

$$\theta + \beta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Inductively, we obtain $n\theta + \beta \in [\pi/2, 3\pi/2]$ for all $n \in \mathbb{N}_0$. Since $\theta \neq 0$, this implies $|n\theta + \beta| \to \infty \quad (n \to \infty)$, which is again a contradiction.

(ii) Assume that there are only finitely many $n \in \mathbb{N}_0$, say $n_1, \ldots, n_L$, such that

$$\cos(n\theta + \beta) > 0 \quad (n = n_1, \ldots, n_L).$$

Thus,

$$\cos\left(((n_L + 1)\theta + \beta) + k\theta\right) = \cos\left((n_L + 1 + k)\theta + \beta\right) \leq 0$$

for all $k \in \mathbb{N}_0$, which contradicts the preceding lemma when β is replaced by $(n_L + 1)\theta + \beta \bmod (-\pi, \pi]$. $\square$

Next, we derive more information by invoking upon a result in Diophantine approximation known as the Dirichlet approximation theorem.

Theorem 3.5 Let $\theta, \beta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.

(a) If $\theta = s\pi/t$ ($s \in \mathbb{Z},\ t \in \mathbb{N},\ \gcd(s,t) = 1$) is a rational multiple of π, then the sequence $(\cos(n\theta + \beta))_{n\in\mathbb{N}_0}$ is purely periodic.

Moreover, if s is even, then the least period of the sequence $(\cos(n\theta + \beta))_{n\in\mathbb{N}_0}$ is t, while if s is odd, then its least period is $2t$.

(b) If θ is not a rational multiple of π, then as n varies over $\mathbb{N}_0$ the range of values of $\cos(n\theta + \beta)$ is dense in $[-1, 1]$.

Proof. (a) Since the cosine function is periodic of period 2π, the sequence

$$\cos(n\theta + \beta) = \cos\left(\frac{ns\pi}{t} + \beta\right) \quad (n \in \mathbb{N}_0)$$

is purely periodic with at most $2t$ values in a period, corresponding to $n = 0, 1, 2, \ldots,$ $2t - 1$, namely,

$$\cos\beta,\ \cos\left(\frac{s\pi}{t} + \beta\right),\ \cos\left(\frac{2s\pi}{t} + \beta\right), \ldots,\ \cos\left(\frac{s(2t-1)\pi}{t} + \beta\right).$$

If $t = 1$, then either $\theta = 2k\pi$ or $\theta = (2k+1)\pi$ for some $k \in \mathbb{Z}$. In either case, $\theta \notin [-\pi, \pi) \setminus \{-\pi, 0\}$, and so there is nothing to consider.

Let now $t \geq 2$. If s is even, say $s = 2k$, then t must be odd and the sequence

$$\cos(n\theta + \beta) = \cos\left(\frac{2nk\pi}{t} + \beta\right) \quad (n \in \mathbb{N}_0)$$

is periodic of period t with values in each period being

$$\cos\beta,\ \cos\left(\frac{2k\pi}{t} + \beta\right),\ \cos\left(\frac{4k\pi}{t} + \beta\right), \ldots,\ \cos\left(\frac{2k(t-1)\pi}{t} + \beta\right).$$

If t is not the least period, let its least period be ℓ. Thus $\ell \mid t,\ \ell < t$ and

$$\cos\beta = \cos\left(\frac{2k\ell\pi}{t} + \beta\right),\ \ \cos\left(\frac{2k\pi}{t} + \beta\right) = \cos\left(\frac{2k(\ell+1)\pi}{t} + \beta\right), \ldots$$

The first equality yields two possibilities

$$\frac{2k\ell\pi}{t} + \beta = 2N\pi \pm \beta \text{ for some } N \in \mathbb{Z}.$$

The possibility $2k\ell\pi/t + \beta = 2N\pi + \beta$ yields $N\pi = k\ell\pi/t$, which is impossible because $t \nmid k\ell,\ t \geq 3$ and so only the other possibility can hold which yields $\beta = N\pi - k\ell\pi/t$. Similarly, the second inequality yields $\beta = M\pi - 2k\pi/t - k\ell\pi/t$ for some $M \in \mathbb{Z}$. Equating the two values of β gives $(N-M)\pi = -2k\pi/t$ which is not tenable because $t \nmid (2k)$ for $t \geq 3$. Consequently, t is the least period.

If $s = 2k+1$ is odd, then the sequence

$$\cos(n\theta + \beta) = \cos\left(\frac{n(2k+1)\pi}{t} + \beta\right) \quad (n \in \mathbb{N}_0)$$

is periodic of period $2t$ with values in each period being

$$\cos\beta,\ \cos\left(\frac{(2k+1)\pi}{t} + \beta\right),\ \cos\left(\frac{2(2k+1)\pi}{t} + \beta\right), \\ \ldots, \cos\left(\frac{(2t-1)(2k+1)\pi}{t} + \beta\right).$$

It remains to show that $2t$ is the least period in this case. If $2t$ is not the least period, let the least period be m. Thus $m \mid 2t,\ m < 2t$ and

$$\cos\beta = \cos\left(\frac{(2k+1)m\pi}{t} + \beta\right), \\ \cos\left(\frac{(2k+1)\pi}{t} + \beta\right) = \cos\left(\frac{(2k+1)(m+1)\pi}{t} + \beta\right), \ldots$$

The first equality yields two possibilities

$$\frac{(2k+1)m\pi}{t} + \beta = 2N\pi \pm \beta \text{ for some } N \in \mathbb{Z}.$$

The possibility $(2k+1)m\pi/t + \beta = 2N\pi + \beta$ yields $N\pi = (2k+1)m\pi/2t$, which is impossible because $2t \nmid (2k+1)m,\ t \geq 2$ and so only the other possibility can hold which yields $2\beta = 2N\pi - (2k+1)m\pi/t$. Similarly, the second inequality yields $2\beta = 2M\pi - 2(2k+1)\pi/t - (2k+1)m\pi/t$ for some $M \in \mathbb{Z}$. Equating the two values of β gives $(N-M)\pi = -(2k+1)\pi/t$ which is not tenable because $t \nmid (2k+1)$, for $t \geq 2$. Consequently, $2t$ is the least period.

(b) Assume that θ is not a rational multiple of π, say $\theta = \vartheta \cdot 2\pi$, where $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. Writing

$$\vartheta = [\vartheta] + \xi,$$

where $[\vartheta]$ denotes its integer part, and $\xi := \{\vartheta\} \in (0,1)$ denotes its fractional part which must be irrational, for $n \in \mathbb{N}_0$, we have

$$\cos(n\theta + \beta) = \cos(n\vartheta 2\pi + \beta) = \cos(2n\xi\pi + \beta) = \cos\left(\{n\xi\}\, 2\pi + \beta\right).$$

Since ξ is irrational, by the Dirichlet's approximation theorem, see e.g. Corollary 2.3.2, we know the set $\{\{n\xi\}\ ;\ k \in \mathbb{N}\}$ is dense in [0,1]. Consequently, the set $\{\{n\xi\}2\pi\ ;\ n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$, implying that the range of values of $\cos\left(\{n\xi\}\, 2\pi + \beta\right)$ $(n \in \mathbb{N})$ is dense in $[-1, 1]$. $\square$

When θ is a rational multiple of π, the values in a period can be distinct or otherwise as seen in the next example.

Example 1. Take $\theta = s\pi/t = 2\pi/3,\ \beta = 0$. The sequence

$$(u_n)_{n\in\mathbb{N}_0} = \left(\cos\left(\frac{2n\pi}{3}\right)\right)_{n\in\mathbb{N}_0}$$

is periodic of length 3 with each period being

$$\cos(0) = 1,\ \ \cos\left(2\pi/3\right) = -1/2,\ \ \cos\left(4\pi/3\right) = -1/2,$$

showing that there are exactly $2 < 3 = t$ distinct sequence values.

Example 2. Take $\theta = s\pi/t = 2\pi/3,\ \beta = \pi/2$. The sequence

$$(u_n)_{n\in\mathbb{N}_0} = \left(\cos\left(\frac{\pi}{2} + \frac{2n\pi}{3}\right)\right)_{n\in\mathbb{N}_0}$$

is periodic of length 3 with each period being

$$\cos\left(\frac{\pi}{2}\right) = 0,\ \ \cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ \ \cos\left(\frac{\pi}{2} + \frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

showing that there are exactly $3 = t$ distinct sequence values.

2.2 Linear combination of cosine functions

In this section, we consider a linear combination of cosine functions. Such expression is always periodic as we now see.

Theorem 3.6 Let $\theta_1 = s_1/t_1, \ldots, \theta_d = s_d/t_d$ $(s_r, t_r \in \mathbb{N},\ \gcd(s_r, t_r) = 1)$ be rational numbers in the open interval $(0, 1)$ and let α_r, β_r be real numbers such that the purely periodic sequence

$$u_n = \sum_{r=1}^{d} \alpha_r \cos(2\pi\theta_r n + \beta_r)$$

is not identically zero. Let $T := l.c.m.(t_1, \ldots, t_d)$. Then the least period of the sequence (u_n) is a divisor of T.

Proof. As seen in the proof of Theorem 3.5, the sequence

$$u_n = \sum_{r=1}^{d} \alpha_r \cos\left(2n\pi\frac{s_r}{t_r} + \beta_r\right) \quad (n \in \mathbb{N}_0)$$

is purely periodic with a period T and the values in such a period correspond to $n \in \{0, 1, 2, \ldots, T-1\}$, namely,

$$\sum_{r=1}^{d} \alpha_r \cos\beta_r,\ \sum_{r=1}^{d} \alpha_r \cos\left(2\pi\frac{s_r}{t_r} + \beta_r\right),\ \sum_{r=1}^{d} \alpha_r \cos\left(4\pi\frac{s_r}{t_r} + \beta_r\right),$$
$$\ldots, \sum_{r=1}^{d} \alpha_r \cos\left(2(T-1)\pi\frac{s_r}{t_r} + \beta_r\right).$$

$\square$

From Theorem 3.6, there arises the natural question of determining when T is the least period. A rather surprising fact that T is always the least period of such sequence is now proved. To proceed further, let us note the following useful fact. Since

$$\sum_{n=0}^{T-1} \exp\left(2i\pi n s_r/t_r\right) = 0,$$

it follows that $\sum_{n=0}^{T-1} u_n = 0$.

Theorem 3.7 For $r \in \{1, 2, \ldots, d\}$, let

$$s_r/t_r \quad (s_r, t_r \in \mathbb{N},\ \ \gcd(s_r, t_r) = 1)$$

be rational numbers in the unit interval $(0, 1)$ and let $\alpha_r (\neq 0), \beta_r$ be real numbers. If the purely periodic non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r \cos\left(\frac{2\pi n s_r}{t_r} + \beta_r\right),$$

then its least period is equal to $T :=$ l.c.m.$(t_1, t_2, \ldots, t_d)$.

Proof. Let ℓ be the least period of (u_n). Since $\ell \mid T$, let $T = \ell L$ for some $L \in \mathbb{N}$. We proceed to prove the theorem by induction on d.

The case $d = 1$ is contained in Part (a) of Theorem 3.5. Assume now that $d \geq 2$ and that the theorem holds up to $d - 1$.

If $t_r \mid \ell$ for every $r \in \{1, 2, \ldots, d\}$, then $\ell = T$ and we are done. Suppose then that not all of the t_r's divide ℓ. If there are some t_r's that divide ℓ and some t_r's that do not divide ℓ. Without loss of generality, assume that ℓ is divisible by $t_1, t_2, \ldots, t_m$, but not divisible by $t_{m+1}, t_{m+2}, \ldots, t_d$ for some $m \in \{1, 2, \ldots, d-1\}$. For $k, j \in \mathbb{N}_0$, we have

$$\begin{aligned}
Lu_j =& \sum_{k=0}^{L-1} u_{k\ell+j} = \sum_{k=0}^{L-1} \sum_{r=1}^{d} \alpha_r \cos\left(\frac{2\pi s_r}{t_r}(k\ell + j) + \beta_r\right) \\
=& \sum_{r=1}^{m} \alpha_r \sum_{k=0}^{L-1} \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right) \\
&+ \sum_{r=m+1}^{d} \alpha_r \operatorname{Re}\left(\sum_{k=0}^{L-1} \exp\left(\frac{2i\pi s_r}{t_r}(k\ell + j) + i\beta_r\right)\right) \\
=& L \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right) \\
&+ \sum_{r=m+1}^{d} \alpha_r \operatorname{Re}\left(\exp\left(\frac{2i\pi s_r}{t_r} j + i\beta_r\right) \frac{1 - \exp\left(2i\pi \ell L s_r / t_r\right)}{1 - \exp\left(2i\pi \ell s_r / t_r\right)}\right) \\
=& L \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right).
\end{aligned}$$

Thus, $u_j = \sum_{r=1}^{m} \alpha_r \cos\left(\frac{2\pi s_r}{t_r} j + \beta_r\right)$, which shows that u_n has $m \leq d-1$ terms, the induction hypothesis finishes this case.

If none of the t_r's divides ℓ, the same argument as in the last steps shows that $u_n \equiv 0$, which is untenable. $\square$

We remark that the above proof can clearly be applied, with appropriate adjustments, to certain other trigonometric functions and/or periodic functions.

3 Results related to Subproblem III

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P. For θ ($\neq 0$) and b both belonging to $[0, P)$, the distribution of the values of $f(n\theta+b)$ for certain specific functions f, as n varies over the nonnegative integers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, has been of much interest recently. In the course of establishing the decidability of the positivity problem for binary sequences, Halava, Harju and Hirvensalo have shown that the inequality $\cos(n\theta+b) \geq 0$ cannot hold for all $n \in \mathbb{N}_0$ [Lemma 2.5]. In above section, it is shown that the sequence $(\cos(n\theta + b))_{n\in\mathbb{N}_0}$ takes infinitely many positive and negative values.

Here, we derive similar information about the values of $f(n\theta+b)$ for general f. First, when θ is a rational multiple of P, we show that the sequence $(f(n\theta + b))_{n\in\mathbb{N}_0}$ is purely periodic. When θ is not a rational multiple of P and f is continuous, through the use of a theorem of Dirichlet in Diophantine approximation, we show that the set of values of $f(n\theta + b)$ is dense in the range of f. Next, we deal with a linear combination of $f(n\theta + b)$'s with rational θ. Let $\theta_1 = s_1/t_1, \ldots, \theta_d = s_d/t_d$ be rational numbers, written in reduced fractions, in the open interval $(0, 1)$, and let $\alpha_i \in \mathbb{R},\ b_i \in [0, P)$ be such that the purely periodic sequence $(u_n)_{n\in\mathbb{N}_0}$ defined via

$$u_n = \sum_{i=1}^{d} \alpha_i f(P\theta_i n + b_i)$$

is not identically zero. When f is the cosine function, the least period of $(u_n)_{n\in\mathbb{N}_0}$ is shown in the above section to be equal to T, the least common multiple of the denominators of $t_1, \ldots, t_d$. This is, however, not true in general as seen in the following example.

Take $f(x) = \cos^2 x$, a function with least period π. The sequence

$$(f(\pi n/2) + f(\pi n/2 + \pi/2))_{n\in\mathbb{N}_0}$$

is constant and so is purely periodic of period $1 \neq 2 = \text{l.c.m.}(2, 2)$. It is thus natural to ask when the least period of $(u_n)_{n\in\mathbb{N}_0}$ is equal to T. Under the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0, \tag{26}$$

for all $\tau \in \mathbb{N}$, $\tau \geq 2$ and all $\beta \in [0, P)$, we show that the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is exactly equal to T. The same conclusion holds if

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0 \tag{27}$$

for all $\tau \in \mathbb{N}$, $\tau \geq 2$, $\tau \mid T$ and all $\beta \in [0, P)$. Note that (26) is a condition on f itself whereas (27) depends on both the function f and the rational numbers $\theta_1, \theta_2, \ldots, \theta_d$. In the last section, examples are given to illustrate the use of our main results.

3.1 A single function

We begin with some consequences of the two conditions (26) and (27).

Lemma 3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P and let $b \in [0, P)$. If $s, t \in \mathbb{N}$, $\gcd(s, t) = 1$, then

$$\sum_{n=0}^{t-1} f\left(\frac{Pn}{t} + b\right) = \sum_{n=0}^{t-1} f\left(\frac{Psn}{t} + b\right).$$

Proof. Since $\gcd(s, t) = 1$, the set $\{0, s, 2s, \ldots, s(t-1)\}$ is a complete residue system modulo t. By the periodicity of f, it follows that

$$\begin{aligned}&\left\{f(b), f\left(\frac{P}{t} + b\right), \ldots, f\left(\frac{(t-1)P}{t} + b\right)\right\}\\ &\quad = \left\{f(b), f\left(\frac{sP}{t} + b\right), \ldots, f\left(\frac{(t-1)sP}{t} + b\right)\right\}\end{aligned}$$

for any $s \in \mathbb{N}$ with $\gcd(s, t) = 1$, and the result follows. □

We remark that if (26) holds for some fixed $\tau = t$ and fixed $\beta = b$, then Lemma 3.8 and the periodicity of f together show that

$$\sum_{n=0}^{kt-1} f\left(\frac{Psn}{t} + b\right) = 0 \tag{28}$$

for any $k \in \mathbb{N}$.

Lemma 3.9 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P, $b \in [0, P)$ and $t, T \in \mathbb{N}$. If

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{t} + b\right) = 0$$

holds for some fixed $t \geq 2$ and $t \mid T$, then

$$\sum_{n=0}^{T-1} f\left(\frac{Psn}{t} + b\right) = 0 \tag{29}$$

holds for all $s \in \mathbb{N}$ with $\gcd(s, t) = 1$.

Proof. Let $T = tm \ (m \in \mathbb{N})$. By the periodicity of f, we have

$$m \sum_{n=0}^{t-1} f\left(\frac{Pn}{t} + b\right) = \sum_{n=0}^{T-1} f\left(\frac{Pn}{t} + b\right) = 0.$$

Lemma 3.8 thus implies that $\sum_{n=0}^{t-1} f\left(Psn/t + b\right) = 0$ where for $s \in \mathbb{N}$ with $\gcd(s, t) = 1$. The desired result follows at once by the preceding remark. $\square$

Theorem 3.10 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P and let $b \in [0, P)$.

(a) If $\theta = sP/t$ ($s \in \mathbb{Z}$, $t \in \mathbb{N}$, $\gcd(s, t) = 1$) is a rational multiple of P, then the sequence $\left(f(n\theta + b)\right)_{n \in \mathbb{N}_0}$ is purely periodic.

Moreover, if f satisfies the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0, \tag{30}$$

for all $\tau \in \mathbb{N}$, $\tau \geq 2$ and all $\beta \in [0, P)$, and the sequence $\left(f(n\theta + \beta)\right)$ is not identically zero, then its least period is equal to t.

(b) If θ is not a rational multiple of P and f is continuous, then as n varies over $\mathbb{N}_0$, the set of values of $f(n\theta + \beta)$ is dense in the range of f.

Proof. (a) Let $u_n = f(nsP/t + b)$. For each $k, m \in \mathbb{N}_0$, since

$$u_{kt+m} = f\left(\frac{(kt+m)sP}{t} + b\right) = f\left(\frac{msP}{t} + b\right) = u_m,$$

the sequence $(u_n)_{n \in \mathbb{N}_0}$ is purely periodic with period t.

Let ℓ be the least period of $(u_n)_{n \in \mathbb{N}_0}$ and assume that $\ell < t,\ t \geq 2$. Being the least period, it is easily checked that $\ell \mid t$ and since $\gcd(t/\ell, s) = 1$, the validity of (30) implies that of (28) which in turn shows that

$$tu_j = \sum_{k=0}^{t-1} u_{k\ell+j} = \sum_{k=0}^{t-1} f\left(\frac{(k\ell+j)Ps}{t} + b\right) = \sum_{k=0}^{t-1} f\left(\frac{kPs}{t/\ell} + \frac{jPs}{t} + b\right) = 0$$

for any $j \in \mathbb{N}_0$. contradicting the fact that the sequence is nonzero.
(b) Assume now that θ is not a rational multiple of P, and put $\theta = \nu P$ so that $\nu \in \mathbb{R}\backslash\mathbb{Q}$. Writing $\nu = [\nu]+\xi$, where $[\nu]$ denotes its integer part, and $\xi := \{\nu\} \in (0,1)$ denotes its fractional part which must be irrational, for $n \in \mathbb{N}_0$ we have

$$f(n\theta + b) = f(n\nu P + b) = f(n\xi P + b) = f(\{n\xi\}P + b).$$

Since ξ is irrational, by Kronecker's approximation theorem, Corollary 2.3, the set $\{\{n\xi\};\ n \in \mathbb{N}_0\}$ is dense in $[0,1]$. Consequently, the set $\{\{n\xi\}P + b\ ;\ n \in \mathbb{N}_0\}$ is dense in the interval $[b, P+b)$. By the continuity and periodicity of f, the set $\{f(\{n\xi\}P + b)\ ;\ n \in \mathbb{N}_0\}$ is thus dense in the range of f. $\square$

Regarding the result about least period in the last part of Theorem 3.10 (a), apart from the condition (30), there are also other condition(s) ensuring the least period being t. We give another one in the next theorem.

Theorem 3.11 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P, $b \in [0, P)$ and let $s, t \in \mathbb{N}$ with $\gcd(s,t) = 1$. Assume that for each y in the range of f, there are only finitely many, say $x_1, \ldots, x_R \in [0, P)$ such

$$f(x_i) = y \quad (i = 1, \ldots, R) \tag{31}$$

and that $x_i - x_j$ is not a rational multiple of P for all $i, j \in \{1, \ldots, R\}$, $i \neq j$. If the sequence $(f(nsP/t + b))_{n\in\mathbb{N}_0}$ is not identically zero, then it is purely periodic with least period equal to t.

Proof. That the sequence is purely periodic with period t follows directly from the proof of the first part of Theorem 3.10 (a). To show that t is the least period, suppose that its least period $t'(\in \mathbb{N})$ is $< t$, $t \geq 2$. Thus, $f(t'sP/t + b) = f(b)$. Writing

$$\frac{st'}{t} = \left[\frac{st'}{t}\right] + \left\{\frac{st'}{t}\right\},$$

where $[st'/t]$ denotes its integer part, and $\{st'/t\} \in (0,1) \cap \mathbb{Q}$ denotes its fractional part, we obtain

$$f\left(\left\{\frac{st'}{t}\right\} P + b\right) = f(b).$$

By hypothesis, let f take the value $f(b)$ at $x \in \{b = b_1, \ldots, b_R\} \subset [0, P)$. Since $\{st'/t\}P + b \in (0, 2P)$, we deduce that

$$\left\{\frac{st'}{t}\right\} P + b \in \{b_1, \ldots, b_R, P + b_1 \ldots, P + b_R\},$$

which is a contradiction because $\{t's/t\}P \notin \{0, P\}$ and each $b_i - b_j$ $(i \neq j)$ is not a rational multiple of P. $\square$

3.2 Linear Combination of Functions

Our next theorem deals with a linear combination of $f(n\theta + b)$'s, generalizing Theorem 3.10 (a).

Theorem 3.12 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P. Assume f satisfies the condition

$$\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0, \tag{32}$$

for all $\tau \in \mathbb{N}$, $\tau \geq 2$ and all $\beta \in [0, P)$. For $d \in \mathbb{N}$, $r \in \{1, 2, \ldots, d\}$, let

$$\frac{s_r}{t_r} \quad (s_r, t_r \in \mathbb{N}, \ \gcd(s_r, t_r) = 1)$$

be rational numbers in the unit interval $(0, 1)$, and let $\alpha_r \in \mathbb{R} \setminus \{0\}$, $b_r \in [0, P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n \in \mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r} + b_r\right),$$

then its least period is equal to $T := \text{l.c.m.}(t_1, t_2, \ldots, t_d)$.

Proof. Clearly, the sequence (u_n) is purely periodic with period T. Let ℓ be its least period so that $\ell \mid T$. We proceed to prove the theorem by induction on d. The case $d = 1$ is contained in Theorem 3.10 (a). Assume now that $d \geq 2$ and that the theorem holds up to $d - 1$.

If $t_r \mid \ell$ for every $r \in \{1, 2, \ldots, d\}$, then $\ell = T$ and we are done.

Suppose then that not all of the t_r's divide ℓ. We treat first the case where there are some t_r's that divide ℓ and some t_r's that do not divide ℓ. Without loss of generality, assume that ℓ is divisible by $t_1, t_2, \ldots, t_m$, but not divisible by

$t_{m+1}, t_{m+2}, \ldots, t_d$ for some $m \in \{1, 2, \ldots, d-1\}$. For $k, j \in \mathbb{N}_0$, we have

$$\begin{aligned}
Tu_j &= \sum_{k=0}^{T-1} u_{k\ell+j} = \sum_{k=0}^{T-1}\sum_{r=1}^{d} \alpha_r f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right) \\
&= \sum_{r=1}^{m} \alpha_r \sum_{k=0}^{T-1} f\left(\frac{Ps_r}{t_r}j+b_r\right) + \sum_{r=m+1}^{d} \alpha_r \sum_{k=0}^{T-1} f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right) \\
&= T\sum_{r=1}^{m} \alpha_r f\left(\frac{Ps_r}{t_r}j+b_r\right) + \sum_{r=m+1}^{d} \alpha_r \sum_{k=0}^{T-1} f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right). \qquad (33)
\end{aligned}$$

In the second sum on the right hand side of the last expression, writing

$$\sum_{k=0}^{T-1} f\left(\frac{Ps_r}{t_r}(k\ell+j)+b_r\right) = \sum_{k=0}^{T-1} f\left(\frac{Ps_r k\ell_r'}{t_r'}+\gamma_r\right), \qquad (34)$$

where $\gcd(\ell_r', t_r') = 1$, $t_r' \mid T$ (as $t_r \mid T$) and γ_r is equal to $\frac{Ps_r j}{t_r} + b_r$ reduced mod P. The condition (32) implies that (28) holds which in turn implies

$$\sum_{k=0}^{T-1} f\left(\frac{Ps_r k\ell_r'}{t_r'}+\gamma_r\right) = 0.$$

Thus, (33) gives $u_j = \sum_{r=1}^{m} \alpha_r f\left(\frac{Ps_r}{t_r}j+b_r\right)$, which shows that $(u_j)_{j\in\mathbb{N}_0}$ has $m \le d-1$ terms, the induction hypothesis finishes this case.

If none of the t_r's divides ℓ, the same arguments as in the last steps show that $u_n \equiv 0$, which is untenable. $\square$

Apart from (26), there are other sufficient conditions for the least period to be T. We provide one in the next theorem.

Theorem 3.13 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P. For $d \in \mathbb{N}$, $r \in \{1, 2, \ldots, d\}$, let s_r/t_r ($s_r, t_r \in \mathbb{N}$, $\gcd(s_r, t_r) = 1$) be rational numbers in the unit interval $(0, 1)$, $\alpha_r \in \mathbb{R}\backslash\{0\}$, $b_r \in [0, P)$ and $T :=$ l.c.m.$(t_1, t_2, \ldots, t_d)$. Assume f satisfies the condition

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{\tau}+\beta\right) = 0 \qquad (35)$$

for all $\tau \in \mathbb{N}$, $\tau \ge 2$, $\tau \mid T$ and all $\beta \in [0, P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r}+b_r\right),$$

then its least period is equal to T.

Proof. Proceed exactly as in the proof of Theorem 3.12 up to (34). Now use the condition (35) and Lemma 3.9 to deduce that (29) holds which in turn gives

$$\sum_{k=0}^{T-1} f\left(\frac{Ps_r k\ell'_r}{t'_r} + \gamma_r\right) = 0.$$

The same conclusion as at the end of the proof of Theorem 3.12 finishes the proof.

□

3.3 Examples

In this subsection, we work out some examples to illustrate the use and the necessity of the conditions of our results. The first shows that analogous result in above section is a special case when the condition (26) is fulfilled.

Example 1. Let $d \in \mathbb{N}$; $\rho_1, \rho_2 \in \mathbb{R}$; $\lambda \in \mathbb{R}^+ := \{x \in \mathbb{R} \ ; \ x > 0\}$ and $\gamma_1, \gamma_2 \in [0, 2\pi/\lambda)$. Consider $f(x) = \rho_1 \cos(\lambda x + \gamma_1) + \rho_2 \sin(\lambda x + \gamma_2) \not\equiv 0$, which is a periodic continuous function with least period $2\pi/\lambda$. Define the sequence $(u_n)_{n\in\mathbb{N}_0}$ by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{2\pi n s_r}{\lambda t_r} + b_r\right),$$

where $s_r, t_r \in \mathbb{N}$, $t_r \geq 2$, $\gcd(s_r, t_r) = 1$, $\alpha_r \in \mathbb{R} \setminus \{0\}$ $(r = 1, \ldots, d)$, $b_r \in [0, 2\pi/\lambda)$ and $T := \text{l.c.m.}(t_1, t_2, \ldots, t_d)$. By Theorem 3.12, to see that the sequence $(u_n)_{n\in\mathbb{N}_0}$ has least period T, we need only check that f satisfies the condition (26), i.e.,

$$\sum_{n=0}^{\tau-1} \left\{\rho_1 \cos\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) + \rho_2 \sin\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right)\right\} = 0 \qquad (36)$$

for all $\tau \geq 2$, and $\beta \in [0, 2\pi/\lambda)$. Since

$$\sum_{n=0}^{\tau-1} \cos\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) = \sum_{n=0}^{\tau-1} \Re\left\{\exp\ i\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right)\right\} = 0,$$

and similarly, $\sum_{n=0}^{\tau-1} \sin\left(\frac{2n\pi}{\tau} + \lambda\beta + \gamma_1\right) = 0$, the condition (36) is verified.

Moreover, the function f in Example 1 satisfies the conditions of both Theorems 3.10 and 3.11.

Our second example provides an uncountable class of well-behaved functions satisfying the condition (27).

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and periodic with least period P. Then, f possesses a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{2m\pi x}{P} + b_m \sin \frac{2m\pi x}{P} \right) \qquad (x \in \mathbb{R}). \tag{37}$$

Define the sequence $(u_n)_{n\in\mathbb{N}_0}$ via

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r} + B_r\right),$$

where $d, s_r, t_r \in \mathbb{N}$, $t_r \geq 2$, $\gcd(s_r, t_r) = 1$, $\alpha_r \in \mathbb{R} \setminus \{0\}$, $B_r \in [0, P)$. Let $T := \text{l.c.m.}(t_1, t_2, \ldots, t_d)$. Consider the set, S, of such functions f for which their Fourier coefficients are subject to the conditions

$$a_0 = a_m = b_m = 0 \text{ for all } m \mid T,\ m \in \mathbb{N}.$$

For $f \in S$, the condition (27) is fulfilled, and so Theorem 3.13 confirms that the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is T.

The next illustration gives an example where the condition (26) does not hold but the condition (27) does.

Example 3. Let A and P be positive real numbers and consider

$$f(x) = \begin{cases} 4Ax/P & \text{if } 0 \leq x < P/4, \\ -4Ax/P + 2A & \text{if } P/4 \leq x < 3P/4, \\ 4Ax/P - 4A & \text{if } 3P/4 \leq x < P \end{cases}$$

The function f is continued throughout $\mathbb{R}$ by periodicity with least period P. For $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and $b_1, b_2 \in [0, P)$, define the sequence $(u_n)_{n\in\mathbb{N}_0}$ by

$$u_n = \alpha_1 f\left(\frac{Pn}{2} + b_1\right) + \alpha_2 f\left(\frac{3Pn}{4} + b_2\right).$$

Since $\sum_{n=0}^{2} f(Pn/3 + P/2) \neq 0$, the function f does not satisfy the condition (26). To verify whether the condition (27) holds, we need only prove that

$$\sum_{n=0}^{3} f\left(Pn/t + \beta\right) = 0$$

when $t = 2$ or $t = 4$ and $\beta \in [0, P)$. For $t = 2$, we have

$$\sum_{n=0}^{3} f\left(\frac{Pn}{2} + \beta\right) = 2\left(f(\beta) + f\left(\frac{P}{2} + \beta\right)\right) = 0.$$

Applying this to the case $t = 4$, we obtain

$$\sum_{n=0}^{3} f\left(\frac{Pn}{4} + \beta\right) = \left(f(\beta) + f\left(\frac{P}{2} + \beta\right)\right) + \left(f\left(\frac{P}{4} + \beta\right) + f\left(\frac{3P}{4} + \beta\right)\right) = 0.$$

By Theorem 3.13, the least period of the sequence $(u_n)_{n\in\mathbb{N}_0}$ is equal to l.c.m.$(2, 4) = 4$.

Observe that the function f in Example 3 satisfies neither the condition (30) of Theorem 3.10 (a) nor the finite value-attainment condition related to (31) of Theorem 3.11.

Our final illustration provides an example of a wildly behaved non-continuous function.

Example 4. Let

$$f(x) = \begin{cases} \sin 2\pi x & \text{if } \ x \in \mathbb{Q}, \\ 0 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Clearly, $f(x + 1) = f(x) \quad (x \in \mathbb{R})$. To show that 1 is the least period of f, we first note that any irrational number $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ cannot be a period of f because if we take $x = 1/3 - \gamma$, then $f(x + \gamma) = \sin 2\pi/3 \neq 0 = f(x)$. Next, let $\alpha \in \mathbb{Q}$ and $0 < \alpha < 1$ and suppose that $f(x + \alpha) = f(x)$. Then, for each $x \in \mathbb{Q}$, we get

$$\sin(2\pi x + 2\pi\alpha) = \sin 2\pi x. \tag{38}$$

Taking $x = 0$, we must have $\alpha = 1/2$. Substituting $\alpha = 1/2$ into (38) shows $\sin 2\pi x = 0$ for each $x \in \mathbb{Q}$, which is a contradiction. We conclude then that 1 is the least period of f.

Consider the sequence $(u_n)_{n\in\mathbb{N}_0}$ defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{ns_r}{t_r} + b_r\right),$$

where $s_r, t_r \in \mathbb{N}$, $t_r \geq 2$, $\gcd(s_r, t_r) = 1$, $\alpha_r \in \mathbb{R} \setminus \{0\}$, $b_r \in [0, 1)$. Let $T :=$ l.c.m.$(t_1, t_2, \ldots, t_d)$. To see that the sequence $(u_n)_{n\in\mathbb{N}_0}$ has the least period T, we verify the condition (26), i.e., $\sum_{n=0}^{\tau-1} f\left(n/\tau+\beta\right)=0$, for all $\tau\in\mathbb{N},\ \tau\geq 2$, and all $\beta\in[0,1)$. If $\beta\in\mathbb{Q}$, then

$$\sum_{n=0}^{\tau-1} f\left(\frac{n}{\tau}+\beta\right)=\sum_{n=0}^{\tau-1}\sin\left(\frac{2\pi n}{\tau}+2\pi\beta\right)=\sum_{n=0}^{\tau-1}\Im\left(\exp i\left(\frac{2\pi n}{\tau}+2\pi\beta\right)\right)=0.$$

If $\beta\in\mathbb{R}\setminus\mathbb{Q}$, then by the definition of f, each term $f\left(n/\tau+\beta\right)$ is 0. Thus,

$$\sum_{n=0}^{\tau-1} f\left(n/\tau+\beta\right)=0.$$

By Theorem 3.12, the least period of $(u_n)_{n\in\mathbb{N}_0}$ is equal to T.

This last example also gives us a function satisfying the condition (30) of Theorem 3.10 (a) but does not satisfy the condition (31) of Theorem 3.11.

4 Results related to the main problem

4.1 Auxiliary Lemmas

In this subsection, we collect those auxiliary lemmas essential for the two proofs of our main theorem.

Lemma 3.14 Let $\varphi,\theta\in[-\pi,\pi)$ with $\theta\notin\{-\pi,0\}$.

I. If $\theta=s\pi/t$ is a rational multiple of π where $s,t\ (>0)\in\mathbb{Z}\setminus\{0\}$, $\gcd(s,t)=1$, then as n varies over $\mathbb{N}\cup\{0\}$, the function $\cos(\varphi+n\theta)$ is periodic and takes at most $2t$ explicitly computable distinct values corresponding to $n=0,1,\ldots,2t-1$.

II. If θ is not a rational multiple of π, then as n varies over $\mathbb{N}\cup\{0\}$, the range of values of $\cos(\varphi+n\theta)$ is dense in $[-1,1]$.

III. The function $\cos(\varphi+n\theta)$ takes both positive and negative values for infinitely many $n\in\mathbb{N}\cup\{0\}$.

Proof. Part I is contained in the statement and proof of Lemma 2.9 (a), part II is Lemma 2.9 (b), and part III is Propositions 3.3 and 3.4. □

The results in the next lemma are shapes of sequence elements that have already been shown to be decidable; those proved for second order recurrence sequences from Section 4 of Chapter 2 are in Part I, and those proved for third order recurrence sequences from Section 5 of Chapter 2 are in Part II.

Lemma 3.15 The Positivity Problem for the following forms of sequence elements are decidable.

Part I.

(HHH1) $u_n = A\lambda_1^n + B\lambda_2^n \quad (A, B \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\})$,

(HHH2) $u_n = (A + Bn)\lambda^n \quad (A, B \in \mathbb{R};\ \lambda \in \mathbb{R}\setminus\{0\})$,

(HHH3) $u_n = A\lambda^n + \bar{A}\bar{\lambda}^n \quad (A \in \mathbb{C};\ \lambda \in \mathbb{C}\setminus\mathbb{R})$.

Part II.

(LT1) $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\})$,

(LT2) $u_n = (A + Bn + Cn^2)\lambda^n \quad (A, B, C \in \mathbb{R};\ \lambda \in \mathbb{R}\setminus\{0\})$,

(LT3) $u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\})$,

(LT4) $u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (A \in \mathbb{R};\ \lambda_1 \in \mathbb{R}\setminus\{0\};\ B \in \mathbb{C};\ \lambda_2 \in \mathbb{C}\setminus\mathbb{R})$.

4.2 First (elementary) Proof of the Main Theorem

The first proof of our main theorem is long yet purely elementary. For cases whose proofs also appear in the second proof, we merely refer to the details provided in the second proof.

We subdivide the proof into two cases depending on whether the characteristic polynomial $\mathrm{Char}(z)$ has only real roots.

Case 1. Proof of the Main Theorem when Char(z) has only real roots.

In this case, the general term of the sequence is

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0,\ m \leq 4),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct nonzero real numbers and

$$P_i(n) = A_{i,1} + A_{i,2}n + \cdots + A_{i,\ell_i}n^{\ell_i - 1} \in \mathbb{R}[n] \quad (\ell_i \in \mathbb{N};\ i = 1, 2, \ldots, m;\ A_{i,\ell_i} \neq 0),$$

with $\ell_1 + \ell_2 + \cdots + \ell_m = 4$. We have two possibilities to consider.

1.1 There are two roots λ_i, λ_j $(i, j \in \{1, 2, \ldots, m;\ i \neq j)$ such that $|\lambda_i| = |\lambda_j|$.

For a detailed proof, see Subcase 1.1 in the next subsection.

1.2 All roots have different absolute values.

Without loss of generality, assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_m|$. Here,

$$u_n = \lambda_1^n \left\{P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n\right\} \quad (n \geq 0).$$

We treat two subcases depending on the sign of λ_1.

- $\lambda_1 < 0$.

Since $\mathrm{sign}(P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n) = \mathrm{sign}(A_{1,\ell_1})$ when n is large enough and $\mathrm{sign}(\lambda_1^n)$ oscillates, the sequence (u_n) is nonnegative only when $A_{1,\ell_1} = 0$, which contradicts the definition of $A_{1,\ell_1} = 0$

- $\lambda_1 > 0$.

For a detailed proof, see Subcase 1.2 in the next subsection.

Case 2. Proof of the Main Theorem when Char(z) has non-real roots.

2.1 C($z_1\bar{z}_1z_2\bar{z}_2$).

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0), \tag{39}$$

where $A, B \in \mathbb{C}$ and $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_1 = |\lambda_1|e^{i\theta_1}$, $\lambda_2 = |\lambda_2|e^{i\theta_2}$, $A = |A|e^{i\varphi_1}$ and $B = |B|e^{i\varphi_2}$ where $\theta_1, \theta_2, \varphi_1, \varphi_2 \in [-\pi, \pi]$, $\theta_1, \theta_2 \notin \{-\pi, 0\}$ so that

$$u_n = 2\left\{|\lambda_1|^n|A|\cos(\varphi_1 + n\theta_1) + |B||\lambda_2|^n\cos(\varphi_2 + n\theta_2)\right\}.$$

Without loss of generality, we need only treat two possibilities $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| = |\lambda_2|$.

2.1.1 $|\lambda_1| > |\lambda_2|$ (> 0).

Here,

$$u_n = 2|\lambda_1|^n\left\{|A|\cos(\varphi_1 + n\theta_1) + |B|\left(|\lambda_2/\lambda_1|\right)^n\cos(\varphi_2 + n\theta_2)\right\}.$$

Thus, (u_n) is nonnegative if and only if

$$|A|\cos(\varphi_1 + n\theta_1) + |B|\left(|\lambda_2|/|\lambda_1|\right)^n\cos(\varphi_2 + n\theta_2) \geq 0. \tag{40}$$

By Lemma 3.14, $|A|\cos(\varphi_1+n\theta_1)$ takes some fixed positive and negative values for infinitely many n provided $A\neq 0$, and since

$$|B|\,(|\lambda_2/\lambda_1|)^n\cos(\varphi_2+n\theta_2)\to 0\ (n\to\infty),$$

the requirement (40) holds if and only if $A=0$, and thus $u_n=B\lambda_2^n+\bar{B}\bar{\lambda}_2^n\ \ (n\geq 0)$, which is of the form (HHH3) in Lemma 3.15.

2.1.2 $|\lambda_1|=|\lambda_2|$.

Here,

$$u_n=2|\lambda_1|^n\left\{|A|\cos(\varphi_1+n\theta_1)+|B|\cos(\varphi_2+n\theta_2)\right\}.$$

Notice that the arguments used in the preceding sub-case do not work here. This is the first situation where our analysis has to get back to the shape of the roots of the characteristic polynomial. Tracing back to the proofs of Theorems 3.1 and 3.2, we see that the case $\mathrm{C}(z_1\bar{z}_1z_2\bar{z}_2)$ occurs in Case I: $\mathcal{D}>0$, with two distinct complex conjugate pairs, which are labeled I.1.1.2, I.1.2, I.2.1.2 and I.2.2. Using the notation of Theorem 3.1, the corresponding information is as follows:

I.1.1.2 $\beta=0,\ \alpha^2-4\gamma\geq 0$ and $\alpha>0$. We have $x_1=\frac{1}{2}(\Box+\heartsuit)i,\ x_2=-\frac{1}{2}(\Box+\heartsuit)i=\bar{x}_1,\ x_3=\frac{1}{2}(\Box-\heartsuit)i,\ x_4=-\frac{1}{2}(\Box-\heartsuit)i=-\bar{x}_3$, with $\Box\neq\pm\heartsuit$ being two nonzero real numbers. The roots of the characteristic polynomial, denoted for unambiguity by $\tilde{\lambda}$, are $\tilde{\lambda}_1=\frac{a_1}{4}+\frac{1}{2}(\Box+\heartsuit)i,\ \tilde{\lambda}_2=\frac{a_1}{4}-\frac{1}{2}(\Box+\heartsuit)i=\bar{\tilde{\lambda}}_1,\ \tilde{\lambda}_3=\frac{a_1}{4}+\frac{1}{2}(\Box-\heartsuit)i,\ \tilde{\lambda}_4=\frac{a_1}{4}-\frac{1}{2}(\Box-\heartsuit)i=-\bar{\tilde{\lambda}}_3$. Combining with the condition and notation of our on-going case, we get

$$|\tilde{\lambda}_1|^2=|\tilde{\lambda}_3|^2\Longleftrightarrow\left(\frac{a_1}{4}\right)^2+\frac{1}{4}(\Box+\heartsuit)^2=\left(\frac{a_1}{4}\right)^2+\frac{1}{4}(\Box-\heartsuit)^2\Longleftrightarrow\Box\heartsuit=0,$$

which is a contradiction and we are done in this case.

I.1.2 $\beta=0,\ \alpha^2-4\gamma<0$. We have $x_1=\frac{1}{2}(v'+\heartsuit i),\ x_2=\frac{1}{2}(-v'-\heartsuit i)=-x_1,\ x_3=\frac{1}{2}(v'-\heartsuit i)=\bar{x}_1,\ x_4=\frac{1}{2}(-v'+\heartsuit i)=-\bar{x}_1$. Then $\tilde{\lambda}_1=\left(\frac{a_1}{4}+\frac{v'}{2}\right)+\frac{\heartsuit}{2}i,\ \tilde{\lambda}_2=\left(\frac{a_1}{4}-\frac{v'}{2}\right)-\frac{\heartsuit}{2}i,\ \tilde{\lambda}_3=\left(\frac{a_1}{4}+\frac{v'}{2}\right)-\frac{\heartsuit}{2}i=\bar{\tilde{\lambda}}_1,\ \tilde{\lambda}_4=\left(\frac{a_1}{4}-\frac{v'}{2}\right)+\frac{\heartsuit}{2}i=\bar{\tilde{\lambda}}_2$. Combining with the condition and notation of our on-going case, we get

$$\begin{aligned}|\tilde{\lambda}_1|^2=|\tilde{\lambda}_2|^2&\Longleftrightarrow\left(\frac{a_1}{4}+\frac{v'}{2}\right)^2+\left(\frac{\heartsuit}{2}\right)^2=\left(\frac{a_1}{4}-\frac{v'}{2}\right)^2+\left(\frac{\heartsuit}{2}\right)^2\\&\Longleftrightarrow\frac{a_1v'}{2}=0\ \ (v'\neq 0)\Longleftrightarrow a_1=0.\end{aligned}$$

Thus, $\tilde{\lambda}_1 = \frac{v'}{2} + \frac{\heartsuit}{2}i, \quad \tilde{\lambda}_2 = -\frac{v'}{2} - \frac{\heartsuit}{2}i = -\tilde{\lambda}_1$.

I.2.1.2 $\beta \neq 0,\ \alpha^2 - 4\gamma \geq 0$ and $\alpha > 0$. We have

$$x_1 = \frac{1}{2}(u' + i\heartsuit) \in \mathbb{C} \setminus \mathbb{R},\ x_2 = \bar{x}_1,\ x_3 = -x_1,\ x_4 = -\bar{x}_1$$

yielding $\frac{a_1}{4} + \frac{u'}{2} + i\frac{\heartsuit}{2} = \tilde{\lambda}_1 = \bar{\tilde{\lambda}}_2, \quad \frac{a_1}{4} - \frac{u'}{2} - i\frac{\heartsuit}{2} = \tilde{\lambda}_3 = \bar{\tilde{\lambda}}_4$. Combining with the condition and notation of our on-going case, we get

$$\begin{aligned} |\tilde{\lambda}_1|^2 = |\tilde{\lambda}_3|^2 &\iff \left(\frac{a_1}{4} + \frac{u'}{2}\right)^2 + \left(\frac{\heartsuit}{2}\right)^2 = \left(\frac{a_1}{4} - \frac{u'}{2}\right)^2 + \left(\frac{\heartsuit}{2}\right)^2 \\ &\iff \frac{a_1 u'}{2} = 0 \ \ (u' \neq 0) \iff a_1 = 0. \end{aligned}$$

Thus, $\tilde{\lambda}_1 = \frac{u'}{2} + \frac{\heartsuit}{2}i, \quad \tilde{\lambda}_3 = -\frac{u'}{2} - \frac{\heartsuit}{2}i = -\tilde{\lambda}_1$.

I.2.2 $\beta \neq 0,\ \alpha^2 - 4\gamma < 0$. The roots of $p(x + a_1/4)$ are of the same shape as those in the subcase I.2.1.2.

Collecting together all possibilities in I.1.1.2, I.1.2, I.2.1.2 and I.2.2, the two roots in (39) can only be of the form $\lambda_2 = -\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, and the general term of the sequence in (39) is thus of the form $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n + B(-\lambda_1)^n + \bar{B}(-\bar{\lambda}_1)^n$, i.e., for $k \geq 0$,

$$u_{2k} = (A + B)\lambda_1^{2k} + (\bar{A} + \bar{B})\bar{\lambda}_1^{2k}, \quad u_{2k+1} = (A - B)\lambda_1^{2k+1} + (\bar{A} - \bar{B})\bar{\lambda}_1^{2k+1}.$$

Both u_{2k} and u_{2k+1} are of the form (HHH3) and so are decidable by Lemma 3.15.

2.2 C($z^2\bar{z}^2$).

In this case, the general term of the sequence is

$$u_n = (A + Bn)\lambda_1^n + (\bar{A} + \bar{B}n)\bar{\lambda}_1^n \ (n \geq 0),$$

where $A, B \in \mathbb{C}$ and $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_1 = |\lambda_1|e^{i\theta},\ A = |A|e^{i\varphi_1},\ B = |B|e^{i\varphi_2}$ where $\theta, \varphi_1, \varphi_2 \in [-\pi, \pi),\ \theta \neq \{-\pi, 0\}$. Thus,

$$u_n = 2|\lambda_1|^n \left\{|A|\cos(\varphi_1 + n\theta) + n|B|\cos(\varphi_2 + n\theta)\right\}.$$

Then the sequence (u_n) is nonnegative if and only if for all n,

$$|A|\cos(\varphi_1 + n\theta) + n|B|\cos(\varphi_2 + n\theta) \geq 0. \tag{41}$$

Since $\mathrm{sign}(|A|\cos(\varphi_1+n\theta)+n|B|\cos(\varphi_2+n\theta)) = \mathrm{sign}(\cos(\varphi_2+n\theta))$ when n is large enough, provided $B\cos(\varphi_2+n\theta) \neq 0$. By Lemma 3.15, $\cos(\varphi_2+n\theta)$ takes some positive and some negative values for infinitely many $n \in \mathbb{N}$. Thus, (41) holds only when $B = 0$, and so $u_n = A\lambda_1^n + \bar{A}\bar{\lambda}_1^n$, which is of the form (HHH3) in Lemma 3.15.

2.3 C($r_1^2 z\bar{z}$).

In this case, the general term of the sequence is

$$u_n = (A+Bn)\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \ (n \geq 0),$$

where $A, B, \lambda_1(\neq 0) \in \mathbb{R}$, $C \in \mathbb{C}$ and $\lambda_3 \in \mathbb{C}\setminus\mathbb{R}$. Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = (A+Bn)\lambda_1^n + 2|C||\lambda_3|^n\cos(\varphi+n\theta).$$

We subdivide into three sub-cases depending on the absolute values of $|\lambda_1|$ and $|\lambda_3|$.

2.3.1 $|\lambda_1| = |\lambda_3|$.

There are two further possibilities.

2.3.1(1) $\lambda_1 < 0$.

Here,

$$u_n = |\lambda_1|^n\{(-1)^n(A+Bn) + 2|C|\cos(\varphi+n\theta)\} \ (n \geq 0).$$

Since $(-1)^n(A+Bn) + 2|C|\cos(\varphi+n\theta) \to \pm\infty \quad (n \to \infty)$ according as n is even or odd provided $B \neq 0$, the sequence (u_n) is nonnegative only when $B = 0$ and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) in Lemma 3.15.

2.3.1(2) $\lambda_1 > 0$.

For a detailed proof, see Subcase 1 of $C(r_1^2 z\bar{z})$ in the next subsection.

2.3.2 $|\lambda_1| > |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + Bn + 2|C|(|\lambda_3|/\lambda_1)^n\cos(\varphi+n\theta)\} \quad (n \geq 0).$$

We consider two possibilities corresponding to the signs of λ_1.

2.3.2(1) $\lambda_1 < 0$.

Since $\operatorname{sign}(A + Bn + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)) = \operatorname{sign}(B)$ when n is large enough provided $B \neq 0$, and $\operatorname{sign}(\lambda_1^n)$ oscillates, the sequence (u_n) is nonnegative only when $B = 0$, and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) in Lemma 3.15.

2.3.2(2) $\lambda_1 > 0$.

For detailed proof, see subcase 2 of $C(r_1^2 z\bar{z})$ in next subsection.

2.3.3 $|\lambda_1| < |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = |\lambda_3|^n \left\{(A + Bn)(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi + n\theta)\right\} \quad (n \geq 0).$$

Observe that $(A + Bn)(\lambda_1/|\lambda_3|)^n \to 0$ $(n \to \infty)$. By Lemma 3.14, $\cos(\varphi + n\theta)$ takes some fixed positive and some fixed negative values for infinitely many n. Thus, the sequence (u_n) is nonnegative only when $C = 0$, yielding $u_n = (A + Bn)\lambda_1^n$, which is of the form (HHH2) in Lemma 3.15.

2.4 C($r_1 r_2 z\bar{z}$).

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \ (n \geq 0),$$

where $A, B \in \mathbb{R},\ C \in \mathbb{C},\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$.
Let $\lambda_3 = |\lambda_3| e^{i\theta},\ C = |C| e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi),\ \theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + B\lambda_2^n + 2|C||\lambda_3|^n \cos(\varphi + n\theta).$$

We split our consideration into three possibilities.

1. The three λ's have the same absolute values, i.e., $|\lambda_1| = |\lambda_2| = |\lambda_3|$.
2. There are two λ_i's having the same absolute value, i.e., $|\lambda_1| = |\lambda_2|$ or $|\lambda_1| = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.
3. All three roots λ_1, λ_2 and λ_3 have different absolute values.

2.4.1 $|\lambda_1| = |\lambda_2| = |\lambda_3|$.

For a detailed proof, see Subcase 1, of C($r_1 r_2 z\bar{z}$) in the next subsection.

2.4.2 $|\lambda_1| = |\lambda_2|$ or $|\lambda_1| = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.

We need only treat the first two cases as the third is similar to the second.

2.4.2(1) $|\lambda_1| = |\lambda_2|$.

For a detailed prooof, see Subcase 3.1, of C($r_1 r_2 z\bar{z}$) in the next subsection.

2.4.2(2) $|\lambda_1| = |\lambda_3|$.

Here,

$$u_n = A\lambda_1^n + 2|C|\cos(\varphi + n\theta)|\lambda_1|^n + B\lambda_2^n \quad (n \geq 0).$$

We subdivide into two further sub-cases depending on whether $|\lambda_1| > |\lambda_2|$.

2.4.2(2.1) $|\lambda_1| > |\lambda_2|$.

We consider two possibilities corresponding to the sign of λ_1.

- $\lambda_1 < 0$.

Rewrite the general term of the sequence as

$$u_n = |\lambda_1|^n \left\{(-1)^n A + 2|C|\cos(\varphi + n\theta) + B(\lambda_2/|\lambda_1|)^n\right\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if for all $k \in \mathbb{N} \cup \{0\}$ we must have

$$-2|C|\cos(\varphi + 2k\theta) - B(\lambda_2/|\lambda_1|)^{2k} \leq A \leq 2|C|\cos(\varphi + (2k+1)\theta) + B(\lambda_2/|\lambda_1|)^{2k+1} \tag{42}$$

Since $B(\lambda_2/|\lambda_1|)^n \to 0 (n \to \infty)$, and by Lemma 3.14 as k varies over $\mathbb{N} \cup \{0\}$, both $\cos(\varphi + 2k\theta)$ and $\cos(\varphi + (2k+1)\theta)$ take some fixed positive and negative values infinitely often, the relations (42) hold only when $A = C = 0$. Thus, $u_n = B\lambda_2^n$, and so is nonnegative only when $B \geq 0$ and $\lambda_2 > 0$.

- $\lambda_1 > 0$.

For a detailed proof, see Subcase 3.3 in C($r_1 r_2 z\bar{z}$) in the next subsection.

2.4.2(2.2) $|\lambda_1| < |\lambda_2|$.

We consider two possibilities corresponding to the sign of λ_2.

- $\lambda_2 < 0$.

Here,

$$u_n = \lambda_2^n \left\{A(\lambda_1/\lambda_2)^n + 2|C|(|\lambda_1|/\lambda_2)^n \cos(\varphi + n\theta) + B\right\}, (n \geq 0).$$

Observe that $A(\lambda_1/\lambda_2)^n + 2|C|(|\lambda_1|/\lambda_2)^n \cos(\varphi + n\theta) + B \to B$ $(n \to \infty)$, provided $B \neq 0$. Since λ_2^n oscillates between $\pm|\lambda_2|$, then the sequence (u_n) is nonnegative only when $B = 0$, and so $u_n = A\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) in Lemma 3.15.

- $\lambda_2 > 0$.

For a detailed proof, see Subcase 3.2 in $\mathrm{C}(r_1 r_2 z\bar{z})$ in the next subsection.

2.4.3 All three roots λ_1, λ_2 and λ_3 have different absolute values.

Without loss of generality, assume $|\lambda_1| > |\lambda_2|$. Here,

$$u_n = \lambda_1^n \left(A + B(\lambda_2/\lambda_1)^n\right) + 2|C||\lambda_3|^n \cos(\varphi + n\theta) \quad (n \geq 0).$$

We subdivide into two further sub-cases depending on whether $|\lambda_1| > |\lambda_3|$.

2.4.3(1) $|\lambda_1| > |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n \left\{A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\right\} \quad (n \geq 0).$$

We consider two possibilities corresponding to the sign of λ_1.

- $\lambda_1 < 0$.

Since $\mathrm{sign}(A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)) = \mathrm{sign}(A)$ when n is large enough provided $A \neq 0$, and $\mathrm{sign}(\lambda_1^n)$ oscillates the sequence (u_n) is nonnegative only when $A = 0$ and so $u_n = B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4) in Lemma 3.15.

- $\lambda_1 > 0$.

For a detailed proof, see Subcase 2 in $\mathrm{C}(r_1 r_2 z\bar{z})$ in the next subsection.

2.4.3(2) $|\lambda_1| < |\lambda_3|$.

Rewrite the general term of the sequence as

$$u_n = |\lambda_3|^n \left\{(A + B(\lambda_2/\lambda_1)^n)(\lambda_1/|\lambda_3|)^n + 2|C|\cos(\varphi + n\theta)\right\} \quad (n \geq 0).$$

Observe that $(A + B(\lambda_2/\lambda_1)^n)(\lambda_1/|\lambda_3|)^n \to 0 \quad (n \to \infty)$. Next, by Lemma 3.14, $\cos(\varphi + n\theta)$ takes some fixed positive and some fixed negative values infinitely often.

Thus, the sequence (u_n) is nonnegative only when $C = 0$ yielding $u_n = A\lambda_1^n + B\lambda_2^n$, which is of the form (HHH1) in Lemma 3.15. $\square$

4.3 Non-elementary Proof of the Main Theorem

Again, we subdivide the proof into two cases depending on whether the characteristic polynomial $\mathrm{Char}(z)$ has only real roots.

Case 1: Char(z) has only real roots.

In this case, the general term of the sequence is

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0,\ m \leq 4),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct nonzero real numbers and

$$P_i(n) = A_{i,1} + A_{i,2}n + \cdots + A_{i,\ell_i}n^{\ell_i - 1} \in \mathbb{R}[x] \quad (\ell_i \in \mathbb{N};\ i = 1, 2, \ldots, m;\ A_{i,\ell_i} \neq 0),$$

with $\ell_1 + \ell_2 + \cdots + \ell_m = 4$. We have two possibilities to consider.

1.1 There are two roots λ_i, λ_j $(i, j \in \{1, 2, \ldots, m;\ i \neq j)$ such that $|\lambda_i| = |\lambda_j|$.

Without loss of generality, let the two roots be $\lambda_1 > 0$ and $\lambda_2 = -\lambda_1$. Here,

$$u_n = \{P_1(n) + (-1)^n P_2(n)\}\lambda_1^n + P_3(n)\lambda_3^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0),$$

$$\begin{aligned} \text{i.e.,} \qquad u_{2k} =& (P_1(2k) + P_2(2k))\lambda_1^{2k} + P_3(2k)\lambda_3^{2k} + \cdots + P_m(2k)\lambda_m^{2k}, \\ u_{2k+1} =& (P_1(2k+1) - P_2(2k+1))\lambda_1^{2k+1} + P_3(2k+1)\lambda_3^{2k+1} + \cdots \\ &+ P_m(2k+1)\lambda_m^{2k+1} \ (k \geq 0). \end{aligned}$$

The sequence (u_n) is nonnegative if and only if both the sequences (u_{2k}) and (u_{2k+1}) are nonnegative. The two sequences (u_{2k}) and (u_{2k+1}) are decidable as they satisfy recurrence relations of lower order, i.e., they must be one of the forms stated in Lemma 3.15.

1.2 All roots have different absolute values.

By Theorem 2.11, we need only treat the case where there is a positive dominating root, say, $\lambda_1 > 0$.

Without loss of generality, assume $\lambda_1 > |\lambda_2| > \cdots > |\lambda_m|$. Here,

$$u_n = \lambda_1^n \{P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n\} \quad (n \geq 0),$$

and so (u_n) is nonnegative if and only if

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \geq 0 \quad \text{for all} \quad n \geq 0.$$

- If $A_{1,\ell_1} < 0$, then

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n < 0$$

for all sufficiently large n, and so this case is untenable.

- If $A_{1,\ell_1} > 0$, since

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \to \infty \quad (n \to \infty),$$

there is an explicitly computable least $M_0 \in \mathbb{N} \cup \{0\}$ such that

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \geq 0 \quad \text{for all} \quad n \geq M_0,$$

and so the sequence (u_n) is nonnegative if and only if $M_0 = 0$.

Case 2: Char(z) has non-real roots.

The possible shapes of the four roots are

1. two complex conjugate pairs, either distinct or identical, denoted by $C(z_1\bar{z}_1z_2\bar{z}_2)$;
2. two identical real numbers and one complex conjugate pair, denoted by $C(r_1^2z\bar{z})$;
3. two distinct real numbers and one complex conjugate pair, denoted by $C(r_1r_2z\bar{z})$.

The first possibility $C(z_1\bar{z}_1z_2\bar{z}_2)$ is decidable by Theorem 2.11 as Char(z) has no positive dominating roots. The remaining two possible cases are now treated.

2.1 C($r_1^2 z\bar{z}$).

In this case, the general term of the sequence is

$$u_n = (A + Bn)\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n \quad (n \geq 0),$$

where $A, B, \lambda_1(\neq 0) \in \mathbb{R},\ C \in \mathbb{C},\ \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ and the bar denotes complex conjugate. Let $\lambda_2 = |\lambda_2|e^{i\theta},\ C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi),\ \theta \notin \{-\pi, 0\}$ so that

$$u_n = (A + Bn)\lambda_1^n + 2|C||\lambda_2|^n \cos(\varphi + n\theta).$$

By Theorem 2.11, we need only treat the case where there is a positive dominating root, say, $\lambda_1 > 0$. There are two possibilities.

2.1.1 $\lambda_1 = |\lambda_2|$.

Write

$$u_n = \lambda_1^n\{A + Bn + 2|C|\cos(\varphi + n\theta)\} \qquad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if

$$A \geq -Bn - 2|C|\cos(\varphi + n\theta) \ \text{ for all } n \geq 0. \tag{43}$$

- If $B < 0$, then (43) cannot be fulfilled.
- If $B = 0$, then $u_n = A\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n$, which is of the form (LT4).
- If $B > 0$, then $-Bn - 2|C|\cos(\varphi + n\theta) \to -\infty \quad (n \to \infty)$. Thus, there exists an explicitly computable $N_0 \in \mathbb{N}$ depending on B, C, φ, θ such that

$$\max_{n \in \mathbb{N}\cup\{0\}} \{-Bn - 2|C|\cos(\varphi + n\theta)\} = -BN_0 - 2|C|\cos(\varphi + N_0\theta).$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$A \geq -BN_0 - 2|C|\cos(\varphi + N_0\theta).$$

2.1.2 $\lambda_1 > |\lambda_2|$.

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + Bn + 2|C|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta)\} \quad (n \geq 0).$$

Since $\operatorname{sign}(A + Bn + 2|C|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta)) = \operatorname{sign}(B)$ when n is sufficiently large provided $B \neq 0$, for the sequence (u_n) to be nonnegative we must have $B \geq 0$ and

$$A \geq -Bn - 2|C|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta) \quad (n \geq 0).$$

By Lemma 3.14, there is a least integer $N_L \in \mathbb{N} \cup \{0\}$ such that $\cos(\varphi + N_L\theta) < 0$. Since $(|\lambda_2|/\lambda_1)^n \to 0$ $(n \to \infty)$, there is an $N_M > N_L$ such that for all $n \geq N_M$ we have

$$-2|C|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta) < -2|C|(|\lambda_2|/\lambda_1)^{N_L} \cos(\varphi + N_L\theta).$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$B \geq 0 \text{ and } A \geq \max\{-Bn - 2|C|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta) \; ; \; N_L \leq n \leq N_M\}.$$

2.2 Case C($r_1r_2z\bar{z}$).

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0),$$

where $A, B \in \mathbb{R}$, $C \in \mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$.
Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + B\lambda_2^n + 2|C||\lambda_3|^n \cos(\varphi + n\theta).$$

By Theorem 2.11, we need only decide the situation where $\operatorname{Char}(z)$ has a positive dominating root, say, $\lambda_1 > 0$ and so $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|\}$.
There are three possibilities.

1. The three λ's have the same absolute value, i.e., $\lambda_1 = |\lambda_2| = |\lambda_3|$.

2. All three roots λ_1, λ_2 and λ_3 have different absolute values.

3. There are exactly two λ_i's having the same absolute value,
   i.e, $\lambda_1 = |\lambda_2|$ or $\lambda_1 = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.

2.2.1 $\lambda_1 = |\lambda_2| = |\lambda_3|$**.**
Here, $0 < \lambda_1 = -\lambda_2 = |\lambda_3|$. The sequence term is of the form

$$u_n = \{A + (-1)^n B\}\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda_3}^n \qquad (n \geq 0),$$

i.e.,
$$u_{2k} = \{A + B\}\lambda_1^{2k} + C\lambda_3^{2k} + \bar{C}\bar{\lambda_3}^{2k},$$
$$u_{2k+1} = \{A - B\}\lambda_1^{2k+1} + C\lambda_3^{2k+1} + \bar{C}\bar{\lambda_3}^{2k+1} \quad (k \geq 0).$$

The two sequences (u_{2k}) and (u_{2k+1}) are decidable because they are of the form (LT4).

2.2.2 All three roots λ_1, λ_2 **and** λ_3 **have different absolute values.**
Rewrite the general term of the sequence as

$$u_n = \lambda_1^n \{A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if

$$A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \geq 0 \quad (n \geq 0). \tag{44}$$

- If $A = 0$, then $u_n = B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4).

- If $A < 0$, then

$$A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \to A < 0 \quad (n \to \infty),$$

showing that (44) cannot be fulfilled.

- If $A > 0$, then there is an explicitly computable least integer $N_0 \in \mathbb{N} \cup \{0\}$ such that (44) holds for all $n \geq N_0$. Consequently, in this case the sequence (u_n) is nonnegative if and only if $N_0 = 0$.

2.2.3 $\lambda_1 = |\lambda_2|$ **or** $\lambda_1 = |\lambda_3|$ **or** $\lambda_2 = |\lambda_3|$**.**

2.2.3(1) $\lambda_1 = |\lambda_2| = -\lambda_2 > |\lambda_3|$**.** Here,

$$u_n = \{A + (-1)^n B\}\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda_3}^n \qquad (n \geq 0),$$

i.e.,
$$u_{2k} = \{A + B\}\lambda_1^{2k} + C\lambda_3^{2k} + \bar{C}\bar{\lambda_3}^{2k},$$
$$u_{2k+1} = \{A - B\}\lambda_1^{2k+1} + C\lambda_3^{2k+1} + \bar{C}\bar{\lambda_3}^{2k+1} \quad (k \geq 0).$$

The two sequences (u_{2k}) and (u_{2k+1}) are both decidable because they are of the form (LT4).

2.2.3(2) $\lambda_1 > |\lambda_2| = |\lambda_3| > 0$**.**

• If $\lambda_2 > 0$, then $|\lambda_3| = |\lambda_2| = \lambda_2$, and so

$$u_n = \lambda_1^n \left\{A + (B + 2\,|C|\cos(\varphi + n\theta))\,(\lambda_2/\lambda_1)^n\right\}. \tag{45}$$

Since $(B + 2|C|\cos(\varphi + n\theta))\,(\lambda_2/\lambda_1)^n \to 0$ $(n \to \infty)$, either

$$-\,(B + 2|C|\cos(\varphi + n\theta))\,(\lambda_2/\lambda_1)^n < 0 \quad \text{ for all } \; n \geq 0$$

or there is an explicit $N_1 \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} 0 \leq \max_{n \geq 0} &\left\{-\,(B + 2|C|\cos(\varphi + n\theta))\,(\lambda_2/\lambda_1)^n\right\} \\ &= -\,(B + 2|C|\cos(\varphi + N_1\theta))\,(\lambda_2/\lambda_1)^{N_1}. \end{aligned}$$

The sequence (u_n) is thus nonnegative if and only if

$$A \geq \max\left\{0, -\,(B + 2|C|\cos(\varphi + N_1\theta))\,(\lambda_2/\lambda_1)^{N_1}\right\}.$$

• If $\lambda_2 < 0$, then $|\lambda_3| = |\lambda_2| = -\lambda_2$, and so

$$u_n = A\lambda_1^n + \{(-1)^n B + 2\,|C|\cos(\varphi + n\theta)\}\,|\lambda_3|^n.$$

The two subsequences corresponding to even and odd subscripts, i.e,

$$\begin{aligned} u_{2k} &= A\lambda_1^{2k} + \{B + 2\,|C|\cos(\varphi + 2k\theta)\}\,|\lambda_3|^{2k} \quad (k \geq 0), \\ u_{2k+1} &= A\lambda_1^{2k+1} + \{-B + 2\,|C|\cos(\varphi + (2k+1)\theta)\}\,|\lambda_3|^{2k+1} \quad (k \geq 0) \end{aligned}$$

are of the form (45) and so similar arguments show that both are decidable.

2.2.3(3) $\lambda_1 = |\lambda_3| > |\lambda_2| > 0$**.**

Write

$$u_n = \lambda_1^n \left\{A + 2|C|\cos(\varphi + n\theta) + B(\lambda_2/\lambda_1)^n\right\}.$$

The sequence (u_n) is nonnegative if and only if

$$A \geq -2|C|\cos(\varphi + n\theta) - B(\lambda_2/\lambda_1)^n \quad (n \geq 0). \tag{46}$$

• If θ is a rational multiple of π, say, $\theta = s\pi/t$ where $s,\ t\ (>0) \in \mathbb{Z}\backslash\{0\}$ and $\gcd(s,t) = 1$. By Lemma 3.14, $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ distinct explicit (positive and negative) values at $n \in \{0, 1, \ldots, 2t-1\} (\bmod\ 2t)$; among these values, let c_t be the least (negative).
Since $-B(\lambda_2/\lambda_1)^n \to 0\ (n \to \infty)$, (46) holds if and only if

$$A \geq \begin{cases} -2c_t\,|C| & \text{if "}B = 0\text{" or "}B > 0, \lambda_2 > 0\text{"}, \\ \max_{\,n=0,1,\ldots,2t-1}\{-2\,|C| \cos(\varphi + n\theta) - B(\lambda_2/\lambda_1)^n\} & \text{otherwise.} \end{cases}$$

• If θ is not a rational multiple of π, rewrite the terms of the sequence as

$$u_n = |\lambda_2|^n \left\{(A + 2|C| \cos(\varphi + n\theta)) \left(\lambda_1/\,|\lambda_2|\right)^n + B(\lambda_2/\,|\lambda_2|)^n\right\}.$$

The sequence (u_n) is nonnegative if and only if

$$\{A + 2|C| \cos(\varphi + n\theta)\} \left(\lambda_1/\,|\lambda_2|\right)^n + B(\lambda_2/\,|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{47}$$

If $C = 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$ which is of the form (HHH1). Assume henceforth that $C > 0$.

(a) If $A \leq 0$, since the values of $\cos(\varphi + n\theta)$ are dense in the closed interval $[-1, 1]$ and $(\lambda_1/\,|\lambda_2|)^n \to \infty \quad (n \to \infty)$, there is an explicitly computable $N_L \in \mathbb{N} \cup \{0\}$ such that

$$\{A + 2|C| \cos(\varphi + N_L\theta)\}(\lambda_1/\,|\lambda_2|)^{N_L} + B(\lambda_2/\,|\lambda_2|)^{N_L} < 0,$$

and so (47) cannot be fulfilled.

(b) If $0 < A < 2|C|$, let $2\,|C| - A = \Delta > 0$ so that

$$\begin{aligned} &\{A + 2|C| \cos(\varphi + n\theta)\} \left(\lambda_1/\,|\lambda_2|\right)^n + B\left(\lambda_2/\,|\lambda_2|\right)^n \\ &\quad = \{2\,|C|\,(1 + \cos(\varphi + n\theta)) - \Delta\} \left(\lambda_1/\,|\lambda_2|\right)^n + B\left(\lambda_2/\,|\lambda_2|\right)^n. \end{aligned} \tag{48}$$

Taking a subsequence (n_k) for which $\cos(\varphi + n_k\theta) \to -1 \quad (k \to \infty)$, the left-hand expression in (48) tends to $-\infty$ as $k \to \infty$, showing that (47) cannot be fulfilled.

(c) If $A > 2|C| > 0$, let $\delta = A - 2\,|C| > 0$. Since

$$\begin{aligned} &\{A + 2|C| \cos(\varphi + n\theta)\} \left(\lambda_1/\,|\lambda_2|\right)^n + B(\lambda_2/\,|\lambda_2|)^n \\ &\qquad \geq \delta(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \to \infty \quad (n \to \infty), \end{aligned}$$

there is an explicitly computable least $N_0 \in \mathbb{N} \cup \{0\}$, depending on $A, B, C, \varphi, \theta,$ λ_1, λ_2 such that

$$\{A + 2|C| \cos(\varphi + n\theta)\} (\lambda_1 / |\lambda_2|)^n + B(\lambda_2 / |\lambda_2|)^n \geq 0$$

for all $n \geq N_0$. Consequently, (47) holds if and only if $N_0 = 0$.

(d) If $A = 2\,|C|$, then (47) becomes

$$2\,|C| \{1 + \cos(\varphi + n\theta)\} (\lambda_1 / |\lambda_2|)^n + B(\lambda_2 / |\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{49}$$

We pause to prove two important claims.

Claim 1. There is at most one integer $N_0 \in \mathbb{N} \cup \{0\}$ such that

$$1 + \cos(\varphi + N_0\theta) = 0. \tag{50}$$

Proof of Claim 1. If there were two distinct such integers N_0 and N_0', then there would exist two integers k_0, k_0' such that

$$\varphi + N_0\theta = (2k_0 + 1)\pi, \quad \varphi + N_0'\theta = (2k_0' + 1)\pi.$$

Subtracting the two equations, we find that θ is a multiple of π, which is a contradiction. $\square$

Claim 2. If $(n_k) \subset \mathbb{N} \cup \{0\}$ is an increasing sequence of positive integers such that

$$1 + \cos(\varphi + n_k\theta) \to 0 \quad (k \to \infty),$$

then

$$2\,|C| \{1 + \cos(\varphi + n_k\theta)\} (\lambda_1 / |\lambda_2|)^{n_k} + B(\lambda_2 / |\lambda_2|)^{n_k} \to \infty \quad (k \to \infty).$$

Proof of Claim 2. By Claim 1, we know that from a certain k onward all the values of $1 + \cos(\varphi + n_k\theta)$ are nonzero. Since the function

$$2\,|C| \{1 + \cos(\varphi + x\theta)\} (\lambda_1 / |\lambda_2|)^x \pm B(\lambda_2 / |\lambda_2|)^x$$

is differentiable up to arbitrary orders, the desired result follows from a number of well-known theorems in real analysis, such as L' Hoṕital's rule. $\square$

Returning to (47), if there is an $N_0 \in \mathbb{N} \cup \{0\}$ such that (50) holds, which must be unique by Claim 1, then for (47) to hold we must have $B(\lambda_2/|\lambda_2|)^{N_0} \geq 0$. Moreover, using Claim 2 and the fact that for n large enough, the values of $1 + \cos(\varphi + n\theta)$ are positive and bounded by 1, we deduce that

$$2\,|C|\,\{1 + \cos(\varphi + n\theta)\}\,(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N} \cup \{0\}$, depending on $B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$2\,|C|\,\{1 + \cos(\varphi + n\theta)\}\,(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \geq 0 \quad \text{for all } n \geq N_1.$$

Using all the obtained information, we conclude that (47) holds if and only if $N_1 = 0$.

CHAPTER 4

SUMMARY AND OPEN PROBLEMS

The research work contained in this dissertation deals with The Positivity Problem. We summarize our results in Section 1. In Section 2, some open problems are listed.

1 Summary

This section summarizes our work in this dissertation. The results can be divided into four parts. In the first part, we find all possible shapes of the roots of a fourth degree polynomial with integer coefficients, and proceed to determine the positivity by case analysis.

In the second part, for $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$, we show that as n runs through the nonnegative integers the nonzero sequence $(\cos(n\theta + \beta))$:

1. takes infinitely many positive and negative values ;

2. if $\theta = s/t$ is a rational multiple of π, the sequence is purely periodic with least period equal to t when s is even and equal to $2t$ when s is odd;

3. if θ is not a rational multiple of π, the range of values of the sequence is dense in the unit interval $(0, 1)$.

Any sequence of the form $\left(\sum_r^d \alpha_r \cos\left(2n\pi s_r/t_r\right) + \beta_r\right)$, with rational s_r/t_r belonging to the unit interval $(0, 1)$, is shown to be purely periodic with least period equal to the least common multiple of $t_1, \ldots, t_d$, as given below:

1. Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$. There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) < 0$.

2. Let $\theta, \beta \in [-\pi, \pi)$ and $\theta \notin \{-\pi, 0\}$.

   (i) The relation $\cos(n\theta + \beta) \leq 0$ cannot hold for all $n \in \mathbb{N}_0$.

   (ii) There are infinitely many $n \in \mathbb{N}_0$ such that $\cos(n\theta + \beta) > 0$.

3. Let $\theta, \beta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.

(a) If $\theta = s\pi/t$ ($s \in \mathbb{Z}$, $t \in \mathbb{N}$, $\gcd(s,t) = 1$) is a rational multiple of π, then the sequence $(\cos(n\theta + \beta))_{n\in\mathbb{N}_0}$ is purely periodic.

Moreover, if s is even, then the least period of the sequence $(\cos(n\theta + \beta))_{n\in\mathbb{N}_0}$ is t, while if s is odd, then its least period is $2t$.

(b) If θ is not a rational multiple of π, then as n varies over $\mathbb{N}_0$ the range of values of $\cos(n\theta + \beta)$ is dense in $[-1, 1]$.

4. Let $\theta_1 = s_1/t_1, \ldots, \theta_d = s_d/t_d$ ($s_r, t_r \in \mathbb{N}$, $\gcd(s_r, t_r) = 1$) be rational numbers in the open interval $(0, 1)$, and let α_r, β_r be real numbers such that the purely periodic sequence $u_n = \sum_{r=1}^{d} \alpha_r \cos(2\pi\theta_r n + \beta_r)$ is not identically zero. Let $T := l.c.m.(t_1, \ldots, t_d)$. Then the least period of the sequence (u_n) is a divisor of T.

5. For $r \in \{1, 2, \ldots, d\}$, let s_r/t_r $(s_r, t_r \in \mathbb{N},\ \gcd(s_r, t_r) = 1)$ be rational numbers in the unit interval $(0, 1)$, and let $\alpha_r (\neq 0), \beta_r$ be real numbers. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r \cos\left(\frac{2\pi n s_r}{t_r} + \beta_r\right),$$

then its least period is equal to $T := \text{l.c.m.}(t_1, t_2, \ldots, t_d)$.

In the third part, let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P. For $\theta (\neq 0)$ and b both in the interval $[0, P)$, we show that when n runs through the nonnegative integers, the nonzero sequence $(f(n\theta + b))$ is purely periodic if θ is a rational multiple of P. While if θ is not a rational multiple of P and f is continuous, the sequence $(f(n\theta + b))$ is dense in the range of f. Moreover, under appropriate conditions, a sequence of the form $\left(\sum_{r=1}^{d} \alpha_r f\left(Pns_r/t_r + b_r\right)\right)$ with rationals s_r/t_r is shown to be purely periodic with least period being the least common multiple of $t_1, \ldots, t_d$, as given below:

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero purely periodic function with least period P and let $b \in [0, P)$. If $s, t \in \mathbb{N}$, $\gcd(s, t) = 1$, then

$$\sum_{n=0}^{t-1} f\left(\frac{Pn}{t} + b\right) = \sum_{n=0}^{t-1} f\left(\frac{Psn}{t} + b\right).$$

2. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period P, $b\in[0,P)$ and $t,T\in\mathbb{N}$. If $\sum_{n=0}^{T-1} f\left(\frac{Pn}{t}+b\right)=0$ holds for some fixed $t\geq 2$ and $t\mid T$, then $\sum_{n=0}^{T-1} f\left(\frac{Psn}{t}+b\right)=0$ holds for all $s\in\mathbb{N}$ with $\gcd(s,t)=1$.

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period P and let $b\in[0,P)$.

(a) If $\theta=sP/t$ $(s\in\mathbb{Z},\ t\in\mathbb{N},\ \gcd(s,t)=1)$ is a rational multiple of P, then the sequence $(f(n\theta+b))_{n\in\mathbb{N}_0}$ is purely periodic.

Moreover, if f satisfies the condition $\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau}+\beta\right)=0$, for all $\tau\in\mathbb{N}$, $\tau\geq 2$ and all $\beta\in[0,P)$, and the sequence $(f(n\theta+\beta))$ is not identically zero, then its least period is equal to t.

(b) If θ is not a rational multiple of P and f is continuous, then as n varies over $\mathbb{N}_0$, the set of values of $f(n\theta+\beta)$ is dense in the range of f.

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period P, $b\in[0,P)$ and let $s,t\in\mathbb{N}$ with $\gcd(s,t)=1$. Assume that for each y in the range of f, there are only finitely many, say $x_1,\ldots,x_R\in[0,P)$ such that $f(x_i)=y$ $(i=1,\ldots,R)$ and that x_i-x_j is not a rational multiple of P for all $i,j\in\{1,\ldots,R\}$, $i\neq j$. If the sequence $(f(nsP/t+b))_{n\in\mathbb{N}_0}$ is not identically sero, then it is purely periodic with least period equal to t.

5. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period P. Assume f satisfies the condition $\sum_{n=0}^{\tau-1} f\left(\frac{Pn}{\tau}+\beta\right)=0$, for all $\tau\in\mathbb{N}$, $\tau\geq 2$ and all $\beta\in[0,P)$. For $d\in\mathbb{N}$, $r\in\{1,2,\ldots,d\}$, let s_r/t_r $(s_r,t_r\in\mathbb{N},\ \gcd(s_r,t_r)=1)$ be rational numbers in the unit interval $(0,1)$, and let $\alpha_r\in\mathbb{R}\backslash\{0\}$, $b_r\in[0,P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n=\sum_{r=1}^{d}\alpha_r f\left(\frac{Pns_r}{t_r}+b_r\right),$$

then its least period is equal to $T:=\text{l.c.m.}(t_1,t_2,\ldots,t_d)$.

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a nonzero purely periodic function with least period P. For $d\in\mathbb{N}$, $r\in\{1,2,\ldots,d\}$, let s_r/t_r $(s_r,t_r\in\mathbb{N},\ \gcd(s_r,t_r)=1)$ be rational numbers in the unit interval $(0,1)$, $\alpha_r\in\mathbb{R}\backslash\{0\}$, $b_r\in[0,P)$ and

$T := \text{l.c.m.}(t_1, t_2, \ldots, t_d)$. Assume f satisfies the condition

$$\sum_{n=0}^{T-1} f\left(\frac{Pn}{\tau} + \beta\right) = 0 \tag{51}$$

for all $\tau \in \mathbb{N},\ \tau \geq 2,\ \tau \mid T$ and all $\beta \in [0, P)$. If the purely periodic, non-identically zero sequence $(u_n)_{n\in\mathbb{N}_0}$ is defined by

$$u_n = \sum_{r=1}^{d} \alpha_r f\left(\frac{Pns_r}{t_r} + b_r\right),$$

then its least period is equal to T.

In the final part, we show that the problem of determining whether each element of a sequence satisfying a fourth order linear recurrence with integer coefficients is nonnegative, referred to as the Positivity Problem for fourth order linear recurrence sequence, is decidable.

2 Open Problems

It seems natural to ask whether our two proofs of the fourth order Positivity problem are applicable to proving the general case. Should this be so, the general positivity problem would be solved, which in turns implies that a long unsolved conjecture of Skolem would be settled. Though, a good deal of the above analysis, such as the case where all roots of $\text{Char}(z)$ are real, does indeed work in the general situation, in the case of fifth order recurrence, there are instances, such as combinations of cosine values with the presence of more irrational multiples, which we are not yet able to settle their decidability. It is believed that the attack via case analysis is the right approach. With appropriate reduction processes, the number of cases to be considered should then be reduced to be a manageable task and crucial results in Diophantine approximation should prove to be vital.

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