



รายงานวิจัยฉบับสมบูรณ์

โครงการ การศึกษาและคำนวณหาควอซีนอร์มอลโมด

Calculation of Quasi-normal Modes of Black Holes

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Abstract

We calculate analytically quasi-normal modes of AdS Schwarzschild black holes including first-order corrections. We consider massive scalar, gravitational and electromagnetic perturbations. Our results are in agreement with numerical calculations. In the case of electromagnetic perturbations, ours is the first calculation to provide an analytic expression for quasi-normal frequencies, because the effective potential vanishes at the zeroth order. We show that the first-order correction is logarithmic.

Also we study the perturbative behaviour of topological black holes with scalar hair. We calculate analytically the quasi-normal modes of the electromagnetic perturbations. In the case of small black holes we find clear evidence off a second-order phase transition of a topological black hole to a hairy configuration. We also find evidence of a second-order phase transition of the AdS vacuum solution to a topological black hole.

งานวิจัยนี้เป็นการคำนวณหาควาซอร์มอลโหมดเพอร์เทอร์เบชันลำดับที่หนึ่ง ของหลุมดำในปริภูมิเวลา แอนไทดิเทอร์ ชวาร์ชชาย โดยเป็นการเพอร์เทอร์เบชันแบบ สเกลาร์มีมวล ความโน้มถ่วง และแม่เหล็กไฟฟ้า ผลการคำนวณที่ได้มีค่าตรงกับผลการคำนวณเชิงตัวเลข ในกรณีการเพอร์เทอร์เบชันแม่เหล็กไฟฟ้า ศักย์ของหลุมดำเป็นศูนย์ในการเพอร์เทอร์เบชันลำดับที่ศูนย์ งานวิจัยนี้ได้ทำการคำนวณหาความถี่จากเพอร์เทอร์เบชันลำดับที่หนึ่ง โดยพจน์ที่ได้เป็นลอการิทึม

อีกทั้งยังได้ทำงานวิจัยเกี่ยวกับหลุมดำทอพอโลยีที่มีสเกลาร์แฮร์ ด้วยวิธีการคำนวณแบบเพอร์เทอร์เบชันของสมการแม่เหล็กไฟฟ้า โดยพบว่าในกรณีหลุมดำขนาดเล็กมีการเปลี่ยนเฟสลำดับที่สองจากหลุมดำทอพอโลยีเป็นหลุมดำที่มีโครงสร้างแฮร์ อีกทั้งพบการเปลี่ยนเฟสลำดับที่สองของผลเฉลยสุญญากาศเอดีเอส ไปยังหลุมดำทอพอโลยี

Acknowledgments (กิตติกรรมประกาศ)

This research work could not have been done without the kindness help and advice from my advisor, Professor George Siopsis, at the department of Physics and Astronomy, the University of Tennessee, Knoxville, Tennessee. I'm deeply grateful for his kindness and helpfulness.

I would like to thank Srinakharinwirot University, Office of the Commission for Higher Education and Thailand Research Fund (TRF) for the research fund granted to me. I also would like to thank the department of Physics, Srinakharinwirot University for supporting my work with the time, office space, etc.



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Chapter 1 Introduction (บทนำ)

Physics is the study of the physical nature and its principals. The two main important Physics theories, successfully explaining many physical phenomena are the theory of relativity and quantum theory. However both theories describe completely different situations. Relativity is suitable for the macroscopic systems or is applied in the systems in which its particle speed considerable to the speed of light. Quantum theory is, however, used to describe the microscopic system, the atom-size systems. There are attempts to search for new theories that can describe all kinds of situations. Call such kind of these theories 'Quantum-Gravity Theory'. The main leading theories currently are String Theory and Loop Quantum Gravity Theory

Black Holes are the regions where the gravitation is extremely high. It is so high that even light can not escape when enters through the horizon. This excruciating gravitation can crash down any macroscopic particles into elementary particles and accelerate the speed of these microscopic particles closer to the speed of light. To understand the physical nature of the black holes we need a new theory that embraces both relativity and quantum. In other word the study of black hole physics can be used to verify the candidate quantum-gravity theories, e.g. String theory and Quantum Loop Gravity.

One of the important puzzles of the black holes is its thermodynamics. According to classical relativity theory, black hole entropy is very large (near infinity), but its temperature is zero because it consumes everything near them without giving anything back.

This classical black hole entropy contradicts with the first law of thermodynamics

$$dE = TdS$$

And the third law of thermodynamics, larger than zero

$$T > 0$$

In 1972 Bekenstein [1] pointed out the similarity between the non-decreasing area theorem and the second law of thermodynamics. He proposed that the area of the black hole horizon should be proportional to its entropy. This idea contradicted the traditional idea that entropy should be proportional to the volume of the system.

In 1973, Bardeen, Carter and Hawking [4] provided a rigorous proof of the first law of black holes,

$$\delta M = \frac{\kappa}{8\pi} \delta A$$

the second law, the non-decreasing of the horizon area of a black hole, is

$$\delta A > 0$$

and the third law $T > 0$, where M is the mass of the black hole, A , the surface area of the black holes, is

$$S = \frac{A}{4}$$

and κ is the surface gravity. The temperature is, Hawking temperature

$$T = \frac{\kappa}{2\pi}$$

Their work is in agreement with Bekenstein's idea. This new black hole entropy and temperature are the consequence of the quantum effect of the system.

The above research has become the foundation of quantum gravity and can be used to verify any new quantum-gravity theory candidates, e.g. Loop Quantum Gravity Theory and (Super)String Theory

In loop quantum gravity theory, the space-times is discontinuous and have been quantized, where the smallest length is The Plank's length. In 1996, Rovelli [4] calculated a black hole surface by counting the quantization of the area. He found the area is in agreement with the entropy as Bekenstein has predicted. In 1998, Hod [5] noticed that the

area of Schwarzschild black hole horizon equals to the frequency of the quasi-normal modes of the black holes.

String theory is currently the most successful in the calculation of the entropy and scattering cross section of the black holes, where both in macroscopic and microscopic pictures obtain the same result [6]. This calculation agreement has led to the conjecture of the correspondence between Anti de Sitter space-time gravity and Conformal Field Theory (AdS/CFT) [7]

AdS/CFT correspondence has generated the extensive research on the connection between the low effective energy string theory and conformal field theory. One of the important connections is that the poles of the propagator, which is the Green's functions in quantum field theory, turn out to be the quasi-normal frequencies of the AdS black holes.

Quasi-normal mode of black holes is the phenomena that the black holes absorb and radiated particles in certain ranges of frequency, similar to the Bohr's atom model. In other word, the black hole radiation is quantized. We expect that the quasi-normal mode research can be used to verify or/and test new coming quantum-gravity theory candidates.

The steps of the calculation is : a) finding the solutions to the interested black hole wave equation in some regions, b) fitting the boundary conditions to the solution, only ingoing wave at the horizon and outgoing wave at the infinity, c) taking the limit of the far away-zone solution and near but outside horizon-zone to the intermediate zone limit, d) only few solutions and frequencies satisfy the boundary conditions, called them quasi-normal modes and frequencies. Example of fitting the boundary condition is in [10].

There are many models of black holes, e.g. de sitter(dS), AdS, Schwarzschild, Reissner-Nordstrøm, Kerr, etc. The research in this filed are both numerical and analytical. Our work is analytical one.

Chapter 2 Literature and Related Research (เอกสารและงานวิจัยที่เกี่ยวข้อง)

We will describe some related research as the following. In 2000, Horowitz and Hubeny [8] numerically calculated the lowest lying quasi-normal frequencies in the many-dimensional Schwarzschild AdS space-times. In 2002, Starinets [9] numerically calculated the higher overtones of the frequencies in the five-dimensional Schwarzschild AdS space-times. The higher frequency overtones, (ω_n) , are related to the lowest lying quasi-normal frequencies as

$$\omega_n = \omega_0 \pm 2n(1 \mp i)$$

In 2001, Cardoso and Lemos [11] calculated the quasi-normal modes in three-dimensional BTZ space-times. To make the solutions satisfy the boundary conditions they set the argument of the gamma function in the solutions to be zero.

In 2003 Musiri and Siopsis [10] analytically calculated the quasi-normal modes and frequencies in three and five-dimensional AdS space-times. In three dimension space-times, we found the exact solutions. In the five dimension space-times, we analytically calculated the lowest lying quasi-normal frequencies. We also perform the first-order perturbative calculation of the modes and frequencies. Our analytical results are in agreement with the numerical results by Horowitz and Hubeny [8] and Starinets [9].

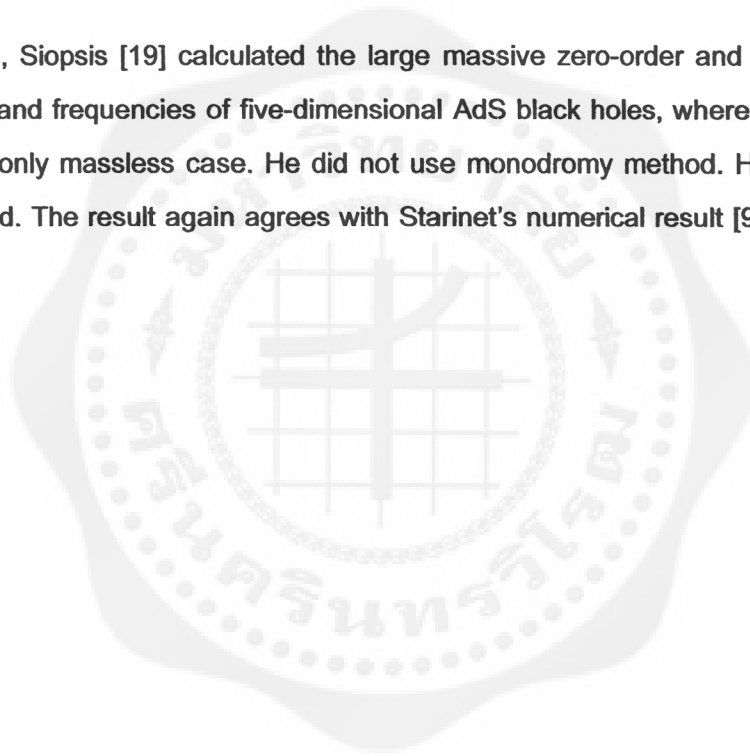
The method of connection the far away-zone solution and the near horizon-zone was the standard procedure of the calculation in this field, where the calculation is performed outside the black holes. Until in 2003 Motl [12] did not only extend the inside-black hole solution to the far away zone-solution, but he also extended the real variable radius, r , to be a complex variable. The extension to the complex variable allows Motl to rotate his solution inside the black holes to some new appropriate regions. However all the physical quantity, e.g. frequencies, should not change according to the rotation. He could be able to obtain a new constraint equation of the frequencies and solved it, called his method 'Monodromy'. His result, zero-order perturbative frequencies, is in agreement with

numerical results by Hod [5] and Nollert [13]. This is for the first time in Schwarzschild space-time that analytical result agrees with the numerical work.

In 2003, Musiri and Siopsis [10] took Motl's method, Monodromy [12], to extend the calculation to the first-order perturbative frequencies of large Schwarzschild black holes. Our first-order perturbation result is in agreement with Nollert's numerical result [13].

In 2003, Musiri and Siopsis [15] used monodromy method to calculate the higher tones of frequencies of large AdS black holes. Again the result agrees with Starinet's numerical result [9].

In 2004, Siopsis [19] calculated the large massive zero-order and first-order quasi-normal modes and frequencies of five-dimensional AdS black holes, where all the analytical work before is only massless case. He did not use monodromy method. He perturbs Heun equation instead. The result again agrees with Starinet's numerical result [9].



Chapter 3 Research Procedures (วิธีดำเนินการวิจัย), Research Result(ผลการวิจัย), Summary and Suggestion (สรุป อภิปรายผล และข้อเสนอแนะ)

There are three research plans in this project.

1. Calculate the scalar massive zero-order and first-order quasi-normal modes and frequencies of five-dimensional AdS black holes. We will use the method in [19], where they calculated only the large mass expansion. But our work the mass does not to be large.
2. Calculate the scalar, gravitational and electromagnetic quasi-normal modes and frequencies of any dimensional AdS black holes. We will calculate both zero order and first perturbation by using monodromy method [12]. In the electromagnetic case, there is a problem of the vanishing of the potential in the zero order, causing no zero-order in this case. We will compare our work in four dimensions with the numerical results in [20]
3. Calculate the scalar quasi-normal modes and frequencies of four-dimensional topological black holes with scalar hair, [21], by monodromy method. We compare our analytical result with the numerical result from the co-authors in our second paper

The first two plans listed above have done and published in 'Physical Reviews D73, 064001 (2006). The third plan is also finished and published in 'Journal of High Energy Physics 10, 006 (2006). Both publications have been attached as the following

Perturbative calculation of quasinormal modes of AdS Schwarzschild black holes

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(Received 15 November 2005; published 1 March 2006)

We calculate analytically quasinormal modes of AdS Schwarzschild black holes including first-order corrections. We consider massive scalar, gravitational and electromagnetic perturbations. Our results are in good agreement with numerical calculations. In the case of electromagnetic perturbations, ours is the first calculation to provide an analytic expression for quasinormal frequencies, because the effective potential vanishes at zeroth order. We show that the first-order correction is logarithmic.

DOI: 10.1103/PhysRevD.73.064001

PACS numbers: 04.50.+h, 04.70.Bw, 11.25.-w, 11.27.+d

I. INTRODUCTION

Recently there has been a lot of research studying quasinormal modes of black holes in asymptotically AdS space-times [1]. Understanding these modes may give some insight into the AdS/CFT correspondence.

Here we present a fairly comprehensive study of quasinormal modes of AdS Schwarzschild black holes with a metric in d dimensions given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \quad (1)$$

$$f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}},$$

and derive analytical expressions including first-order corrections. The results are in good agreement with the results of the numerical analysis.

In the case of massive perturbations, which we discuss in Sec. II, we extend the approach of [2] to include black holes of arbitrary size. We perform an expansion in $1/m$, where m is the mass of the perturbation. The calculation involves a large amount of cancellations between various terms resulting in a sensible perturbative expansion of asymptotic expressions for quasinormal frequencies. In Sec. III, we discuss gravitational perturbations and obtain the first-order corrections to the analytic expressions derived in [3,4] (adapting the monodromy argument proposed in [5] and extended to first order in [6]). We find good agreement with numerical results [7]. In Sec. IV, we extend the discussion to electromagnetic perturbations. In this case, the zeroth-order effective potential vanishes rendering the analytic derivation of quasinormal frequencies impossible. We show that including the first-order correction leads to an analytic expression in agreement with numerical results. Unlike other types of perturbation, the

correction in electromagnetic modes is logarithmic. We summarize our conclusions in Sec. V.

II. MASSIVE SCALAR PERTURBATIONS

In this section we calculate quasinormal frequencies for massive scalar perturbations of *finite* black holes in AdS generalizing a procedure introduced in [2]. We consider explicitly the five-dimensional case in which the wave equation reduces to a Heun equation. Generalizing to higher dimensions is straightforward albeit tedious due to the increase in singular points.

Using the line element (1) in $d = 5$, we obtain the horizon radius

$$\frac{r_H^2}{R^2} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu}{R^2}}. \quad (2)$$

The wave equation for a massive scalar of mass m is

$$\frac{1}{r^3} \partial_r (r^3 f(r) \partial_r \Phi) - \frac{1}{f(r)} \partial_t^2 \Phi + \frac{1}{r^2} \nabla_\Omega^2 \Phi = m^2 \Phi. \quad (3)$$

It is convenient to transform to a dimensionless coordinate

$$y = s \left(\frac{2r^2}{R^2} + 1 \right), \quad s = \frac{1}{\frac{2r_H^2}{R^2} + 1}, \quad (4)$$

in terms of which the factor $f(r)$ [Eq. (1) with $d = 5$] reads

$$f[r(y)] = \frac{y^2 - 1}{2s(y - s)}. \quad (5)$$

We see that s is a parameter describing the size of the black hole. When $s \rightarrow 0$, we approach the large black-hole limit ($r_H \rightarrow \infty$) and expect to arrive at the results of [2].

Separating variables,

$$\Phi = e^{-i\omega t} Y_{\ell\tilde{m}}(\Omega) \Psi(y), \quad (6)$$

we obtain the radial wave equation expressed in terms of y ,

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$$(y-s)(y^2-1)\Psi'' + (3y^2-1-2sy)\Psi' + \left[\frac{\hat{\omega}^2}{4} \frac{(y-s)^2}{y^2-1} - \frac{\hat{L}^2}{4} - (y-s)\hat{m}^2 \right] \Psi = 0, \quad (7)$$

where we introduced the dimensionless parameters

$$\hat{\omega}^2 = 2s\omega^2 R^2, \quad \hat{L}^2 = 2s\ell(\ell+2), \quad \hat{m} = \frac{mR}{2}. \quad (8)$$

The singularities of the wave equation are given by

$$y = \pm 1, s, \quad (9)$$

where $y = 1$ is the horizon, $y = s$ is the black-hole singularity and $y = -1$ is an unphysical singularity. In order to bring (7) into a manageable form, we need to study the behavior of the wave function near the singularities. Two independent solutions of (7) are obtained by examining the behavior near the horizon ($y \rightarrow 1$),

$$\Psi_{\pm} \sim (y-1)^{\pm i(\hat{\omega}/4)\sqrt{1-s}}, \quad (10)$$

where Ψ_+ , Ψ_- represent outgoing and ingoing waves, respectively. We will choose Ψ_- for quasinormal modes.

Near the singularity $y \rightarrow -1$ we obtain a different set of independent solutions

$$\Psi \sim (y+1)^{\pm(\hat{\omega}/4)\sqrt{1+s}}. \quad (11)$$

Since this is an unphysical singularity there is no physical choice. By studying the behavior at large r ($y \rightarrow \infty$), we find another set of independent solutions which determine the scaling behavior and are given by

$$\Psi \sim y^{-h_{\pm}}, \quad h_{\pm} = 1 \pm \sqrt{1+\hat{m}^2}. \quad (12)$$

For quasinormal modes we want the solution to vanish for large r ($y \rightarrow \infty$), leading us to choose

$$\Psi \sim y^{-h_+}. \quad (13)$$

We may write the solution of (7) in the form

$$\Psi = (y-1)^{-i(\hat{\omega}/4)\sqrt{1-s}}(y+1)^{-(\hat{\omega}/4)\sqrt{1+s}}F(y). \quad (14)$$

Substituting this expression into the wave equation (7), we obtain an equation for $F(y)$,

$$(y^2-1)F'' + \left\{ \left(3 - (\sqrt{1+s} + i\sqrt{1-s})\frac{\hat{\omega}}{2} \right)y + s + (\sqrt{1+s} - i\sqrt{1-s})\frac{\hat{\omega}}{2} \right\} F' + \left\{ \frac{\hat{\omega}}{2} \left[(s + i\sqrt{1-s^2})\frac{\hat{\omega}}{4} - (\sqrt{1+s} + i\sqrt{1-s}) \right] - \hat{m}^2 \right\} F + \frac{1}{y-s} \left\{ (s^2-1)F' - \frac{\hat{L}^2}{4}F + [(1-s)\sqrt{1+s} - i(1+s)\sqrt{1-s}]\frac{\hat{\omega}}{4}F \right\} = 0. \quad (15)$$

If we are interested in the limit of large frequencies $\hat{\omega}$, we may focus on the region of large y [2]. In this case, the last term on the left-hand side of (15) is negligible compared with the other terms and the wave equation simplifies to a hypergeometric equation,

$$(y^2-1)F'' + \left\{ \left(3 - (\sqrt{1+s} + i\sqrt{1-s})\frac{\hat{\omega}}{2} \right)y + s + (\sqrt{1+s} - i\sqrt{1-s})\frac{\hat{\omega}}{2} \right\} F' + \left\{ \frac{\hat{\omega}}{2} \left[(s + i\sqrt{1-s^2})\frac{\hat{\omega}}{4} - (\sqrt{1+s} + i\sqrt{1-s}) \right] - \hat{m}^2 \right\} F = 0. \quad (16)$$

Two linearly independent solutions of (16) are

$$\begin{aligned} \mathcal{F}_1 &= F(a_+, a_-; c; -x), \\ \mathcal{F}_2 &= x^{1-c}F(1+a_+-c, 1+a_--c; 2-c; -x), \\ x &= \frac{y-1}{2}, \end{aligned} \quad (17)$$

where

$$a_{\pm} = h_{\pm} - (\sqrt{1+s} + i\sqrt{1-s})\frac{\hat{\omega}}{4}, \quad (18)$$

$$c = \frac{3}{2} + \frac{1}{2}(s - i\sqrt{1-s}\hat{\omega}). \quad (19)$$

Using the transformation properties of hypergeometric functions, we may reexpress the solutions (17) in terms of a new set of independent solutions which match the scaling behavior (12) for large r ($x \rightarrow \infty$),

$$\mathcal{K}_{\pm} = (x+1)^{-a_{\pm}}F(a_{\pm}, c-a_{\pm}; a_{\pm}-a_{\pm}+1; 1/(x+1)). \quad (20)$$

We ought to choose $\mathcal{K}_+$, since it leads to $\Psi \rightarrow 0$ as $x \rightarrow \infty$. $\mathcal{K}_+$ may be expressed as a linear combination of $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\mathcal{K}_+ = \mathcal{A}_0\mathcal{F}_1 + \mathcal{B}_0\mathcal{F}_2, \quad (21)$$

where

$$\begin{aligned} \mathcal{A}_0 &= \frac{\Gamma(1-c)\Gamma(1-a_-+a_+)}{\Gamma(1-a_-)\Gamma(1-c+a_+)}, \\ \mathcal{B}_0 &= \frac{\Gamma(c-1)\Gamma(1+a_+-a_-)}{\Gamma(a_+)\Gamma(c-a_-)}. \end{aligned} \quad (22)$$

For the correct behavior at the horizon, we demand

$$\mathcal{B}_0 = 0,$$

which leads to two conditions

$$c - a_- = 1 - n, \quad n = 1, 2, 3, \dots \quad (23)$$

or

$$a_+ = 1 - n, \quad n = 1, 2, 3, \dots \quad (24)$$

Equation (23) leads to the zeroth-order frequencies,

$$\hat{\omega}_n = -2(\sqrt{1+s} + i\sqrt{1-s}) \left[n + h_+ - \frac{3}{2} + \frac{s}{2} \right]. \quad (25)$$

Notice that the phase approaches $\pi/4$ in the large black-hole limit ($r_H \rightarrow \infty$ or $s \rightarrow 0$), as expected [2].

Using (24), we find a second set of frequencies given by

$$\hat{\omega}_n = 2(\sqrt{1+s} - i\sqrt{1-s})(n + h_+ - 1). \quad (26)$$

Both sets of frequencies, (25) and (26), at leading order agree on the imaginary part and have opposite real parts. We shall work with (25) without loss of generality. Notice also that the two sets of quasinormal frequencies match the results of [2] in the large black-hole limit ($s \rightarrow 0$).

To find the first-order correction to the zeroth-order expression for quasinormal frequencies (25), we shall solve the Heun equation (15) perturbatively. To this end, let us bring it to the form

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0, \quad (27)$$

where [cf. Eq. (16)]

$$\begin{aligned} \mathcal{H}_0 = & \partial_y^2 + \frac{1}{y^2 - 1} \left\{ \left(3 - (\sqrt{1+s} + i\sqrt{1-s}) \frac{\hat{\omega}}{2} \right) y \right. \\ & + s + (\sqrt{1+s} - i\sqrt{1-s}) \frac{\hat{\omega}}{2} \Big\} \partial_y \\ & + \frac{1}{y^2 - 1} \left\{ \frac{\hat{\omega}}{2} \left[(s + i\sqrt{1-s^2}) \frac{\hat{\omega}}{4} \right. \right. \\ & \left. \left. - (\sqrt{1+s} + i\sqrt{1-s}) \right] - \hat{m}^2 \right\}, \end{aligned} \quad (28)$$

and the correction (to be treated as a perturbation) is given by

$$\mathcal{H}_1 = \frac{1}{(y^2 - 1)(y - s)} \left[(s^2 - 1)\partial_y + ((1 - s)\sqrt{1+s} - i(1 + s)\sqrt{1-s}) \frac{\hat{\omega}}{4} \right]. \quad (29)$$

We have neglected the angular momentum contribution for simplicity. We may expand the wave function as

$$F = F_0 + F_1 + \dots, \quad (30)$$

where F_0 obeys the zeroth-order equation [Eq. (16)]

$$\mathcal{H}_0 F_0 = 0. \quad (31)$$

Solving this equation leads to the zeroth-order expressions

for quasinormal frequencies (25). The first-order equation is

$$\mathcal{H}_1 F_0 + \mathcal{H}_0 F_1 = 0. \quad (32)$$

We may solve for F_1 by using a variation of parameters,

$$F_1 = \mathcal{K}_- \int_x^\infty \frac{\mathcal{K}_+ \mathcal{H}_1 F_0}{\mathcal{W}} - \mathcal{K}_+ \int_x^\infty \frac{\mathcal{K}_- \mathcal{H}_1 F_0}{\mathcal{W}}, \quad (33)$$

where $\mathcal{K}_\pm$ are the two linearly independent solutions (20) of Eq. (16) and $\mathcal{W}$ is their Wronskian given by

$$\mathcal{W} = (a_+ - a_-)x^{-c}(1+x)^{c-a_+-a_- - 1}. \quad (34)$$

To study the behavior near the horizon ($x \rightarrow 0$), we may analytically continue the parameters in (33) without affecting the singularity. For $x \sim 0$, we obtain

$$F_1 \sim \mathcal{A}_1 + x^{1-c} \mathcal{B}_1, \quad (35)$$

where

$$\mathcal{B}_1 = \beta_- \int_0^\infty \frac{\mathcal{K}_+ \mathcal{H}_1 F_0}{\mathcal{W}} - \beta_+ \int_0^\infty \frac{\mathcal{K}_- \mathcal{H}_1 F_0}{\mathcal{W}}, \quad (36)$$

and

$$\beta_\pm = \frac{\Gamma(c-1)\Gamma(1+a_\pm - a_\mp)}{\Gamma(a_\pm)\Gamma(c-a_\mp)}. \quad (37)$$

With our choice (23), we find

$$\mathcal{B}_1 = \beta_- \int_0^\infty \frac{\mathcal{K}_+ \mathcal{H}_1 F_0}{\mathcal{W}}. \quad (38)$$

Therefore the quasinormal frequencies, to first order, are found as solutions of

$$\mathcal{B}_0 + \mathcal{B}_1 = 0, \quad (39)$$

where $\mathcal{B}_0$ is given by (22).

We can now find explicit expressions for the first-order correction to quasinormal frequencies. Writing to first order

$$\hat{\omega}_n = -2(\sqrt{1+s} + i\sqrt{1-s}) \left[n + h_+ - \frac{3}{2} + \frac{s}{2} - \epsilon_n \right], \quad (40)$$

we aim at calculating ϵ_n . Let us start with the case of $n = 1$. Our quantization condition (23) becomes $c = a_-$. This truncates the expansion of the hypergeometric solution (20) to

$$F_0 = \mathcal{K}_+ = (1+x)^{-a_+}. \quad (41)$$

After some algebra, we find

$$\mathcal{B}_1 = \frac{\beta_-}{2(a_+ - a_-)} \sum_{k=0}^1 \alpha_k \int_0^\infty dx \frac{x^c (1+x)^{-(c+a_+-a_- - k)}}{2x+1-s}, \quad (42)$$

where the coefficients, α_k ($k = 0, 1$), are given by

$$\begin{aligned}\alpha_0 &= -a_+(s^2 - 1), \\ \alpha_1 &= [(s^2 - 1) + is\sqrt{1 - s^2}][a_+ - a_- + 1 + s].\end{aligned}\quad (43)$$

Using

$$\int_0^\infty dx \frac{x^\lambda(1+x)^{-\mu}}{1+\delta x} = B(\lambda+1, \mu-\lambda) \times F(1, \lambda+1; \mu+1; 1-\delta), \quad (44)$$

we find

$$\begin{aligned}\mathcal{B}_1 &= \frac{B(a_- - 1, a_+ - a_- + 1)}{1-s} \left(\frac{a_-}{a_+ - a_-} \right) \\ &\times \left[-\frac{\alpha_0}{2a_+} F\left(1, a_- + 1; a_+ + 1; \frac{s+1}{s-1}\right) \right. \\ &\left. - \frac{\alpha_1}{2(a_+ - a_- - 1)} F\left(1, a_- + 1, a_+; \frac{s+1}{s-1}\right) \right].\end{aligned}\quad (45)$$

Expanding in $1/h_+$ (large mass expansion), we obtain

$$\begin{aligned}\mathcal{B}_1 &= B(a_- - 1, a_+ - a_- + 1) \left[\frac{s}{4} (s^2 - 1 + is\sqrt{1 - s^2}) \right. \\ &\left. - \frac{i}{8h_+} (1 + o(s)) \right],\end{aligned}\quad (46)$$

where we made use of the expansion of a hypergeometric function

$$\begin{aligned}F(1, \alpha; \beta; z) &= \left(1 - \frac{\alpha}{\beta}z\right)^{-1} + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \frac{\alpha^2 z^2}{\beta^2} \left(1 - \frac{\alpha}{\beta}z\right)^{-3} \\ &+ \dots\end{aligned}\quad (47)$$

which is valid for large α and β [$\sim o(h_+)$]. We obtain from (22) and (40)

$$\mathcal{B}_0 = \epsilon_1 B(a_- - 1, a_+ - a_- + 1) + \dots \quad (48)$$

By using (39) we find the first-order correction for $n = 1$,

$$\epsilon_1 = -\frac{s}{4} (s^2 - 1 + is\sqrt{1 - s^2}) + \frac{i}{8h_+} (1 + o(s)). \quad (49)$$

For a finite-size black hole ($s \neq 0$), this is a $o(h_+^0)$ correction to $n = 1$ quasinormal frequencies. The correction is $o(1/h_+)$ for an infinite-size black hole ($s = 0$) [2]. It should be pointed out that the calculation of ϵ_1 involved the cancellation of $o(h_+)$ terms.

The above result can be straightforwardly extended to $n > 1$. However, the calculation of the relevant integral [Eq. (38)] becomes increasingly tedious with increasing n . For a general n , one obtains expressions $o(h_+^n)$. Thus, naively, the first-order corrections appear of higher order than the zeroth-order result. For $n = 1$, we saw that this was not the case, because nontrivial cancellations occurred between terms contributing to $o(h_+)$ involving hypergeometric functions. For a general n , we expect similar can-

cancellations to occur between terms, leaving an $o(1)$ expression after the dust settles. Because of the complexity of the integrals, instead of performing the calculations by hand, we used the symbolic manipulation program MATHEMATICA, instead. We calculated the first-order correction for $n \leq 10$ and arrived at the general expression

$$\begin{aligned}\epsilon_n &= -\frac{s}{4n} (s^2 - 1 + is\sqrt{1 - s^2}) + \frac{i(2 - 1/n)}{8h_+} (1 + o(s)), \\ n &= 1, 2, 3, \dots\end{aligned}\quad (50)$$

which is $o(h_+^0)$ for finite-size black holes and $o(1/h_+)$ for infinite-size black holes,

$$\epsilon_n = \frac{i}{4} \left(1 - \frac{1}{2n}\right) \frac{1}{h_+}, \quad (51)$$

generalizing our $n = 1$ result (49). We should emphasize that even though we have verified the validity of these expressions analytically for several n (by hand for $n = 1, 2, 3$ and using MATHEMATICA for higher modes), we have been unable to provide a proof of the above results for general n .

III. GRAVITATIONAL PERTURBATIONS

In this section we discuss gravitational perturbations. For massless perturbations, the method discussed in Sec. II is not directly applicable. Instead, we extend the procedure of [3,4] to include first-order corrections to analytical expressions for quasinormal frequencies. Our results are in good agreement with numerical results [7].

The radial wave equation for gravitational perturbations in the black-hole background (1) can be cast into a Schrödinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi, \quad (52)$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)}. \quad (53)$$

The potential V is determined by the type of perturbation and may be deduced from the master equation derived in [8]. For tensor, vector and scalar perturbations, we obtain, respectively, [4]

$$\begin{aligned}V_T(r) &= f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} \right. \\ &\left. + \frac{(d-2)f'(r)}{2r} \right\},\end{aligned}\quad (54)$$

$$\begin{aligned}V_V(r) &= f(r) \left\{ \frac{\ell(\ell + d - 3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} \right. \\ &\left. - \frac{rf'''(r)}{2(d-3)} \right\},\end{aligned}\quad (55)$$

$$\begin{aligned}
V_S(r) = & \frac{f(r)}{4r^2} \left[\ell(\ell + d - 3) - (d - 2) + \frac{(d - 1)(d - 2)\mu}{r^{d-3}} \right]^{-2} \left\{ \frac{d(d - 1)^2(d - 2)^3\mu^2}{R^2 r^{2d-8}} \right. \\
& - \frac{6(d - 1)(d - 2)^2(d - 4)[\ell(\ell + d - 3) - (d - 2)]\mu}{R^2 r^{d-5}} + \frac{(d - 4)(d - 6)[\ell(\ell + d - 3) - (d - 2)]^2 r^2}{R^2} \\
& + \frac{2(d - 1)^2(d - 2)^4\mu^3}{r^{3d-9}} + \frac{4(d - 1)(d - 2)(2d^2 - 11d + 18)[\ell(\ell + d - 3) - (d - 2)]\mu^2}{r^{2d-6}} \\
& + \frac{(d - 1)^2(d - 2)^2(d - 4)(d - 6)\mu^2}{r^{2d-6}} - \frac{6(d - 2)(d - 6)[\ell(\ell + d - 3) - (d - 2)]^2\mu}{r^{d-3}} \\
& - \frac{6(d - 1)(d - 2)^2(d - 4)[\ell(\ell + d - 3) - (d - 2)]\mu}{r^{d-3}} + 4[\ell(\ell + d - 3) - (d - 2)]^3 \\
& \left. + d(d - 2)[\ell(\ell + d - 3) - (d - 2)]^2 \right\}. \quad (56)
\end{aligned}$$

Evidently, the potential always vanishes at the horizon [$V(r_H) = 0$, since $f(r_H) = 0$] regardless of the type of perturbation.

Near the black-hole singularity ($r \sim 0$), the tortoise coordinate (53) may be expanded as

$$r_* = -\frac{1}{(d-2)} \frac{r^{d-2}}{2\mu} - \frac{1}{(2d-5)} \frac{r^{2d-5}}{(2\mu)^2} + \dots, \quad (57)$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$. Using (57), we may expand the potential near the black-hole singularity in the three different cases [Eqs. (54)–(56)], respectively, as

$$\begin{aligned}
V_T = & -\frac{1}{4r_*^2} + \frac{\mathcal{A}_T}{[-2(d-2)\mu]^{1/(d-2)}} r_*^{-(d-1)/(d-2)} + \dots, \\
\mathcal{A}_T = & \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell + d - 3)}{d-2}, \quad (58)
\end{aligned}$$

$$\begin{aligned}
V_V = & \frac{3}{4r_*^2} + \frac{\mathcal{A}_V}{[-2(d-2)\mu]^{1/(d-2)}} r_*^{-(d-1)/(d-2)} + \dots, \\
\mathcal{A}_V = & \frac{d^2 - 8d + 13}{2(2d-15)} + \frac{\ell(\ell + d - 3)}{d-2}, \quad (59)
\end{aligned}$$

and

$$V_S = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_S}{[-2(d-2)\mu]^{1/(d-2)}} r_*^{-(d-1)/(d-2)} + \dots, \quad (60)$$

where

$$\begin{aligned}
\mathcal{A}_S = & \frac{(2d^3 - 24d^2 + 94d - 116)}{4(2d-5)(d-2)} \\
& + \frac{(d^2 - 7d + 14)[\ell(\ell + d - 3) - (d - 2)]}{(d-1)(d-2)^2}. \quad (61)
\end{aligned}$$

We have included only the terms which contribute to the order we are interested in. We may summarize the behavior

of the potential near the origin by

$$V = \frac{j^2 - 1}{4r_*^2} + \mathcal{A} r_*^{-(d-1)/(d-2)} + \dots, \quad (62)$$

where $j = 0(2)$ for scalar and tensor (vector) perturbations and the constant coefficient $\mathcal{A}$ can be found from Eqs. (58)–(61) in the various cases. Throughout the calculation, we shall pretend that j is not an integer. At the end of the calculation, we shall let $j \rightarrow 0, 2$, as appropriate.

After rescaling the tortoise coordinate ($z = \omega r_*$), the Schrödinger-like wave equation (52) with the potential (62) becomes

$$\begin{aligned}
-\frac{d^2\Psi}{dz^2} + \left[\frac{j^2 - 1}{4z^2} - 1 \right] \Psi = & -\mathcal{A} \omega^{-(d-3)/(d-2)} \\
& \times z^{-(d-1)/(d-2)} \Psi. \quad (63)
\end{aligned}$$

In the large frequency limit, we may treat the right-hand side of (63) as a correction. This will allow us to solve the equation perturbatively. We may reexpress (63) as

$$(\mathcal{H}_0 + \omega^{-(d-3)/(d-2)} \mathcal{H}_1) \Psi = 0, \quad (64)$$

where

$$\begin{aligned}
\mathcal{H}_0 = & \frac{d^2}{dz^2} - \left[\frac{j^2 - 1}{4z^2} - 1 \right], \\
\mathcal{H}_1 = & -\mathcal{A} z^{-(d-1)/(d-2)}. \quad (65)
\end{aligned}$$

By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-(d-3)/(d-2)} \Psi_1(z) + \dots \quad (66)$$

and solve (64) perturbatively. The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0, \quad (67)$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z} J_{j/2}(z) + A_2 \sqrt{z} N_{j/2}(z). \quad (68)$$

For large z , it behaves as

$$\begin{aligned}\Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} [A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+)] \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} \\ &\quad + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz},\end{aligned}\quad (69)$$

where $\alpha_{\pm} = \frac{\pi}{4}(1 \pm j)$.

Next, we study the behavior of the wave function at large r . In this region, the tortoise coordinate (53) may be expanded as

$$r_* - \bar{r}_* = -\frac{R^2}{r} + \frac{1}{3} \frac{R^4}{r^3} + \dots \quad (70)$$

The integration constant is readily deduced from the definition (53) of the tortoise coordinate,

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)}. \quad (71)$$

The potential [Eqs. (54)–(56)] for large r may be expanded as

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots, \quad (72)$$

where $j_\infty = d - 1$, $d - 3$ and $d - 5$ for tensor, vector and scalar perturbations, respectively. The Schrödinger-like wave equation (52) in the region of large r becomes

$$-\frac{d^2\Psi}{dr_*^2} + \left[\frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} - \omega^2 \right] \Psi = 0. \quad (73)$$

Since the potential does not vanish as $r \rightarrow \infty$, the wave function ought to vanish there. Imposing this boundary condition yields the acceptable solution to Eq. (73),

$$\Psi(r_*) = B \sqrt{\omega(r_* - \bar{r}_*)} J_{j_\infty/2}(\omega(r_* - \bar{r}_*)). \quad (74)$$

Notice that $\Psi \rightarrow 0$ as $r_* \rightarrow \bar{r}_*$, as desired. Asymptotically, it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}} B \cos[\omega(r_* - \bar{r}_*) + \beta], \quad \beta = \frac{\pi}{4}(1 + j_\infty). \quad (75)$$

By matching this expression to the asymptotic behavior (69) of the solution in the vicinity of the black-hole singularity along the Stokes line $\text{Im}z = \text{Im}(\omega r_*) = 0$, we find a constraint on the coefficients A_1, A_2 ,

$$A_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2 = 0. \quad (76)$$

A second constraint is obtained by imposing the boundary condition

$$\Psi(z) \sim e^{iz}, \quad z \rightarrow -\infty, \quad (77)$$

at the horizon. To this end, we need to analytically continue

the wave function near the origin to negative values of z . A rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r plane, on account of (57). Since near the origin, $J_\nu(z) \sim z^\nu$ (multiplied by an even holomorphic function of z) and using the identity

$$N_\nu(z) = \cot \pi \nu J_\nu(z) - \csc \pi \nu J_{-\nu}(z), \quad (78)$$

we deduce

$$J_\nu(e^{-i\pi}z) = e^{-i\pi\nu} J_\nu(z), \quad (79)$$

$$N_\nu(e^{-i\pi}z) = e^{i\pi\nu} N_\nu - 2i \cos \pi \nu J_\nu(z).$$

Thus for $z < 0$, the wave function (68) changes to

$$\begin{aligned}\Psi_0(z) &= e^{-i\pi(j+1)/2} \sqrt{-z} \{ [A_1 - i(1 + e^{i\pi j})A_2] J_{j/2}(-z) \\ &\quad + A_2 e^{i\pi j} N_{j/2}(-z) \},\end{aligned}\quad (80)$$

whose asymptotic behavior is given by

$$\begin{aligned}\Psi &\sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} [A_1 - i(1 + 2e^{i\pi j})A_2] e^{-iz} \\ &\quad + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} [A_1 - iA_2] e^{iz}.\end{aligned}\quad (81)$$

Imposing the boundary condition (77) at the horizon, we deduce the constraint

$$A_1 - i(1 + 2e^{i\pi j})A_2 = 0. \quad (82)$$

The two constraints (76) and (82) are compatible provided

$$\left| \begin{array}{cc} 1 & -i(1 + 2e^{i\pi j}) \\ \tan(\omega \bar{r}_* - \beta - \alpha_+) & -1 \end{array} \right| = 0, \quad (83)$$

which yields the quasinormal frequencies [4]

$$\omega \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{i\pi j}} + n\pi. \quad (84)$$

These are zeroth-order expressions deduced from the zeroth-order wave equation (67).

Next, we calculate the first-order correction to the asymptotic expressions (84) for quasinormal frequencies. We begin by focusing on the region near the black-hole singularity ($r \sim 0$). To first order, the wave equation (64) becomes

$$\mathcal{H}_0 \Psi_1 + \mathcal{H}_1 \Psi_0 = 0, \quad (85)$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are given in Eq. (65). The solution is

$$\begin{aligned}\Psi_1(z) &= \sqrt{z} N_{j/2}(z) \int_0^z dz' \frac{\sqrt{z'} J_{j/2}(z') \mathcal{H}_1 \Psi_0(z')}{\mathcal{W}} \\ &\quad - \sqrt{z} J_{j/2}(z) \int_0^z dz' \frac{\sqrt{z'} N_{j/2}(z') \mathcal{H}_1 \Psi_0(z')}{\mathcal{W}},\end{aligned}\quad (86)$$

written in terms of the two linearly independent solutions (68) of the zeroth-order Eq. (67). $\mathcal{W} = 2/\pi$ is their Wronskian. Using (68) and (86), we may express the solution to the wave equation (64) up to first order

[Eq. (66)] explicitly as

$$\Psi(z) = \{A_1[1 - b(z)] - A_2 a_2(z)\} \sqrt{z} J_{j/2}(z) + \{A_2[1 + b(z)] + A_1 a_1(z)\} \sqrt{z} N_{j/2}(z), \quad (87)$$

where the functions $a_1(z)$, $a_2(z)$ and $b(z)$ are given by

$$a_1(z) = \frac{\pi \mathcal{A}}{2} \omega^{-(d-3)/(d-2)} \int_0^z dz' z'^{-1/(d-2)} J_{j/2}(z') J_{j/2}(z'), \quad (88)$$

$$a_2(z) = \frac{\pi \mathcal{A}}{2} \omega^{-(d-3)/(d-2)} \int_0^z dz' z'^{-1/(d-2)} N_{j/2}(z') N_{j/2}(z'), \quad (89)$$

$$b(z) = \frac{\pi \mathcal{A}}{2} \omega^{-(d-3)/(d-2)} \int_0^z dz' z'^{-1/(d-2)} J_{j/2}(z') N_{j/2}(z'), \quad (90)$$

respectively. The coefficient $\mathcal{A}$ is defined in Eq. (65) and depends on the type of perturbation. The wave function (87) behaves asymptotically as

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}} [A'_1 \cos(z - \alpha_+) + A'_2 \sin(z - \alpha_+)], \quad (91)$$

where

$$A'_1 = [1 - \bar{b}]A_1 - \bar{a}_2 A_2, \quad A'_2 = [1 + \bar{b}]A_2 + \bar{a}_1 A_1 \quad (92)$$

and we introduced the notation

$$\bar{a}_1 = a_1(\infty), \quad \bar{a}_2 = a_2(\infty), \quad \bar{b} = b(\infty). \quad (93)$$

By matching this to the asymptotic expression (75), we obtain

$$A'_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A'_2 = 0 \quad (94)$$

correcting the zeroth-order constraint (76). Using (92), the first-order constraint (94) in terms of A_1 and A_2 reads

$$[(1 - \bar{b}) \tan(\omega \bar{r}_* - \beta - \alpha_+) - \bar{a}_1] A_1 - [1 + \bar{b} + \bar{a}_2 \tan(\omega \bar{r}_* - \beta - \alpha_+)] A_2 = 0. \quad (95)$$

To find the first-order correction to the second constraint (82), we need to approach the horizon. This entails a rotation by $-\pi$ in the z plane. From the small- z behavior of a Bessel function, $J_\nu(z) \sim z^\nu$, and using the identity (78), we deduce after some algebra

$$\begin{aligned} a_1(e^{-i\pi} z) &= e^{-i\pi(d-3)/(d-2)} e^{-i\pi j} a_1(z), \\ a_2(e^{-i\pi} z) &= e^{-i\pi(d-3)/(d-2)} [e^{i\pi j} a_2(z) - 4 \cos^2 \frac{\pi j}{2} a_1(z) \\ &\quad - 2i(1 + e^{i\pi j}) b(z)], \\ b(e^{-i\pi} z) &= e^{-i\pi(d-3)/(d-2)} [b(z) - i(1 + e^{-i\pi j}) a_1(z)]. \end{aligned} \quad (96)$$

From these expressions and Eq. (79), we arrive at a modified expression for the wave function (87) valid for $z < 0$. In the limit $z \rightarrow -\infty$, we obtain

$$\Psi(z) \sim -ie^{-ij\pi/2} B_1 \cos(-z - \alpha_+) - ie^{ij\pi/2} B_2 \sin(-z - \alpha_+), \quad (97)$$

where

$$\begin{aligned} B_1 &= A_1 - A_1 e^{-i\pi(d-3)/(d-2)} [\bar{b} - i(1 + e^{-i\pi j}) \bar{a}_1] - A_2 e^{-i\pi(d-3)/(d-2)} \left[e^{+i\pi j} \bar{a}_2 - 4 \cos^2 \frac{\pi j}{2} \bar{a}_1 - 2i(1 + e^{+i\pi j}) \bar{b} \right] \\ &\quad - i(1 + e^{i\pi j}) [A_2 + A_2 e^{-i\pi(d-3)/(d-2)} [\bar{b} - i(1 + e^{-i\pi j}) \bar{a}_1] + A_1 e^{-i\pi(d-3)/(d-2)} e^{-i\pi j} \bar{a}_1] \\ B_2 &= A_2 + A_2 e^{-i\pi(d-3)/(d-2)} [\bar{b} - i(1 + e^{-i\pi j}) \bar{a}_1] + A_1 e^{-i\pi(d-3)/(d-2)} e^{-i\pi j} \bar{a}_1. \end{aligned} \quad (98)$$

By imposing the boundary condition (77) at the horizon, we obtain

$$[1 - e^{-i\pi(d-3)/(d-2)} (i\bar{a}_1 + \bar{b})] A_1 - [i(1 + 2e^{i\pi j}) + e^{-i\pi(d-3)/(d-2)} ((1 + e^{i\pi j}) \bar{a}_1 + e^{i\pi j} \bar{a}_2 - i\bar{b})] A_2 = 0 \quad (99)$$

correcting the zeroth-order constraint (82). For compatibility of the two first-order constraints, (95) and (99), we need

$$\begin{vmatrix} 1 + \bar{b} + \bar{a}_2 \tan(\omega \bar{r}_* - \beta - \alpha_+) & i(1 + 2e^{i\pi j}) + e^{-i\pi(d-3)/(d-2)} ((1 + e^{i\pi j}) \bar{a}_1 + e^{i\pi j} \bar{a}_2 - i\bar{b}) \\ (1 - \bar{b}) \tan(\omega \bar{r}_* - \beta - \alpha_+) - \bar{a}_1 & 1 - e^{-i\pi(d-3)/(d-2)} (i\bar{a}_1 + \bar{b}) \end{vmatrix} = 0. \quad (100)$$

Solving (100), we arrive at the first-order expression for quasinormal frequencies,

$$\begin{aligned} \omega \bar{r}_* &= \frac{\pi}{4} (2 + j + j_\infty) + \frac{1}{2i} \ln 2 + n\pi - \frac{1}{8} \{6i\bar{b} - 2ie^{-i\pi(d-3)/(d-2)} \bar{b} - 9\bar{a}_1 + e^{-i\pi(d-3)/(d-2)} \bar{a}_1 + \bar{a}_2 \\ &\quad - e^{-i\pi(d-3)/(d-2)} \bar{a}_2\}, \end{aligned} \quad (101)$$

where we took the limit of interest $j \rightarrow 0, 2$ wherever it was unambiguous, in order to simplify the notation. Using

$$\int_0^\infty dx x^{-\lambda} J_\mu(x) J_\nu(x) = \frac{\Gamma(\lambda) \Gamma(\frac{\nu+\mu+1-\lambda}{2})}{2^\lambda \Gamma(\frac{-\nu+\mu+1+\lambda}{2}) \Gamma(\frac{\nu-\mu+1+\lambda}{2}) \Gamma(\frac{\nu+\mu+1+\lambda}{2})}, \quad (102)$$

we obtain explicit expressions for the first-order coefficients,

$$\begin{aligned} \bar{a}_1 &= \frac{\pi \mathcal{A}}{4} \left(\frac{n\pi}{2\bar{r}_*} \right)^{-(d-3)/(d-2)} \frac{\Gamma(\frac{1}{d-2}) \Gamma(\frac{j}{2} + \frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)}) \Gamma(\frac{j}{2} + \frac{d-1}{2(d-2)})}, \\ \bar{a}_2 &= \left[1 + 2 \cot \frac{\pi(d-3)}{2(d-2)} \cot \frac{\pi}{2} \left(-j + \frac{d-3}{d-2} \right) \right] \bar{a}_1, \\ \bar{b} &= -\cot \frac{\pi(d-3)}{2(d-2)} \bar{a}_1, \end{aligned} \quad (103)$$

where we used the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$. We also set $\omega = n\pi/\bar{r}_*$, since corrections contribute to higher than first order. Notice that these expressions are well defined when j becomes an integer. Thus, the first-order correction is $\sim o(n^{-(d-3)/(d-2)})$.

Next, we compare with numerical results in four dimensions [7]. It is convenient to set the AdS radius $R = 1$. From (1), the radius of the horizon r_H is related to the black-hole parameter μ by

$$2\mu = r_H^3 + r_H \quad (104)$$

for $d = 4$. $f(r)$ has two more (complex) roots, r_- and its complex conjugate, where

$$\begin{aligned} r_- &= e^{i\pi/3} \left(\sqrt{\mu^2 + \frac{1}{27}} - \mu \right)^{1/3} \\ &\quad - e^{-i\pi/3} \left(\sqrt{\mu^2 + \frac{1}{27}} + \mu \right)^{1/3}. \end{aligned} \quad (105)$$

The integration constant in the tortoise coordinate (71) is

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} = -\frac{r_-}{3r_-^2 + 1} \ln \frac{r_-}{r_H} - \frac{r_-^*}{3r_-^{*2} + 1} \ln \frac{r_-^*}{r_H}. \quad (106)$$

The right-hand side consists of two terms which appear to be the complex conjugate of each other. However, despite appearances, $\bar{r}_*$ is not a real number, because in performing the integral, we ought to choose the cuts in the complex r plane so that arguments are defined as $0 \leq \arg r < 2\pi$. Then $\ln r_-^*$ is *not* the complex conjugate of $\ln r_-$ and therefore $\bar{r}_*$ is not real.

For scalar perturbations, we find from Eqs. (101) and (103), together with (60)–(62),

$$\begin{aligned} \omega_n \bar{r}_* &= \left(n + \frac{1}{4} \right) \pi + \frac{i}{2} \ln 2 + e^{i\pi/4} \frac{\mathcal{A}_S \Gamma^4(\frac{1}{4})}{16\pi^2} \sqrt{\frac{\bar{r}_*}{2\mu n}}, \\ \mathcal{A}_S &= \frac{\ell(\ell+1)-1}{6}. \end{aligned} \quad (107)$$

Notice that only the first-order correction is ℓ dependent. In

the limit of the large horizon radius [$r_H \approx (2\mu)^{1/3} \gg 1$], we have from (106)

$$\bar{r}_* \approx \frac{\pi(1+i\sqrt{3})}{3\sqrt{3}r_H}. \quad (108)$$

Numerically for $\ell = 2$,

$$\begin{aligned} \frac{\omega_n}{r_H} &= (1.299 - 2.250i)n + 0.573 - 0.419i \\ &\quad + \frac{0.508 + 0.293i}{r_H^2 \sqrt{n}} \end{aligned} \quad (109)$$

which compares well with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H} \right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.581 - 0.41i \quad (110)$$

including both leading order and offset.

For an intermediate black hole, $r_H = 1$, we obtain

$$\begin{aligned} \omega_n &= (1.969 - 2.350i)n + 0.752 - 0.370i \\ &\quad + \frac{0.654 + 0.458i}{\sqrt{n}} \end{aligned} \quad (111)$$

to be compared with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H} \right)_{\text{numerical}} \approx (1.96 - 2.35i)n + 2.01 - 1.5i. \quad (112)$$

In Fig. 1 we compare our analytical result (111) with data from the numerical analysis [7]. We plot the gap

$$\Delta \omega_n = \omega_n - \omega_{n-1} \quad (113)$$

because the offset does not always agree with numerical results. This is evident from a comparison of (111) with (112). The disagreement was observed in zeroth-order analytical results [4]. Including first-order corrections does not alter the analytical value of the offset, because

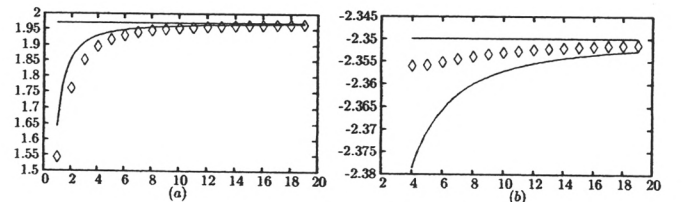


FIG. 1 (color online). The frequency gap (113) for scalar perturbations in $d = 4$ for $r_H = 1$ and $\ell = 2$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (111)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re} \Delta \omega_n$ and $\text{Im} \Delta \omega_n$, respectively, as functions of n .

the corrections are of higher order in $1/n$. This lends reliability to the analytical calculation of the offset supporting the claim made in [4] that numerical estimates of the offset ought to be improved. We shall make a quantitative comparison of analytical and numerical estimates of the offset in the various cases we consider.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.093 + 0.561i}{\sqrt{n}} \quad (114)$$

to be compared with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.61 - 0.6i)n + 2.7 - 0.37i. \quad (115)$$

The two estimates of the offset [analytical (114) and numerical (115)] disagree with each other.

For tensor perturbations, we find from Eqs. (101) and (103), together with (58) and (62),

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2} \ln 2 + e^{i\pi/4} \frac{\mathcal{A}_T \Gamma^4(\frac{1}{4})}{16\pi^2} \sqrt{\frac{\bar{r}_*}{2\mu n}},$$

$$\mathcal{A}_T = \frac{3\ell(\ell+1)+1}{6}. \quad (116)$$

Again, only the first-order correction is ℓ dependent. Numerically for large r_H and $\ell = 0$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.102 + 0.0586i}{r_H^2 \sqrt{n}} \quad (117)$$

in good agreement with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.557 - 0.423i. \quad (118)$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.131 + 0.0916i}{\sqrt{n}} \quad (119)$$

in good agreement with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.97 - 2.35i)n + 0.79 - 0.35i \quad (120)$$

including the offset. In Fig. 2, we plot the gap (113), including both zeroth and first-order contributions and compare with data from the numerical analysis [7].

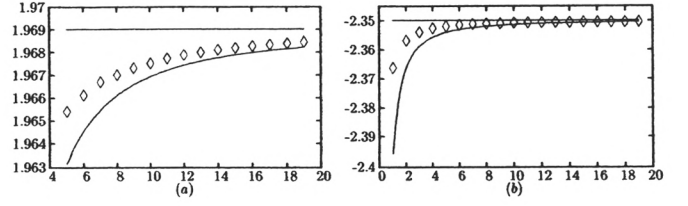


FIG. 2 (color online). The frequency gap (113) for tensor perturbations in $d = 4$ for $r_H = 1$ and $\ell = 0$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (119)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 2.182 - 0.615i + \frac{0.489 + 0.251i}{\sqrt{n}} \quad (121)$$

and compare the gap with data from the numerical analysis in Fig. 3. Asymptotically, the numerical data yield

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.69 - 0.57i)n + 2.29 - 0.46i \quad (122)$$

in good agreement with our analytical result (121), including the offset.

Finally, for vector perturbations, we find from Eqs. (101) and (103), together with (59) and (62),

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2} \ln 2 + e^{i\pi/4} \frac{\mathcal{A}_V \Gamma^4(\frac{1}{4})}{48\pi^2} \sqrt{\frac{\bar{r}_*}{2\mu n}},$$

$$\mathcal{A}_V = \frac{\ell(\ell+1)}{2} + \frac{3}{14}. \quad (123)$$

Numerically for large r_H and $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{8.19 + 6.29i}{r_H^2 \sqrt{n}} \quad (124)$$

to be compared with the result of the numerical analysis [7],

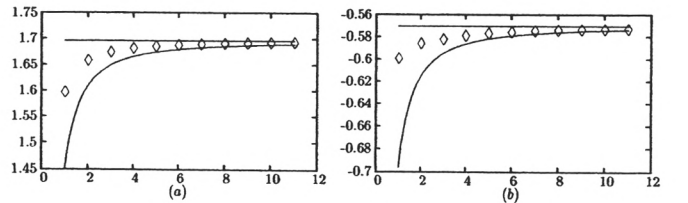


FIG. 3 (color online). The frequency gap (113) for tensor perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 0$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (121)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

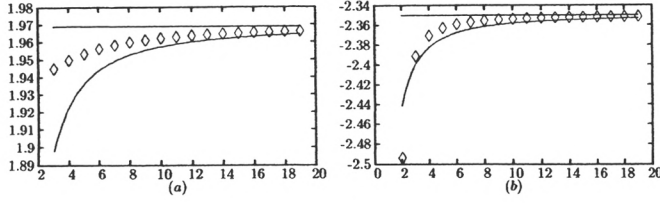


FIG. 4 (color online). The frequency gap (113) for vector perturbations in $d = 4$ for $r_H = 1$ and $\ell = 2$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (126)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.58 - 0.42i. \quad (125)$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.741 + 0.519i}{\sqrt{n}} \quad (126)$$

to be compared with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.97 - 2.35i)n + 0.93 - 0.32i \quad (127)$$

and for a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.239 + 0.6357i}{\sqrt{n}} \quad (128)$$

to be compared with the result of the numerical analysis [7],

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.69 - 0.59i)n + 2.49 - 0.06i. \quad (129)$$

The agreement is, again, good to leading order whereas the estimates of the offset agree for large r_H but diverge as $r_H \rightarrow 0$. A detailed comparison of our analytical results for vector perturbations with corresponding data from numerical analysis can be found in Figs. 4 and 5.

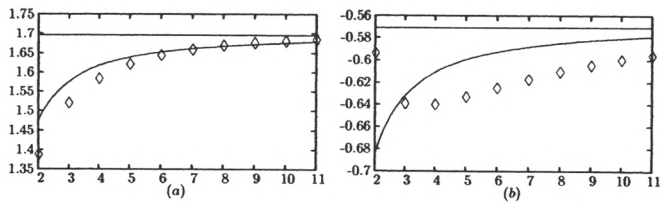


FIG. 5 (color online). The frequency gap (113) for vector perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 2$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (128)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

IV. ELECTROMAGNETIC PERTURBATIONS

In this section we extend the discussion to electromagnetic perturbations. This is a singular case because the potential vanishes at zeroth order. Consequently, the compatibility condition (83) discussed in the previous section has no solutions and no asymptotic expression for quasinormal frequencies may be deduced [4]. Nevertheless, the numerical results are similar to the ones we discussed in the case of gravitational perturbations [7]. We shall show that including first-order corrections leads to analytical asymptotic expressions for quasinormal frequencies in agreement with numerical results. Unlike with gravitational perturbations, where first-order corrections were a power of n [Eqs. (101) and (103)], for electromagnetic perturbations first-order corrections are $o(\ln n)$.

We shall concentrate on the four-dimensional case for definiteness. Generalization to higher dimensions is straightforward. The wave equation reduces to (52) with electromagnetic potential

$$V_{\text{EM}} = \frac{\ell(\ell+1)}{r^2} f(r), \quad (130)$$

where $f(r)$ is given in (1) with $d = 4$. Near the origin, this potential may be expanded in terms of the tortoise coordinate. Using Eq. (57), we obtain

$$V_{\text{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1)r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots, \quad (131)$$

where $j = 1$. This leads to a vanishing potential to zeroth order. Consequently, no analytic expression for quasinormal frequencies is deduced. This is easily seen by substituting $j = 1$ in the zeroth-order expression (84); we obtain a divergent result because $\tan^{-1}i$ is not finite.

This is remedied by including first-order corrections. The compatibility condition (100) of the two first-order constraints (95) and (99) reads

$$\begin{vmatrix} 1 + \bar{b} + \bar{a}_2 \tan \omega \bar{r}_* & -i - \bar{b} + i\bar{a}_2 \\ (1 - \bar{b}) \tan \omega \bar{r}_* - \bar{a}_1 & 1 - \bar{a}_1 + i\bar{b} \end{vmatrix} = 0, \quad (132)$$

where we used $d = 4$, $j = 1$, $j_\infty = d - 3 = 1$, and $\alpha_+ = \beta = \frac{\pi}{2}$. At zeroth order (setting $\bar{a}_1 = \bar{a}_2 = \bar{b} = 0$), we obtain

$$\tan \omega \bar{r}_* = i \quad (133)$$

which has no solution, as expected [4]. At first order, we obtain

$$\tan \omega \bar{r}_* = i + (1 - i)(\bar{a}_1 - \bar{a}_2 - 2\bar{b}) \quad (134)$$

whose first-order solution is

$$\omega \bar{r}_* = n\pi + \frac{1}{2i} \ln \frac{(1 + i)(\bar{a}_1 - \bar{a}_2 - 2\bar{b})}{2}. \quad (135)$$

Using (103) and (131), we deduce explicit expressions for the first-order coefficients,

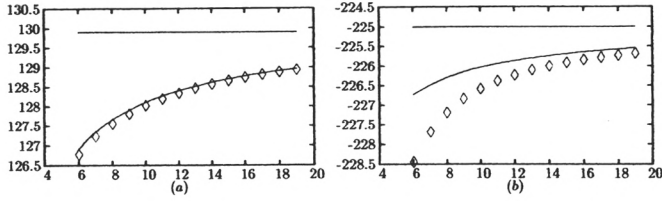


FIG. 6 (color online). The frequency gap (113) for electromagnetic perturbations in $d = 4$ for $r_H = 100$ and $\ell = 1$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (139)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

$$\bar{a}_1 = \mathcal{A}\sqrt{\bar{r}_*}, \quad \bar{a}_2 = \bar{b} = -\bar{a}_1, \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}} \quad (136)$$

and Eq. (135) reads explicitly

$$n\bar{r}_* = n\pi - \frac{i}{4}\ln n + \frac{1}{2i}\ln(2(1+i)\mathcal{A}\sqrt{\bar{r}_*}). \quad (137)$$

Therefore, the correction to the quasinormal frequencies behaves as $\ln n$ in the large ω limit.

To compare with numerical results, set $R = 1$. As with gravitational perturbations, we shall compare the gap, because the offset is not reliable. However, unlike with gravitational perturbations, our analytical estimate of the offset is not reliable, because the first-order corrections we are calculating are $o(\ln n)$, so for a reliable estimate of $o(1)$ contributions, we ought to include second-order perturbative effects.

For the gap, we have from (137)

$$\Delta\omega_n \equiv \omega_n - \omega_{n-1} = \frac{\pi}{\bar{r}_*} \left(1 - \frac{i}{4\pi n} + \dots \right). \quad (138)$$

Both leading and subleading terms are independent of ℓ .

For a large black hole, using (108), we obtain the spectrum

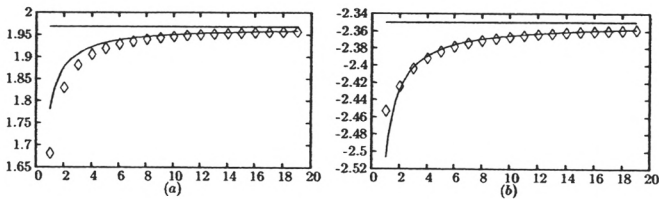


FIG. 7 (color online). The frequency gap (113) for electromagnetic perturbations in $d = 4$ for $r_H = 1$ and $\ell = 1$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (140)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

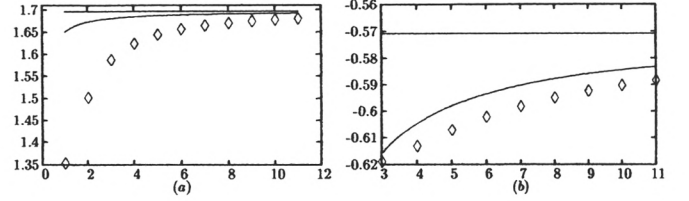


FIG. 8 (color online). The frequency gap (113) for electromagnetic perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 1$: zeroth (horizontal lines) and first order (curved lines) analytical [Eq. (141)] compared with numerical data (diamonds) [7]. (a) and (b) are plots of $\text{Re}\Delta\omega_n$ and $\text{Im}\Delta\omega_n$, respectively, as functions of n .

$$\frac{\Delta\omega_n}{r_H} \approx \frac{3\sqrt{3}(1-i\sqrt{3})}{4} \left(1 - \frac{i}{4\pi n} + \dots \right) = 1.299 - 2.25i - \frac{0.179 + 0.103i}{n} + \dots \quad (139)$$

This analytical result is compared with numerical results [7] for $r_H = 100$ in Fig. 6.

Using Eqs. (104)–(106) and (137), we obtain the spectrum of an intermediate black hole, $r_H = 1$ (see Fig. 7)

$$\omega_n = (1.969 - 2.350i)n - (0.187 + 0.1567i)\ln n + \dots \quad (140)$$

and for a small black hole, $r_H = 0.2$ (see Fig. 8)

$$\omega_n = (1.695 - 0.571i)n - (0.045 + 0.135i)\ln n + \dots \quad (141)$$

All first-order analytical results are in good agreement with numerical results [7].

V. CONCLUSIONS

We studied quasinormal modes for Schwarzschild black holes in asymptotically AdS spaces of arbitrary dimension. We obtained analytical expressions by solving the wave equation perturbatively, including first-order corrections. We studied scalar, electromagnetic and gravitational perturbations. In the case of massive scalar perturbations, we extended the method proposed in [2] for large black holes and derived explicit expressions of quasinormal frequencies for black holes of arbitrary size as a perturbative expansion in $1/m$, where m is the mass of the perturbation.

This method is not directly applicable to massless modes, because the perturbative expansion fails as $m \rightarrow 0$. Instead, we obtained perturbative expansions of gravitational and electromagnetic modes by extending the method proposed in [6] for asymptotically flat spaces. The perturbative expansion was based on zeroth-order results obtained in [3,4]. We showed that our analytical

results were in good agreement with numerical data [7]. In the case of electromagnetic perturbations, zeroth-order expressions do not yield finite quasinormal frequencies, because the effective potential vanishes [4]. By including first-order effects, we were able to arrive at finite analytical expressions with logarithmic subleading contributions.

It would be interesting to extend these results to other types of black holes and also understand their implications on the AdS/CFT correspondence.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-91ER40627. The work of S.M. was partially supported by the Thailand Research Fund. S.M. also gratefully acknowledges the hospitality of the Department of Physics and Astronomy at the University of Tennessee where part of the work was performed.

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- [1] J.S.F. Chan and R.B. Mann, *Phys. Rev. D* **55**, 7546 (1997); G.T. Horowitz and V.E. Hubeny, *Phys. Rev. D* **62**, 024027 (2000); G.T. Horowitz, *Classical Quantum Gravity* **17**, 1107 (2000); B. Wang, C.Y. Lin, and E. Abdalla, *Phys. Lett. B* **481**, 79 (2000); B. Wang, C. Molina, and E. Abdalla, *Phys. Rev. D* **63**, 084001 (2001); V. Cardoso and J.P.S. Lemos, *Phys. Rev. D* **63**, 124015 (2001); **64**, 084017 (2001); *Classical Quantum Gravity* **18**, 5257 (2001); G. Siopsis, *Nucl. Phys. B* **715**, 483 (2005); M. Giammatteo and J. Jing, *Phys. Rev. D* **71**, 024007 (2005); B. Wang, C. Lin, and C. Molina, *Phys. Rev. D* **70**, 064025 (2004); J. Jing, *gr-qc/0502010*; P. Kovtun and A. Starinets, *Phys. Rev. D* **72**, 086009 (2005); R. A. Konoplya, *Phys. Rev. D* **73**, 024009 (2006).
 - [2] G. Siopsis, *Phys. Lett. B* **590**, 105 (2004).
 - [3] V. Cardoso, J. Natario, and R. Schiappa, *J. Math. Phys. (N.Y.)* **45**, 4698 (2004).
 - [4] J. Natário and R. Schiappa, *hep-th/0411267*.
 - [5] L. Motl and A. Neitzke, *Adv. Theor. Math. Phys.* **7**, 307 (2003).
 - [6] S. Musiri and G. Siopsis, *Classical Quantum Gravity* **20**, L285 (2003).
 - [7] V. Cardoso, R. Konoplya, and J. Lemos, *Phys. Rev. D* **68**, 044024 (2003).
 - [8] A. Ishibashi and H. Kodama, *Prog. Theor. Phys.* **110**, 901 (2003).

Quasi-normal modes of electromagnetic perturbations of four-dimensional topological black holes with scalar hair

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ABSTRACT: We study the perturbative behaviour of topological black holes with scalar hair. We calculate both analytically and numerically the quasi-normal modes of the electromagnetic perturbations. In the case of small black holes we find evidence of a second-order phase transition of a topological black hole to a hairy configuration.

KEYWORDS: Classical Theories of Gravity, Black Holes.

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1. Introduction

Quasi-normal modes (QNMs) are well known to play an important role in black hole physics. They determine the late-time evolution of fields in the black hole exterior. After an initial perturbation the black hole starts vibrating into quasi-normal oscillation modes whose frequencies and decay times depend only on the intrinsic features of the black hole itself being insensitive to the details of the initial perturbation. For these reasons, QNMs of black holes in asymptotically flat spacetimes have been extensively studied (for reviews, see [1, 2]).

The Anti-de Sitter - conformal field theory (AdS/CFT) correspondence has led to an intensive investigation of black hole QNMs in asymptotically AdS spacetimes. Quasi-normal modes in AdS spacetime were first computed for a conformally invariant scalar field, whose asymptotic behaviour is similar to flat spacetime [3]. Subsequently, motivated by the AdS/CFT correspondence, Horowitz and Hubeny made a systematic computation of QNMs for scalar perturbations of Schwarzschild-AdS (S-AdS) spacetimes [4]. Their work was extended to gravitational and electromagnetic perturbations of S-AdS black holes in [5]. The study of scalar perturbations was further extended to the case of Reissner-Nordström-AdS (RN-AdS) black holes in [6]. Finally, the QNMs of scalar, electromagnetic and gravitational perturbations of RN-AdS black holes were presented in [7] using the results of [8].

In a parallel development, exact black hole solutions in asymptotically non-flat spacetimes were studied recently and solutions with scalar hair and negative cosmological constant were found. Exact black hole solutions are known in three [9, 10] and four dimensions [11]. For asymptotically flat spacetime, a four-dimensional black hole is also known,

but the scalar field diverges at the horizon [12]. Also spherically symmetric black hole solutions have been found numerically in four [13, 14] and five dimensions [15]. Spaces with a negative cosmological constant also allow for the existence of black holes whose horizon has nontrivial topology in four [16–19] and higher dimensions [20–23] as well as for gravity theories containing higher powers of the curvature [24–27].

Recently, an exact black hole solution in four dimensions with a minimally coupled self-interacting scalar field, in an asymptotically locally anti-de Sitter spacetime, was found [28] (MTZ black hole). The event horizon is a surface of negative constant curvature enclosing the curvature singularity at the origin. It was shown that there is a second-order phase transition at a critical temperature below which a black hole in vacuum undergoes a spontaneous dressing up with a nontrivial scalar field. An extension of the above solution including a charge was presented in [29]. Aspects of the thermodynamics of the MTZ black hole are discussed in [30]. In [31] it was shown that the four-dimensional MTZ black hole can be uplifted to eleven dimensions in supergravity theory.

In this work, we make a perturbative study of the MTZ black hole. We compute the simplest possible QNMs of the MTZ black hole, those of electromagnetic (EM) perturbations. The computation is carried out both analytically and numerically with fairly good agreement. The QNMs provide, near a critical temperature and for small black holes, support for the claim that a vacuum topological black hole (TBH) goes over to a hairy configuration, the MTZ black hole, through a second-order phase transition.

The paper is organized as follows. In section 2, we review the MTZ black hole and its charged extension. In section 3, we discuss the thermodynamics of the MTZ and TBH black holes. In section 4, we calculate the QNMs of the EM perturbations analytically whereas in section 5 we present the numerical computation of the QNMs and compare them with the analytical results of section 4. Finally, section 6 contains our conclusions.

2. Four-Dimensional Topological Black Hole with Scalar Hair

Consider four-dimensional gravity with negative cosmological constant ($\Lambda = -3l^{-2}$) and a scalar field described by the action

$$I = \int d^4x \sqrt{-g} \left[\frac{R + 6l^{-2}}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (2.1)$$

where l is the AdS radius, and G is the Newton's constant. The self-interaction potential is given by

$$V(\phi) = -\frac{3}{4\pi G l^2} \sinh^2 \sqrt{\frac{4\pi G}{3}} \phi, \quad (2.2)$$

which has a global maximum at $\phi = 0$, and has a mass term given by $m^2 = V''|_{\phi=0} = -2l^{-2}$. This mass satisfies the Breitenlohner-Friedman bound that ensures the perturbative stability of AdS spacetime [32, 33]. The field equations are

$$\begin{aligned} G_{\mu\nu} - \frac{3}{l^2} g_{\mu\nu} &= 8\pi G T_{\mu\nu}, \\ \square\phi - \frac{dV}{d\phi} &= 0, \end{aligned} \quad (2.3)$$

where $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, and the stress-energy tensor is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - g_{\mu\nu} V(\phi). \quad (2.4)$$

A static black hole solution with topology $\mathbb{R}^2 \times \Sigma$, where Σ is a two-dimensional manifold of negative constant curvature, is given by [28]

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2} \left[- \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right) dt^2 + \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right)^{-1} dr^2 + r^2 d\sigma^2 \right], \quad (2.5)$$

and the scalar field is

$$\phi = \sqrt{\frac{3}{4\pi G}} \operatorname{Arctanh} \frac{G\mu}{r+G\mu}. \quad (2.6)$$

The position of the horizon is at r_+ , which is the solution of

$$G\mu = \frac{r_+^2}{l} - r_+. \quad (2.7)$$

For $\phi = 0$ we get the vacuum solution (Topological Black Hole (TBH)) [16–19]

$$ds_0^2 = - \left[\frac{r^2}{l^2} - 1 - \frac{2G\mu}{r} \right] dt^2 + \left[\frac{r^2}{l^2} - 1 - \frac{2G\mu}{r} \right]^{-1} dr^2 + r^2 d\sigma^2. \quad (2.8)$$

A charged black hole with scalar hair was presented in [29]. The action is given by

$$I = \int d^4x \sqrt{-g} \left[\frac{R+6l^{-2}}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R\phi^2 - \alpha\phi^4 \right] - \frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \quad (2.9)$$

where α is an arbitrary coupling constant.

The corresponding field equations are

$$\begin{aligned} G_{\mu\nu} - \frac{3}{l^2} g_{\mu\nu} &= 8\pi G (T_{\mu\nu}^\phi + T_{\mu\nu}^{\text{em}}), \\ \square\phi &= \frac{1}{6} R\phi + \alpha\phi^3, \\ \partial_\nu(\sqrt{-g} F^{\mu\nu}) &= 0, \end{aligned} \quad (2.10)$$

and the energy-momentum tensor is given by the sum of

$$T_{\mu\nu}^\phi = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{6} [g_{\mu\nu} \square - \nabla_\mu \nabla_\nu + G_{\mu\nu}] \phi^2 - g_{\mu\nu} \alpha \phi^4, \quad (2.11)$$

and

$$T_{\mu\nu}^{\text{em}} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\gamma\alpha} F_{\delta\beta} g^{\gamma\delta} \right) g^{\alpha\beta}. \quad (2.12)$$

The charged static black hole solution with topology $\mathbb{R}^2 \times \Sigma$ is given by

$$ds^2 = - \left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right] dt^2 + \left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right]^{-1} dr^2 + r^2 d\sigma^2, \quad (2.13)$$

where $-\infty < t < \infty$ and $r > 0$. The scalar field is

$$\phi = \sqrt{\frac{1}{2\alpha l^2}} \frac{G\mu}{(r + G\mu)}, \quad (2.14)$$

with $\alpha > 0$ and the only non-zero component of the electromagnetic field is

$$A_t = -\frac{q}{r}. \quad (2.15)$$

The integration constants q and μ are not independent. They are related by

$$q^2 = -G\mu^2 \left(1 - \frac{2\pi G}{3\alpha l^2} \right). \quad (2.16)$$

They correspond to conserved charges,

$$M = \frac{\sigma}{4\pi} \mu \quad \text{and} \quad Q = \frac{\sigma}{4\pi} q, \quad (2.17)$$

respectively, where σ denotes the area of Σ .

Eq. (2.16) fixes a charge-to-mass ratio for this black hole, which is a function of the constants appearing in the action, G and α . Moreover, eq. (2.16) determines an upper bound for α

$$0 < \alpha \leq \frac{2\pi G}{3l^2}. \quad (2.18)$$

If the upper bound is saturated, the charge vanishes. Then we recover the MTZ black hole. Indeed the form of the self-interacting potential considered in (2.2) can be naturally obtained through the relation between the conformal and Einstein frames [28].

Note that if $\mu = 0$ then both the MTZ black hole (2.5) and the TBH black hole (2.8) go to

$$ds_{AdS}^2 = - \left[\frac{r^2}{l^2} - 1 \right] dt^2 + \left[\frac{r^2}{l^2} - 1 \right]^{-1} dr^2 + r^2 d\sigma^2, \quad (2.19)$$

which is a manifold of negative constant curvature possessing an event horizon at $r = l$. We can say that as $\phi \rightarrow 0$ the MTZ and TBH black holes match continuously at $\mu = 0$ or $r = l$ with (2.19) being a transient configuration as it becomes apparent in the following. In the sequel we set $l = 1$.

3. Thermodynamics

For the MTZ black hole, the temperature, entropy and mass are given respectively by [28]

$$\begin{aligned} T &= \frac{2r_+ - 1}{2\pi}, \\ S &= \frac{\sigma r_+(r_+ + 2G\mu)}{4G(r_+ + G\mu)^2} r_+^2 = \frac{\sigma}{4G} (2r_+ - 1), \\ M &= \frac{\sigma\mu}{4\pi} = \frac{\sigma(r_+^2 - r_+)}{4\pi G}. \end{aligned} \quad (3.1)$$

Notice that the entropy in this case does not satisfy an area law. It is easy to show that the law of thermodynamics $dM = TdS$ holds. Defining the free energy as $F = M - TS$ and using relations (3.1), we obtain

$$F_{MTZ} = -\frac{\sigma}{8\pi G}(2r_+^2 - 2r_+ + 1). \quad (3.2)$$

The free energy (3.2) can be written as

$$F_{MTZ} = -\frac{\sigma}{8\pi G}\left(1 + 2(T - T_0)\pi + 2(T - T_0)^2\pi^2\right), \quad (3.3)$$

where $T_0 = 1/2\pi \approx 0.160$ is the critical temperature. For the vacuum TBH black hole (denoting by ρ_+ the horizon for this case), the temperature, entropy and mass are, respectively,

$$\begin{aligned} T &= \frac{3\rho_+^2 - 1}{4\pi\rho_+}, \\ S &= \frac{\sigma\rho_+^2}{4G}, \\ M &= \frac{\sigma(\rho_+^3 - \rho_+)}{8\pi G}. \end{aligned} \quad (3.4)$$

Then, the free energy of the TBH black hole, using relations (3.4), is

$$F_{TBH} = -\frac{\sigma}{16\pi G}(\rho_+^3 + \rho_+), \quad (3.5)$$

which can be expanded around the critical temperature T_0 as

$$F_{TBH} = -\frac{\sigma}{8\pi G}\left(1 + 2(T - T_0)\pi + 2(T - T_0)^2\pi^2 + (T - T_0)^3\pi^3 + \dots\right). \quad (3.6)$$

Using (3.3) and (3.6), we can calculate the difference between the TBH and MTZ free energies. We obtain

$$\Delta F = F_{TBH} - F_{MTZ} = -\frac{\sigma}{8\pi G}(T - T_0)^3\pi^3 + \dots, \quad (3.7)$$

indicating a phase transition between MTZ and TBH at the critical temperature T_0 . Matching the temperatures of the MTZ black hole and the TBH we get:

$$T = \frac{2r_+ - 1}{2\pi} = \frac{1}{4\pi}\left(3\rho_+ - \frac{1}{\rho_+}\right) \Rightarrow r_+ = \frac{3\rho_+}{4} - \frac{1}{4\rho_+} + \frac{1}{2}. \quad (3.8)$$

It is easily seen that $r_+ \leq \rho_+$, and the inequality is saturated for $r_+ = \rho_+ = 1$. We remark that the temperature T should be non-negative, so $r_+ \geq \frac{1}{2}$ for the MTZ black hole and $\rho_+ \geq \frac{1}{\sqrt{3}}$ for the TBH black hole.

Thermodynamically we can understand this phase transition as follows. Using relations (3.1), (3.4) and (3.8), we find that $S_{TBH} > S_{MTZ}$ and $M_{TBH} > M_{MTZ}$ for the relevant ranges of the horizons r_+ or ρ_+ . If $r_+ > 1$ ($T > T_0$), both black holes have positive mass. As $T > T_0$ implies $F_{TBH} \leq F_{MTZ}$, the MTZ black hole dressed with the scalar field will decay into the bare black hole. In the decay process, the scalar black hole

absorbs energy from the thermal bath, increasing its horizon radius (from r_+ to $\rho_+ > r_+$) and consequently its entropy. Therefore, in a sense the scalar field is absorbed by the black hole.

If $r_+ < 1$ ($T < T_0$), both black holes have negative mass, but now $F_{TBH} > F_{MTZ}$, which means that the MTZ configuration with nonzero scalar field is favorable. As a consequence, below the critical temperature, the bare black hole undergoes a spontaneous “dressing up” with the scalar field. In the process, the mass and entropy of the black hole decrease and the differences in energy and entropy are transferred to the heat bath.

At the critical temperature, the thermodynamic functions of the two phases match continuously, hence, the phase transition is of second order. The order parameter that characterizes the transition can be defined in terms of the value of the scalar field at the horizon; using the solution for the scalar field (2.6) we obtain for $T < T_0$,

$$\lambda_\phi = \left| \tanh \sqrt{\frac{4\pi G}{3}} \phi(r_+) \right| = \left| \frac{r_+ - 1}{r_+} \right| = \frac{T_0 - T}{T_0 + T} . \quad (3.9)$$

For $T > T_0$, λ_ϕ vanishes. Then the relation (3.7) can be written in terms of the order parameter λ_ϕ as

$$\Delta F = F_{TBH} - F_{MTZ} = + \frac{\sigma r_+^3}{8\pi G} \lambda_\phi^3 + \dots \quad (T < T_0). \quad (3.10)$$

The pure AdS space of (2.19) has free energy $F_{AdS} = -\sigma/8\pi G$ as easily can be seen using relations (3.2) or (3.5) with $r_+ = 1$. Then observe that F_{AdS} is the constant term of both F_{MTZ} in (3.3) and F_{TBH} in (3.6). Hence the difference of free energies of MTZ or TBH black holes with the free energy of pure AdS space indicates that the configuration (2.19) is transient between the MTZ and TBH phase transition.

In the next two sections we will find the QNMs of the electromagnetic perturbations of the MTZ black hole and its charged generalization and compare them with the corresponding QNMs of the TBH. This study will provide additional information on the stability of the MTZ black hole and its generalization under electromagnetic perturbations. Note that for the MTZ black hole and its charged generalization, the wave equations, after factoring out the angular parts, are the same. For the TBH, only the function $f(r)$ (eq. (4.2)) changes. There is no change due to the axial or polar character of the perturbation.

4. Analytical Calculation

In this section, we calculate analytically the QNMs of electromagnetic perturbations for both MTZ and topological black holes. Numerical results will be discussed in section 5. Our approach is based on the method discussed in [34–36]. In order to calculate QNMs, in general, one solves the wave equation subject to appropriate boundary conditions at infinity and the horizon. In asymptotically flat spacetime, this can be implemented by a monodromy method [37]. It is then advantageous to follow Stokes lines which go through the black hole singularity. In asymptotically (A)dS spacetimes, the calculation simplifies because the wavefunction vanishes at infinity. Although the monodromy does not enter the

calculation, in several cases it is still advantageous to follow Stokes lines and analytically continue the wavefunction through the black hole singularity [34–36]. However, this is not always necessary, if a general solution (or approximation thereof) of the wave equation as in, e.g., [38] is readily available. In our case, we shall solve the wave equation near the black hole singularity perturbatively and then analytically continue the wavefunction in order to match with the expected behaviour at infinity and the horizon, respectively. This will produce an explicit form of the QNMs for $r_+ > 1$ ($G\mu > 0$). For $r_+ < 1$ ($G\mu < 0$), we do not have explicit analytical expressions. Instead, we obtain a lower bound which is in accord with numerical results.

Electromagnetic perturbations obey the wave equation

$$f(r)\frac{d}{dr}\left(f(r)\frac{d\Psi_\omega}{dr}\right) + \left[\omega^2 - \left(\xi^2 + \frac{1}{4}\right)\frac{f(r)}{r^2}\right]\Psi_\omega = 0, \quad (4.1)$$

where

$$f_{MTZ}(r) = r^2 - \left(1 + \frac{G\mu}{r}\right)^2, \quad f_{TBH}(r) = r^2 - 1 - \frac{2G\mu}{r}. \quad (4.2)$$

It may be cast into a Schrödinger-like form if written in terms of the tortoise coordinate defined by

$$\frac{dx}{dr} = \frac{1}{f(r)}. \quad (4.3)$$

We obtain

$$-\frac{d^2\Psi_\omega}{dx^2} + V(x)\Psi_\omega = \omega^2\Psi_\omega, \quad (4.4)$$

where the potential is

$$V[x(r)] = \left(\xi^2 + \frac{1}{4}\right)\frac{f(r)}{r^2}. \quad (4.5)$$

For QNMs, we impose the boundary condition $\Psi_\omega \rightarrow 0$ as $r \rightarrow \infty$, since the potential does not vanish for large r . At the horizon ($x \rightarrow -\infty$), we demand $\Psi_\omega \sim e^{i\omega x}$ (ingoing wave).

Near the black hole singularity ($r \sim 0$), the tortoise coordinate (4.3) may be approximated by

$$x \approx -\frac{G\mu}{a\lambda} \left(\frac{r}{G\mu}\right)^\lambda, \quad (4.6)$$

where $a = 1$, $\lambda = 3$ for MTZ and $a = 2$, $\lambda = 2$ for TBH. In arriving at (4.6), we choose the integration constant so that $x = 0$ at $r = 0$. The potential near the singularity is

$$V(x) \approx -\frac{\mathcal{A}}{x^{1+1/\lambda}}, \quad \mathcal{A} = \left(\xi^2 + \frac{1}{4}\right) \frac{1}{(aG\mu)^{1-1/\lambda}(-\lambda)^{1+1/\lambda}}. \quad (4.7)$$

We will solve the wave eq. (4.4) by treating the potential (4.7) as a perturbation. The zeroth-order solutions are

$$\Psi_0^\pm = e^{\pm i\omega x} \quad (4.8)$$

and the first-order corrections are

$$\Psi_1^\pm(x) = \frac{1}{2i\omega} \int_0^x dx' e^{i(\omega - i\epsilon)(x-x')} V(x') \Psi_0^\pm(x') - \frac{1}{2i\omega} \int_0^x dx' e^{-i(\omega + i\epsilon)(x-x')} V(x') \Psi_0^\pm(x') \quad (4.9)$$

where we included a small $\epsilon > 0$ to render integrals finite. We shall work with general values of λ in the potential (4.7) and take the limit of interest ($\lambda \rightarrow 2, 3$) at the end of the calculation. The desired solution will be a linear combination of the above wave functions,

$$\Psi_\omega(x) = A_+(\Psi_0^+ + \Psi_1^+) + A_-(\Psi_0^- + \Psi_1^-). \quad (4.10)$$

Asymptotically, it behaves as

$$\Psi_\omega(x) \approx \left(A_+ + ie^{\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_- \right) e^{i\omega x} + \left(A_- - ie^{-\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_+ \right) e^{-i\omega x}. \quad (4.11)$$

At large r ($x \rightarrow +\infty$), the tortoise coordinate (4.3) may be approximated by

$$x \approx x_0 - \frac{1}{r}, \quad x_0 = \int_0^\infty \frac{dr}{f(r)} \quad (4.12)$$

and the potential (4.7) is

$$V(x) \approx \xi^2 + \frac{1}{4}. \quad (4.13)$$

Since we are interested in the asymptotic form of quasi-normal frequencies, we may ignore the potential ($V(x) \lesssim \omega^2$). We obtain the eigenfunction for large r ($x \sim x_0$)

$$\Psi_\omega(x) \sim \sin \omega(x - x_0), \quad (4.14)$$

where we applied the boundary condition $\Psi_\omega \rightarrow 0$ as $r \rightarrow \infty$ ($x \rightarrow x_0$).

This is matched by the linear combination (4.11) provided

$$A_+ + ie^{\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_- = -e^{-2i\omega x_0} \left(A_- - ie^{-\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_+ \right) \quad (4.15)$$

of eigenfunction (eqs. (4.8) and (4.9)) in the vicinity of the singularity.

Next, we approach the horizon ($x \rightarrow -\infty$) by analytically continuing (4.10) to negative x . This amounts to a rotation by $-\pi$ in the complex x -plane. For $x < 0$, we obtain from (4.9), using eqs. (4.7) and (4.8),

$$\Psi_1^\pm(x) = \frac{\mathcal{A}}{2i\omega} e^{i\pi/\lambda} \int_0^{-x} \frac{dx'}{x'^{1+1/\lambda}} \left(e^{i(\omega+i\epsilon)(x+x')} - e^{-i(\omega-i\epsilon)(x+x')} \right) e^{\mp i\omega x'}. \quad (4.16)$$

Taking the limit $x \rightarrow -\infty$, we obtain the behavior near the horizon

$$\Psi_\omega(x) \approx \left(A_+ - ie^{\frac{3i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_- \right) e^{i\omega x} + \left(A_- + ie^{\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_+ \right) e^{-i\omega x}. \quad (4.17)$$

Since we want an ingoing wave ($\Psi_\omega \sim e^{i\omega x}$) at the horizon, we obtain the constraint on the coefficients

$$A_- + ie^{\frac{i\pi}{2\lambda}} \frac{\mathcal{A}\Gamma(-1/\lambda)}{(2\omega)^{1-1/\lambda}} A_+ = 0. \quad (4.18)$$

For compatibility with the other constraint (4.15), we ought to have

$$\begin{vmatrix} e^{-2i\omega x_0} + e^{\frac{i\pi}{2\lambda}} \mathcal{C} & 1 \\ 1 - \mathcal{C} e^{-\frac{i\pi}{2\lambda}} e^{-2i\omega x_0} & \mathcal{C} e^{-\frac{i\pi}{2\lambda}} \end{vmatrix} = 0, \quad (4.19)$$

where $\mathcal{C} = i\mathcal{A}\Gamma(-1/\lambda)(2\omega)^{-1+1/\lambda}$. We deduce

$$e^{2i\omega x_0} = 2e^{-\frac{i\pi}{2\lambda}\mathcal{C}}, \quad (4.20)$$

where we discarded terms which were of order higher than linear in $\mathcal{C}$. Solving for ω , we obtain the quasi-normal frequencies

$$\omega_n = \frac{n\pi}{x_0} + o(\ln n). \quad (4.21)$$

Thus the asymptotic behaviour is completely determined by the parameter x_0 (eq. (4.12)).

In the case of MTZ black holes, $f(r)$ has four roots,

$$r_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 + 4G\mu} \right), \quad \bar{r}_{\pm} = \frac{1}{2} \left(-1 \pm \sqrt{1 - 4G\mu} \right). \quad (4.22)$$

For $G\mu > 0$, the horizon is at r_+ and $r_+ > 1$. We obtain

$$x_0 = \frac{1}{2(r_+ - r_-)} \ln \frac{r_-}{r_+} + \frac{1}{2(\bar{r}_+ - \bar{r}_-)} \ln \frac{\bar{r}_-}{\bar{r}_+}. \quad (4.23)$$

For large $G\mu$, the quasi-normal frequencies are

$$\omega_n \approx 2(1 - i)r_+ n \quad (4.24)$$

matching the behaviour in *five-dimensional* AdS space.

For small (positive) $G\mu$, we find

$$\omega_n \approx -2ni \left(1 + \frac{2G\mu}{\pi i} \ln G\mu \right). \quad (4.25)$$

Continuing to negative values of $G\mu$ is not straightforward. The above discussion is not applicable, because going beyond the horizon (as we approach the singularity) we encounter a potential well at $r_- < r < r_+$ for $G\mu > -1/4$ (horizon $1/2 < r_+ < 1$) which admits bound states and alters the behaviour of quasi-normal frequencies. The minimum of the potential provides a lower bound to the frequencies. By setting $V'(r) = 0$, we find the minimum at

$$r_{min} = -2G\mu. \quad (4.26)$$

At the minimum,

$$V(r_{min}) = - \left(\xi^2 + \frac{1}{4} \right) \left\{ \frac{1}{16(G\mu)^2} - 1 \right\}. \quad (4.27)$$

It is also easily seen that $V''(r_{min}) > 0$, showing that $r = r_{min}$ is indeed a minimum. The eigenfrequencies have imaginary part

$$\omega_I \geq -\sqrt{|V(r_{min})|}, \quad (4.28)$$

which is verified by numerical results. The lowest frequency is close to the lower bound (4.28).

In the TBH case, the horizon is given by

$$r_+ = 2\Re(e^{i\pi/6}s), \quad s = \left(\sqrt{\frac{1}{27} - (G\mu)^2} - iG\mu \right)^{1/3}, \quad (4.29)$$

where $|G\mu| < 3^{-3/2}$. The other two roots of $f(r)$ are also real,

$$r_- = -2\Re(e^{-i\pi/6}s), \quad r'_- = 2\Im s. \quad (4.30)$$

We obtain

$$x_0 = -\frac{r_-}{3r_-^2 - 1} \ln \frac{r_-}{r_+} - \frac{r'_-}{3r_-'^2 - 1} \ln \frac{r'_-}{r_+}. \quad (4.31)$$

For small $G\mu$, we have

$$\omega_n \approx -2ni \left(1 + \frac{4G\mu}{\pi i} \ln G\mu \right) \quad (4.32)$$

to be compared with the behaviour of MTZ QNMs (4.25) near the transition point. The discrepancy is of second order.

For large $G\mu$, the above formulae for the roots read

$$\begin{aligned} r_+ &= \left(G\mu + \sqrt{(G\mu)^2 - \frac{1}{27}} \right)^{1/3} + \left(G\mu - \sqrt{(G\mu)^2 - \frac{1}{27}} \right)^{1/3} \approx (2G\mu)^{1/3}, \\ r_- &= -e^{-\frac{i\pi}{3}} \left(G\mu + \sqrt{(G\mu)^2 - \frac{1}{27}} \right)^{1/3} - e^{+\frac{i\pi}{3}} \left(G\mu - \sqrt{(G\mu)^2 - \frac{1}{27}} \right)^{1/3} \approx -e^{-\frac{i\pi}{3}} r_+, \\ r'_- &= (r_-)^* \approx -e^{+\frac{i\pi}{3}} r_+. \end{aligned} \quad (4.33)$$

Then

$$\omega_n \approx nr_+ \left(\frac{3\sqrt{3}}{4} - i\frac{9}{4} \right). \quad (4.34)$$

For $G\mu < 0$, we encounter a potential well behind the horizon. Arguing as in the MTZ case, we obtain a lower bound

$$\omega_I \geq -\sqrt{|V(r_{min})|} \quad (4.35)$$

In the TBH case, $r_{min} = -3G\mu$ and

$$V(r_{min}) = -\left(\xi^2 + \frac{1}{4} \right) \left\{ \frac{1}{27(G\mu)^2} - 1 \right\} \quad (4.36)$$

5. Numerical Results

5.1 The Method

Another method of studying the same problem is the procedure of Horowitz and Hubeny [4]. We briefly review the method as we applied it to our problem. After performing the transformation $\Psi_\omega(r) = \psi_\omega(r)e^{-i\omega r^*}$, the wave equation (4.1) becomes

$$f(r) \frac{d^2 \psi_\omega(r)}{dr^2} + \left(\frac{df(r)}{dr} - 2i\omega \right) \frac{d\psi_\omega(r)}{dr} = \frac{(\xi^2 + \frac{1}{4})}{r^2} \psi_\omega(r). \quad (5.1)$$

The change of variables $r = 1/x$ (not to be confused with the tortoise coordinate (4.3)) yields an equation of the form

$$s(x) \left[(x - x_+)^2 \frac{d^2 \psi_\omega(x)}{dx^2} \right] + t(x) \left[(x - x_+) \frac{d\psi_\omega(x)}{dx} \right] + u(x) \psi_\omega(x) = 0,$$

where $x_+ = 1/r_+$. It turns out that $s(x), t(x)$ and $u(x)$ are polynomials of third degree for MTZ black holes and of second degree for TBHs. Thus,

$$\begin{aligned} s(x) &= s_0 + s_1(x - x_+) + s_2(x - x_+)^2 + s_3(x - x_+)^3, \\ t(x) &= t_0 + t_1(x - x_+) + t_2(x - x_+)^2 + t_3(x - x_+)^3, \\ u(x) &= u_0 + u_1(x - x_+) + u_2(x - x_+)^2 + u_3(x - x_+)^3. \end{aligned}$$

Expanding the wavefunction around the (inverse) horizon x_+ ,

$$\psi_\omega(x) = \sum_0^\infty a_n(\omega)(x - x_+)^n, \quad (5.2)$$

we arrive at a recurrence formula for the coefficients,

$$a_n(\omega) = -\frac{1}{n(n-1)s_0 + nt_0 + u_0} \sum_{m=n-3}^{n-1} [m(m-1)s_{n-m} + mt_{n-m} + u_{n-m}] a_m(\omega). \quad (5.3)$$

We note that the few coefficients $a_m(\omega)$ with negative index m which will appear for $n < 2$ should be set to zero, while $a_0(\omega)$ is set to one. Since the wave function should vanish at infinity ($r \rightarrow \infty, x = 0$), we deduce

$$\psi_\omega(0) \equiv \sum_0^\infty a_n(\omega)(-x_+)^n = 0. \quad (5.4)$$

The solutions of this equation are precisely the quasi-normal frequencies.

5.2 MTZ Black Holes

We compute the QNMs for the MTZ black holes by solving eq. (5.4) numerically. As a first step we draw the contours $\Re[\psi_\omega(0)] = 0$ and $\Im[\psi_\omega(0)] = 0$ on the complex ω -plane and check the points of intersection. This provides good initial values for the subsequent Müller root-finding technique [39]; in addition it provides an overview of the (approximate) values of the quasi-normal frequencies. In figure 1 we show sample contours for the case $r_+ = 5.0$, $\xi = 1.0$. We note that the parameter ξ does not seem to play a significant role in the behaviour of quasi-normal frequencies, so we set it to a typical value ($\xi = 1.0$) from now on.

From figure 1, it is evident that the QNMs lie on a straight line with *negative* slope and their spacing is more or less constant. In fact the spacing changes a little as we move to the right and eventually attains an asymptotic value, which should be compared with the analytical results of section 4.

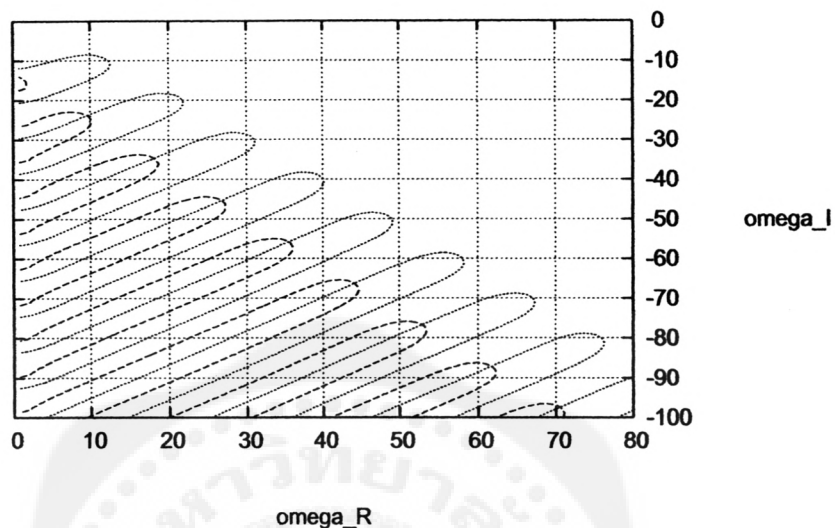


Figure 1: Contours displaying the lines $\Re[\psi_0(\omega)] = 0$ (dashed lines) and $\Im[\psi_0(\omega)] = 0$ (dotted lines) on the complex ω -plane for MTZ Black Holes and $r_+ = 5.0$.

Figure 2 refers to $r_+ = 0.97$, a typical value of the horizon radius smaller than the critical value $r_+ = 1.0$. The striking feature of the QNMs here is that the slope is *positive*. In addition, the intersections no longer lie along a straight line and the spacing changes substantially as we move through the QNMs.

To compare with analytical results, we note that equations (4.21) and (4.23) yield the asymptotic expression for QNMs. We may check that the relative sign of $\Delta\omega_R$ and $\Delta\omega_I$ is negative, therefore a larger ω_R will correspond to an algebraically smaller ω_I , as shown in figure 1.

For small mass ($r_+ \approx 1$), the analytical result (4.25) yields the asymptotic expression for the spacing of QNMs,

$$\Delta\omega_R \approx -\frac{4G\mu}{\pi} \ln G\mu, \quad \Delta\omega_I \approx -2. \quad (5.5)$$

In table 1, we show the results for some indicative values of the horizon radius with $r_+ \geq 1.0$. We observe that the agreement between analytical and numerical results is quite good. The agreement is exact for $r_+ = 1.0$. In fact, the slope tends to minus infinity as we approach $r_+ = 1.0$. We should point out that $\omega_I < 0$, always, so there is no sign of instability.

In table 2 we show the QNMs for $r_+ = 0.97$, as an example of horizon radius smaller than the critical value $r_+ = 1.0$, along with the differences between consecutive values

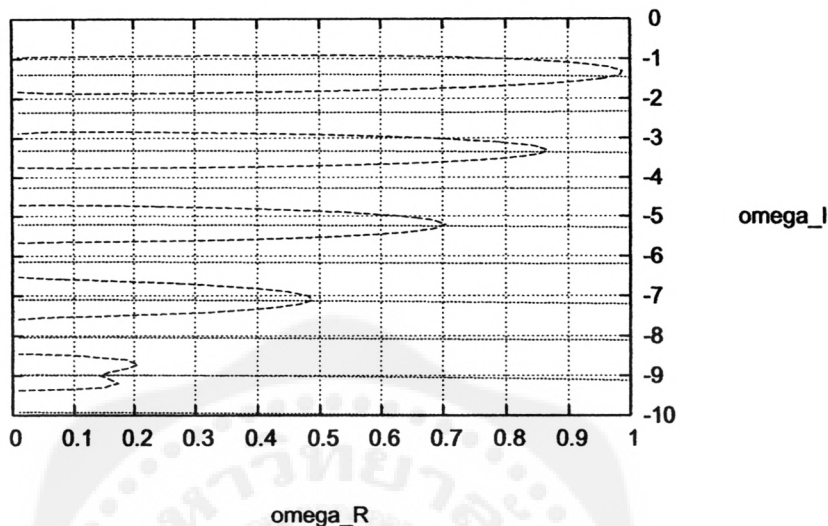
MTZ: E/M perturbations, $r_+=0.97$, $\chi=1.0$


Figure 2: Contours displaying the lines $\Re[\psi_0(\omega)] = 0$ (dashed lines) and $\Im[\psi_0(\omega)] = 0$ (dotted lines) on the complex ω -plane for MTZ Black Holes and $r_+ = 0.97$.

r_+	$G\mu$	T	$\Delta\omega_R^{Ana}$	$\Delta\omega_I^{Ana}$	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
10.000	90.00	3.023	18.96	-20.26	19.00	-20.15
5.000	20.00	1.434	8.91	-10.25	8.77	-10.45
2.000	2.000	0.477	2.75	-4.21	2.70	-4.14
1.050	0.0525	0.175	0.32	-2.15	0.27	-2.16
1.000	0.000	0.160	0.00	-2.00	0.00	-2.00

Table 1: Comparison analytical and numerical values of QNMs of EM perturbations of MTZ Black Holes.

of QNMs. Note that in this case $G\mu = -0.029$, $T = 0.150$. These are the exact QNMs presented in figure 2. It appears that, apart from the change of the sign of the slope, there is a novel phenomenon: the quasi-normal frequencies converge toward the imaginary axis, i.e., their real part decreases. There is only a *finite* number of QNMs for $r_+ < 1.0$. This behaviour has already been predicted analytically in section 4; eq. (4.28) yields a prediction for the lowest possible imaginary part of the frequencies. The result is $\omega_I \geq -9.54i$, which is indeed respected by the imaginary parts appearing in table 2. Although not fully justified, using the analytical result (5.5), we deduce the estimates $|\Delta\omega_R^{Ana}| \approx 0.131$, $|\Delta\omega_I^{Ana}| \approx 2.000$. Checking against the numerical values in table 2, we see that the values in the

ω_R^{Num}	ω_I^{Num}	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
0.973	-1.496	-	-
0.864	-3.351	0.109	1.905
0.701	-5.239	0.163	1.888
0.486	-7.114	0.215	1.875
0.143	-8.980	0.343	1.866

Table 2: Numerical results for quasi-normal frequencies of EM perturbations for MTZ Black Holes with $r_+ = 0.97$. In the last two columns we list the differences of consecutive QNMs.

third column (real part) are of the same order of magnitude as the analytical prediction, whereas the values in the fourth column (imaginary part) are quite close to the analytical estimate 2. However, the variations are significant. If one further decreases the value of the horizon, the number of QNMs decreases until they finally disappear. The behavior of QNMs for such horizons may be viewed as modifications of the critical point $r_+ = 1.0$. In the latter case, the first quasi-normal frequency is $\omega = 1.0 - 1.5i$, followed by frequencies with imaginary parts $\omega_I = -3.5, -5.5, -7.5, \dots$. Unlike the points below criticality, for $r_+ = 1.0$, the real parts do not change and there is no lower bound.

To summarize our findings, for $r_+ > 1$, figure 1 shows that the QNMs have a negative slope and hence large values of ω_R correspond to large values of ω_I . Then as $r_+ \rightarrow 1$ the slope tends to minus infinity. At $r_+ = 1$, $\omega_R = 1$, while the spacing of ω_I is exactly 2. Note that the critical point $r_+ = 1$ corresponds to the pure AdS configuration of (2.19). For $r_+ < 1$, figure 2 shows that the QNMs have a positive slope and hence large values of ω_R correspond to small values of ω_I . We attribute this behavior to a *phase transition*, as one passes through the critical value $r_+ = 1$, of the TBH to the MTZ black hole configuration.

Notice that for any perturbation of a black hole background there are two characteristic parameters that control its behaviour: the oscillation time scale which is given by $\tau_R \equiv 1/\omega_R$ and the damping time scale given by $\tau_I \equiv 1/\omega_I$. In the case of $r_+ > 1$ the scalar field is absorbed by the black hole and the damping time scale τ_I is small, so the perturbations in this case fall off rather rapidly with time. For $r_+ < 1$ the black hole is dressed up with the scalar field and the damping time scale τ_I is large and perturbations last longer. At the critical point of $r_+ = 1$ we have a change of slope, indicating a transient configuration. This behaviour may be associated with the second order phase transition discussed in section 3.

5.3 Topological Black Holes

We now turn to the case of topological black holes (TBHs). The asymptotic value of quasi-normal frequencies for $r_+ \geq 1$ is given by eqs (4.21) and (4.31). The relative sign of $\Delta\omega_R$, and $\Delta\omega_I$ is again negative.

At small masses, eq. (4.32) provides an approximate expression for QNMs in the limit of small horizon (mass) which yields the spacings

$$\Delta\omega_R \approx -\frac{8G\mu}{\pi} \ln G\mu, \quad \Delta\omega_I \approx -2. \quad (5.6)$$

r_+	$G\mu$	T	$\Delta\omega_R^{Ana}$	$\Delta\omega_I^{Ana}$	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
12.692	1015.43	3.023	16.43	-28.55	16.50	-28.50
6.055	107.97	1.434	7.73	-13.60	7.65	-13.70
2.155	3.93	0.477	2.41	-4.78	2.52	-4.81
1.050	0.054	0.175	0.29	-2.16	0.26	-2.16
1.000	0.000	0.160	0.00	-2.00	0.00	-2.00

Table 3: Comparison of analytical and numerical values of QNMs of EM perturbations for TBHs.

ω_R^{Num}	ω_I^{Num}	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
0.972	-1.446	-	-
0.859	-3.352	0.113	1.906
0.689	-5.240	0.170	1.888
0.453	-7.115	0.236	1.875

Table 4: Numerical results for quasi-normal frequencies of EM perturbations for TBHs with $r_+ = 0.97$. In the last two columns we list the differences of consecutive QNMs.

The comparison of these analytical results to the numerical results of section 5 is shown in table 3. We have used values for the horizon which correspond, through eq. (3.8), to the respective values of MTZ black holes that we showed above in table 1 (matching corresponding temperatures). The agreement between analytical and numerical results is very good. In addition, ω_I is again negative, showing no sign of instability.

Next we show the QNMs for $r_+ = 0.97$ (eq. (3.8) yields approximately the same value as in the MTZ case for the same temperature), along with differences between consecutive values of quasi-normal frequencies. Note that in this case, $G\mu = -0.029$, $T = 0.150$.

Similar behaviour to MTZ black holes is observed: the slope below criticality becomes *positive*, while for $r_+ > 1$ it is *negative*. Again, to compare with analytical expressions, eq. (5.6) gives the estimate $|\Delta\omega_R^{Ana}| \approx 0.259$, $|\Delta\omega_I^{Ana}| \approx 2.000$. Checking against the entries in table 4, we find that the values in its third and fourth columns (real and imaginary parts, respectively) are of the same order of magnitude as the analytical estimates, albeit with sizeable variations. The lower bound for the imaginary part, given by eqs. (4.28) and (4.36), reads in this case, $\omega_I \geq -7.33$, which is close to the lowest imaginary part in table 4.

6. Conclusions

We calculated the QNMs of electromagnetic perturbations of the MTZ black hole and topological black holes. We performed this calculation both analytically and numerically with fairly good agreement. We found that there is a change in the slope of the QNMs as we decrease the value of the horizon radius below a critical value, and we argued that this change signals a phase transition of a vacuum topological black hole toward the MTZ black hole with scalar hair.

One may attribute this change in behaviour to the dynamics of the scalar field and associate it with a second-order phase transition [28]. For small black holes ($r_+ < 1$) the scalar field is dressing up the bare topological black hole introducing an order parameter λ_ϕ (see (3.10)) which controls the dynamics of the scalar field. A second-order phase transition for small black holes occurs in more general configurations (including charge, etc) as studies of general scalar, electromagnetic and gravitational perturbations show [40]. We have also found that the vacuum AdS solution at the critical temperature is a transient configuration of the change of phase of the topological black hole to a configuration with scalar hair.

One interesting aspect of small MTZ black holes is that the quasi-normal frequencies converge toward the imaginary axis, i.e., their real part decreases and after the first few quasi-normal frequencies, it vanishes, indicating that for $r_+ < 1$ there are only a finite number of QNMs. We showed that the finite number of such modes for small horizons ($r_+ < 1$) is due to the existence of bound states behind the horizon, which is an unobservable region. This curious phenomenon is worthy of further investigation.

Acknowledgments

We thank Kostas Kokkotas for his valuable assistance in the numerical applications of this work. The work of S.M. was partially supported by the Thailand Research Fund. G.K. and E.P. were supported by (EPEAEK II)-Pythagoras (co-funded by the European Social Fund and National Resources). G.S. was supported in part by the US Department of Energy under grant DE-FG05-91ER40627.

References

- [1] K.D. Kokkotas and B.G. Schmidt, *Quasi-normal modes of stars and black holes*, *Living Rev. Rel.* **2** (1999) 2 [gr-qc/9909058].
- [2] H.-P. Nollert, *Class. and Quant. Grav.* **16** (1999) R159.
- [3] J.S.F. Chan and R.B. Mann, *Scalar wave falloff in asymptotically anti-de Sitter backgrounds*, *Phys. Rev. D* **55** (1997) 7546 [gr-qc/9612026]; *Scalar wave falloff in topological black hole backgrounds*, *Phys. Rev. D* **59** (1999) 064025.
- [4] G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev. D* **62** (2000) 024027 [hep-th/9909056].
- [5] V. Cardoso and J.P.S. Lemos, *Quasi-normal modes of Schwarzschild anti-de sitter black holes: electromagnetic and gravitational perturbations*, *Phys. Rev. D* **64** (2001) 084017 [gr-qc/0105103].
- [6] B. Wang, C.-Y. Lin and E. Abdalla, *Quasinormal modes of reissner-nordstroem anti-de Sitter black holes*, *Phys. Lett. B* **481** (2000) 79 [hep-th/0003295].
- [7] E. Berti and K.D. Kokkotas, *Quasinormal modes of reissner-nordstroem-anti-de Sitter black holes: scalar, electromagnetic and gravitational perturbations*, *Phys. Rev. D* **67** (2003) 064020 [gr-qc/0301052].
- [8] F. Mellor and I. Moss, *Stability of black holes in de Sitter space*, *Phys. Rev. D* **41** (1990) 403.

- [9] C. Martinez and J. Zanelli, *Conformally dressed black hole in 2+1 dimensions*, *Phys. Rev. D* **54** (1996) 3830 [gr-qc/9604021].
- [10] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, *Black holes and asymptotics of 2+1 gravity coupled to a scalar field*, *Phys. Rev. D* **65** (2002) 104007 [hep-th/0201170].
- [11] C. Martinez, R. Troncoso and J. Zanelli, *de Sitter black hole with a conformally coupled scalar field in four dimensions*, *Phys. Rev. D* **67** (2003) 024008 [hep-th/0205319].
- [12] N. Bocharova, K. Bronnikov and V. Melnikov, *Vestn. Mosk. Univ. Fiz. Astron.* **6** (1970) 706; J.D. Bekenstein, *Exact solutions of Einstein conformal scalar equations*, *Ann. Phys. (NY)* **82** (1974) 535; *Black holes with scalar charge*, *Ann. Phys. (NY)* **91** (1975) 75.
- [13] T. Torii, K. Maeda and M. Narita, *Scalar hair on the black hole in asymptotically anti-de Sitter spacetime*, *Phys. Rev. D* **64** (2001) 044007.
- [14] E. Winstanley, *On the existence of conformally coupled scalar field hair for black holes in (anti-)de Sitter space*, *Found. Phys.* **33** (2003) 111–143 [gr-qc/0205092].
- [15] T. Hertog and K. Maeda, *Black holes with scalar hair and asymptotics in $N = 8$ supergravity*, *JHEP* **07** (2004) 051 [hep-th/0404261].
- [16] J.P.S. Lemos, *Cylindrical black hole in general relativity*, *Phys. Lett. B* **353** (1995) 46 [gr-qc/9404041].
- [17] R.B. Mann, *Pair production of topological anti-de Sitter black holes*, *Class. and Quant. Grav.* **14** (1997) L109 [gr-qc/9607071]; *Charged topological black hole pair creation*, *Nucl. Phys. B* **516** (1998) 357 [hep-th/9705223].
- [18] L. Vanzo, *Black holes with unusual topology*, *Phys. Rev. D* **56** (1997) 6475 [gr-qc/9705004].
- [19] D.R. Brill, J. Louko and P. Peldan, *Thermodynamics of (3+1)-dimensional black holes with toroidal or higher genus horizons*, *Phys. Rev. D* **56** (1997) 3600 [gr-qc/9705012].
- [20] D. Birmingham, *Topological black holes in anti-de Sitter space*, *Class. and Quant. Grav.* **16** (1999) 1197 [hep-th/9808032].
- [21] R.-G. Cai and K.-S. Soh, *Topological black holes in the dimensionally continued gravity*, *Phys. Rev. D* **59** (1999) 044013 [gr-qc/9808067].
- [22] B. Wang, E. Abdalla and R.B. Mann, *Scalar wave propagation in topological black hole backgrounds*, *Phys. Rev. D* **65** (2002) 084006 [hep-th/0107243].
- [23] R.B. Mann, *Topological black holes: outside looking in*, gr-qc/9709039.
- [24] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev. D* **62** (2000) 084013 [hep-th/0003271].
- [25] R. Aros, R. Troncoso and J. Zanelli, *Black holes with topologically nontrivial AdS asymptotics*, *Phys. Rev. D* **63** (2001) 084015 [hep-th/0011097].
- [26] R.-G. Cai, Y.S. Myung and Y.-Z. Zhang, *Check of the mass bound conjecture in de Sitter space*, *Phys. Rev. D* **65** (2002) 084019 [hep-th/0110234].
- [27] M.H. Dehghani, *Asymptotically (anti)-de Sitter solutions in gauss-bonnet gravity without a cosmological constant*, *Phys. Rev. D* **70** (2004) 064019 [hep-th/0405206].
- [28] C. Martinez, R. Troncoso and J. Zanelli, *Exact black hole solution with a minimally coupled scalar field*, *Phys. Rev. D* **70** (2004) 084035 [hep-th/0406111].

- [29] C. Martinez, J.P. Staforelli and R. Troncoso, *Topological black holes dressed with a conformally coupled scalar field and electric charge*, *Phys. Rev. D* **74** (2006) 044028 [hep-th/0512022];
C. Martinez and R. Troncoso, *Electrically charged black hole with scalar hair*, *Phys. Rev. D* **74** (2006) 064007 [hep-th/0606130].
- [30] E. Winstanley, *Dressing a black hole with non-minimally coupled scalar field hair*, *Class. and Quant. Grav.* **22** (2005) 2233 [gr-qc/0501096];
E. Radu and E. Winstanley, *Conformally coupled scalar solitons and black holes with negative cosmological constant*, *Phys. Rev. D* **72** (2005) 024017 [gr-qc/0503095];
A.-M. Barlow, D. Doherty and E. Winstanley, *Thermodynamics of de Sitter black holes with a conformally coupled scalar field*, *Phys. Rev. D* **72** (2005) 024008 [gr-qc/0504087].
- [31] I. Papadimitriou, *Non-supersymmetric membrane flows from fake supergravity and multi-trace deformations*, hep-th/0606038.
- [32] P. Breitenlohner and D. Z. Freedman, *Phys. Lett.* **B115** (1982) 197; *Annals Phys.* **144** (1982) 249
- [33] L. Mezincescu and P. K. Townsend, *Annals Phys.* **160** (1985) 406.
- [34] V. Cardoso, J. Natario and R. Schiappa, *Asymptotic quasinormal frequencies for black holes in non-asymptotically flat spacetimes*, *J. Math. Phys.* **45** (2004) 4698 [hep-th/0403132].
- [35] J. Natario and R. Schiappa, *On the classification of asymptotic quasinormal frequencies for d-dimensional black holes and quantum gravity*, *Adv. Theor. Math. Phys.* **8** (2004) 1001 [hep-th/0411267].
- [36] S. Musiri, S. Ness and G. Siopsis, *Perturbative calculation of quasi-normal modes of AdS Schwarzschild black holes*, *Phys. Rev. D* **73** (2006) 064001 [hep-th/0511113].
- [37] L. Motl and A. Neitzke, *Adv. Theor. Math. Phys.* **7** (2003) 307 [hep-th/0301173].
- [38] A. J. M. Medved, D. Martin and M. Visser, *Class. Quant. Grav.* **21** (2004) 2393 [gr-qc/0310097].
- [39] W.-H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery in *Numerical recipes*, Cambridge University Press, Cambridge, England, 1992.
- [40] G. Koutsoumbas, S. Musiri, E. Papantonopoulos and G. Siopsis, in preparation.

Bibliography (หน้าบรรณานุกรม)

- [1] J.D. Bekenstein, Black Holes and the Second Law, *Nuovo Cimento Lett*, **4** (1972) 737;
\\ Black Holes and Entropy, *Phys. Rev.*, **D7** (1973) 2333.
- [2] J.M. Bardeen, B. Carter and S. W. Hawking, The Four Laws of Black Hole Mechanics, *Commun. Math. Phys.*, **31** (1973) 161.
- [3] S. W. Hawking, Particle Creation by Black Holes, *Commun. Math. Phys.*, **43** (1975) 199.
- [4] C. Rovelli, Black Hole Entropy from Loop Quantum Gravity, *Phys.Rev.Letts.*, **14**, (1996) 3288, gr-qc/9603063.
- [5] S. Hod,. Bohr's Correspondence Principle and the Area Spectrum of Quantum Black Holes, *Phys.Rev.Letts.*, **108**, 1063 (1957), gr-qc/9812002.
- [6] FS.S. Gubser, I.R. Klebanov and A.W. Peet, Entropy and Temperature of Black 3-Branes, *Phys. Rev.*, **D54** (1996) 3915, hep-th/9602135 ;
J.M. Maldacena and A. Strominger, Black Hole Greybody Factors and D-Brane Spectroscopy, *Phys. Rev.*, **D55** (1997) 861, hep-th/9609026.
- [7] J. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, *Adv. Theor. Math. Phys.*, **2** (1998) 231, hep-th/9711200.
- [8] G. T. Horowitz and V. E. Hubeny, Quasinormal Modes of AdS Black Holes and the Approach to Thermal Equilibrium, *Phys. Rev.*, **D62** (2000) 024027, hep-th/9909056.
- [9] A. O. Starinets, Quasinormal Modes of Near Extremal Black Branes, *Phys. Rev.*, **D66** (2002) 124013, hep-th/0207133.
- [10] S. Musiri and G. Siopsis, Quasinormal Modes of Large AdS Black Holes, *Phys.Lett.*, **B563** (2003) 102, hep-th/0301081.
- [11] V. Cardoso and J. P.S. Lemos, Scalar, Electromagnetic and Weyl perturbations of BTZ Black Holes: Quasi Normal Modes, *Phys. Rev.*, **D63** (2001) 124015 gr-qc/0101052.
- [12] L. Motl and A. Neitzke, Asymptotic Black Hole Quasinormal Frequencies, *Adv.Theor.Math.Phys.*, **6** (2003) 1135, gr-qc/0212096.
- [13] H.P. Nollert, Quasinormal Modes of Schwarzschild Black Holes: The Determination of Quasinormal Frequencies with very Large Imaginary Parts, *Phys.Rev.* **D47** (1993) 12.
- [14] S. Musiri and G. Siopsis, Perturbative Calculation of Quasi-normal Modes of Schwarzschild Black Holes, *Class.Quant.Grav.*, **20** (2003) L285-L291, hep-th/0308168.

- [15] S. Musiri and G. Siopsis, Asymptotic form of Quasi-normal Modes of Large AdS Black Holes, hep-th/0308196.
- [16] V.Cardoso, J.Nataro and R.Schiappa, Asymtotic Quasinoemal Frequencies for Black Holes in Non-Asymptotically Flat Spacetimes, *J.Math.Phys.*, **45** (2004) 4698-4713. hep-th/0403132
- [17] J.Nataro and R.Schiappa, On the Classification of Asymptotic Quasinormal Frequenies for d-Dimensional Black Holes and Quantum Gravity, hep-th/0411267.
- [18] S. Musiri and G. Siopsis, Perturbative Calculation of Quasinormal Modes of AdS Schwarzschild Black Holes, hep-th/0511113.
- [19] G. Siopsis, Large Mass Expansion of Quasi-normal Modes in AdS5, *Phys.Lett.*, **B590** (2004) 105, hep-th/0402083
- [20]V.Cardoso, R.Konoplya and J. Lemos, Quasinormal Frequencies of Schwarzschild Black Holes in Anti-d- Sitter Spacetimes: A Complete Study on the Asymptotic Behavior, *Phys. Rev.* **D68** (2003) 044024, gr-qc/0305037
- [21] C.Martinez, R.Troncoso and J.Zanelli, Exact Black Holes Solution with a minimally Coupled Scalar Field, *Phys.Rev.*, **D70** (2004) 084035, hep-th/046111.
- [22] S.Hawking, Information Loss in Black Holes, hep-th/0507171.
- [23] A. Ishibashi and H. Kodama, A Master Equation for Gravitational Perturbations of Maximally Symmetric Black Holes in Higher Dimensions, *Prog. Theor. Phys.* **110** (2003) 701, hep-th/0305147

Appendix (หน้าภาคผนวก)

Appendix A : Derivation of Maxwell equations of four-dimensional topology black holes with scalar

Our Research Problem

The metric in eq. (2.6) from Exact black hole solution with a minimally coupled scalar field, Cristian Martinez, Ricardo Troncoso, Jorge Zanelli, Phys.Rev. D70 (2004) 084035, hep-th/0406111, is

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2} \left[- \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right) dt^2 + \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right)^{-1} dr^2 + r^2 \Omega^2 \right] \quad (1)$$

If let $g(r) = \frac{r(r+2G\mu)}{(r+G\mu)^2}$ and $f(r) = \left(\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r} \right)^2 \right)$, the metric can be rewritten as

$$ds^2 = -g(r)f(r)dt^2 + \frac{g}{f}dr^2 + r^2 g d\Omega^2 \quad (2)$$

EM wave equation can be found from Maxwell equation

$$\nabla_\nu F^{\mu\nu} = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3)$$

I will use the same decomposition of A_μ as appear in gr-qc/0105103, i.e.,

$$A_\mu(t, r, \theta, \phi) = \sum_{l,m} \left(\begin{bmatrix} 0 \\ 0 \\ a^{lm}(t, r) \frac{1}{\sin \theta} \partial_\phi Y_{lm} \\ -a^{lm}(t, r) \sin \theta \partial_\theta Y_{lm} \end{bmatrix} + \begin{bmatrix} f^{lm}(t, r) Y_{lm} \\ h^{lm}(t, r) Y_{lm} \\ k^{lm}(t, r) \partial_\theta Y_{lm} \\ k^{lm}(t, r) \partial_\phi Y_{lm} \end{bmatrix} \right) \quad (4)$$

The Maxwell equations are the following,

$$\nabla_\mu F^{t\mu} = \nabla_r F^{tr} + \nabla_\theta F^{t\theta} + \nabla_\phi F^{t\phi} = 0 \quad (5)$$

$$\nabla_\mu F^{r\mu} = \nabla_t F^{rt} + \nabla_\theta F^{r\theta} + \nabla_\phi F^{r\phi} = 0 \quad (6)$$

$$\nabla_\mu F^{\theta\mu} = \nabla_t F^{\theta t} + \nabla_r F^{\theta r} + \nabla_\phi F^{\theta\phi} = 0 \quad (7)$$

$$\nabla_\mu F^{\phi\mu} = \nabla_t F^{\phi t} + \nabla_r F^{\phi r} + \nabla_\theta F^{\phi\theta} = 0 \quad (8)$$

1 The first equation

The Maxwell equation 5

$$\begin{aligned} 0 &= \partial_r F^{tr} + \Gamma_{ra}^t F^{ar} + \Gamma_{ra}^r F^{ta} \\ &+ \partial_\theta F^{t\theta} + \Gamma_{\theta a}^t F^{a\theta} + \Gamma_{\theta a}^\theta F^{ta} \\ &+ \partial_\phi F^{t\phi} + \Gamma_{\phi a}^t F^{a\phi} + \Gamma_{\phi a}^\phi F^{ta} \end{aligned} \quad (9)$$

where all six Γ are

$$\begin{aligned} \Gamma_{ra}^t &= \frac{g^{tt}}{2} [\partial_r g_{ta} + \partial_a g_{tr} - \partial_t g_{ra}] = \frac{g^{tt}}{2} \partial_r g_{tt} = \frac{1}{2g} \partial_r g + \frac{1}{2f} \partial_r f = \Gamma_{ra}^t \\ \Gamma_{ra}^r &= \frac{g^{rr}}{2} [\partial_r g_{ra} + \partial_a g_{rr} - \partial_r g_{ra}] = \frac{g^{rr}}{2} \partial_r g_{rr} = \frac{1}{2g} \partial_r g - \frac{1}{2f} \partial_r f = \Gamma_{ra}^r \end{aligned}$$

$$\begin{aligned}
\Gamma_{\theta a}^t &= \frac{g^{tt}}{2} [\partial_\theta g_{ta} + \partial_a g_{\theta t} - \partial_t g_{\theta\theta}] = 0 \\
\Gamma_{\theta a}^\theta &= \frac{g^{\theta\theta}}{2} [\partial_\theta g_{\theta a} + \partial_a g_{\theta\theta} - \partial_\theta g_{\theta a}] = \frac{1}{2g} \partial_r g + \frac{1}{r} = \Gamma_{\theta r}^\theta \\
\Gamma_{\phi a}^t &= \frac{g^{tt}}{2} [\partial_\phi g_{ta} + \partial_a g_{\theta\phi} - \partial_t g_{\phi\theta}] = 0 \\
\Gamma_{\phi a}^\phi &= \frac{g^{\phi\phi}}{2} [\partial_\phi g_{\phi a} + \partial_a g_{\phi\phi} - \partial_\phi g_{\phi\theta}] \\
\Gamma_{\phi r}^\phi &= \frac{g^{\phi\phi}}{2} \partial_r g_{\phi\phi} = \frac{1}{2g} \partial_r g + \frac{1}{r} \\
\Gamma_{\phi\theta}^\phi &= \frac{g^{\phi\phi}}{2} \partial_\theta g_{\phi\phi} = \cot \theta
\end{aligned}$$

Equation (9) becomes

$$\begin{aligned}
0 &= \partial_r g^{tt} g^{rr} F_{tr} + \left(\frac{1}{2g} \partial_r g + \frac{1}{2f} \partial_r f \right) g^{tt} g^{rr} F_{tr} + \left(\frac{1}{2g} \partial_r g - \frac{1}{2f} \partial_r f \right) g^{tt} g^{rr} F_{tr} \\
&+ \partial_\theta g^{tt} g^{\theta\theta} F_{t\theta} + 0 + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{tt} g^{rr} F_{tr} \\
&+ \partial_\phi g^{tt} g^{\phi\phi} F_{t\phi} + 0 + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{tt} g^{rr} F_{tr} + \cot \theta g^{tt} g^{\theta\theta} F_{t\theta} \\
&= -\partial_r \frac{1}{g^2} F_{tr} - \frac{1}{g^2 r^2 f} \partial_\theta F_{t\theta} - \frac{1}{g^2 r^2 f \sin^2 \theta} \partial_\phi F_{t\phi} \\
&- \frac{2}{g^2} \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) F_{tr} - \frac{1}{g^2 r^2 f} \cot \theta F_{t\theta} \\
&= -\partial_r \frac{1}{g^2} [(\partial_t h^{lm}) Y_{lm} - (\partial_r f^{lm}) Y_{lm}] \\
&- \frac{1}{g^2 r^2 f} \left[(\partial_t a^{lm}) \partial_\theta \left(\frac{1}{\sin \theta} \partial_\phi Y_{lm} \right) + (\partial_t k^{lm}) \partial_\theta^2 Y_{lm} - f^{lm} \partial_\theta^2 Y_{lm} \right] \\
&- \frac{1}{g^2 r^2 f \sin^2 \theta} \left[-(\partial_t a^{lm}) \sin \theta \partial_\theta \partial_\phi Y_{lm} + (\partial_t k^{lm}) \partial_\phi^2 Y_{lm} - f^{lm} \partial_\phi^2 Y_{lm} \right] \\
&- \frac{2}{g^2} \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) [(\partial_t h^{lm}) Y_{lm} - (\partial_r f^{lm}) Y_{lm}] \\
&- \frac{\cos \theta}{\sin \theta} \frac{1}{r^2 g^2 f} \left[(\partial_t a^{lm}) \left(\frac{1}{\sin \theta} \partial_\phi Y_{lm} \right) + (\partial_t k^{lm}) \partial_\theta Y_{lm} - f^{lm} \partial_\theta Y_{lm} \right] \tag{10}
\end{aligned}$$

where I omitt to write down summation notation, $\sum_{lm}$. The coefficient of $\partial_t a^{lm}$ are combined to be zero. The coefficient of $-k^{lm} + f^{lm}$ becomes $l(l+1)Y_{lm}$ where we use the equation

$$\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta Y_{lm} + \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_{lm} = -l(l+1)Y_{lm} \tag{11}$$

Y_{lm} are orthogonal functions, then the coefficient of each Y_{lm} in (10) have to be zero, i.e.

$$\begin{aligned}
0 &= -\partial_r \frac{1}{g^2} [\partial_t h^{lm} - \partial_r f^{lm}] + l(l+1) \frac{1}{r^2 g^2 f} [\partial_t k^{lm} - f^{lm}] - \frac{2}{g^2} \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) [\partial_t h^{lm} - \partial_r f^{lm}], \quad \text{or} \\
0 &= -\partial_r [\partial_t h^{lm} - \partial_r f^{lm}] + l(l+1) \frac{1}{r^2 f} [\partial_t k^{lm} - f^{lm}] - \frac{2}{r} [\partial_t h^{lm} - \partial_r f^{lm}] \\
0 &= -f \partial_r r^2 [\partial_t h^{lm} - \partial_r f^{lm}] + l(l+1) [\partial_t k^{lm} - f^{lm}] \tag{12}
\end{aligned}$$

2 The second Maxwell equation

The Maxwell equation (6)

$$\begin{aligned}
 0 &= \partial_t F^{rt} + \Gamma_{ta}^r F^{at} + \Gamma_{ta}^t F^{ra} \\
 &+ \partial_\theta F^{r\theta} + \Gamma_{\theta a}^r F^{a\theta} + \Gamma_{\theta a}^\theta F^{ra} \\
 &+ \partial_\phi F^{r\phi} + \Gamma_{\phi a}^r F^{a\phi} + \Gamma_{\phi a}^\phi F^{ra}
 \end{aligned} \tag{13}$$

where all six Γ are

$$\begin{aligned}
 \Gamma_{ta}^r &= \Gamma_{tt}^r = -\frac{g^{rr}}{2} \partial_r g_{rr} = \frac{1}{2} \left(f \partial_r f + \frac{f^2}{g} \partial_r g \right) \\
 \Gamma_{ta}^t &= \frac{g^{tt}}{2} [\partial_t g_{ta} + \partial_a g_{tt} - \partial_t g_{ta}] = \Gamma_{tr}^t = \frac{1}{2} \left(\frac{1}{g} \partial_r g + \frac{1}{f} \partial_r f \right) \\
 \Gamma_{\theta a}^r &= \frac{g^{rr}}{2} [\partial_\theta g_{ra} + \partial_a g_{r\theta} - \partial_r g_{\theta a}] = \Gamma_{\theta\theta}^r = -\frac{r^2 f}{2g} \partial_r g - r f \\
 \Gamma_{\theta a}^\theta &= \Gamma_{\theta r}^\theta = \frac{1}{2g} \partial_r g + \frac{1}{r} \\
 \Gamma_{\phi a}^r &= \frac{g^{rr}}{2} [\partial_\phi g_{ra} + \partial_a g_{r\phi} - \partial_r g_{\phi a}] = \Gamma_{\phi\phi}^r = -\frac{r^2 f \sin^2 \theta}{2g} \partial_r g - r f g \sin^2 \theta \\
 \Gamma_{\phi r}^\phi &= \frac{1}{2g} \partial_r g + \frac{1}{r}, \quad \Gamma_{\phi\theta}^\phi = \cot \theta
 \end{aligned} \tag{14}$$

Equation 13 becomes

$$\begin{aligned}
 0 &= -\frac{1}{g^2} \partial_t F_{rt} + 0 + 0 \\
 &+ \frac{f}{r^2 g^2} \partial_\theta F_{r\theta} + 0 + 0 \\
 &+ \frac{f}{r^2 g^2 \sin^2 \theta} \partial_\phi F_{r\phi} + 0 + \cot \theta g^{rr} g^{\theta\theta} F_{r\theta} \\
 &= -\frac{1}{g^2} [\partial_t \partial_r f^{lm} Y_{lm} - \partial_t^2 h^{lm} Y_{lm}] \\
 &+ \frac{f}{r^2 g^2} \left\{ \left[\partial_r a^{lm} \partial_\theta \frac{1}{\sin \theta} \partial_\phi Y_{lm} + \partial_r k^{lm} \partial_\theta^2 Y_{lm} \right] - h^{lm} \partial_\theta^2 Y_{lm} \right\} \\
 &+ \frac{f}{r^2 g^2 \sin^2 \theta} \left\{ \left[-\partial_r a^{lm} \sin \theta \partial_\theta \partial_\phi Y_{lm} + \partial_r k^{lm} \partial_\phi^2 Y_{lm} \right] - h^{lm} \partial_\phi^2 Y_{lm} \right\} \\
 &+ \frac{\cos \theta}{\sin \theta} \frac{f}{r^2 g^2} \left\{ \left[\partial_r a^{lm} \frac{1}{\sin \theta} \partial_\phi Y_{lm} + \partial_r k^{lm} \partial_\theta Y_{lm} \right] - h^{lm} \partial_\theta Y_{lm} \right\}
 \end{aligned} \tag{15}$$

The coefficients of $\partial_r a^{lm}$ are combined to be zero. The coefficients of $\partial_r k^{lm} - h^{lm}$ are combined to become

$$\partial_\theta^2 Y_{lm} + \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_{lm} + \frac{\cos \theta}{\sin \theta} \partial_\theta Y_{lm} = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta Y_{lm} + \frac{1}{\sin^2 \theta} \partial_\phi^2 Y_{lm} = -l(l+1) Y_{lm}$$

And equation (15) becomes, after Y_{lm} and $\frac{1}{g^2}$ have been taken out

$$\partial_t [-\partial_r f^{lm} + \partial_t h^{lm}] - \frac{f}{r^2} [\partial_r k^{lm} - h^{lm}] l(l+1) = 0 \tag{16}$$

3 The third Maxwell equation

The Maxwell equation (7) is

$$\begin{aligned}
0 &= \partial_t F^{\theta t} + \Gamma_{ta}^\theta F^{at} + \Gamma_{ta}^t F^{\theta a} \\
&+ \partial_r F^{\theta r} + \Gamma_{ra}^\theta F^{ar} + \Gamma_{ra}^r F^{\theta a} \\
&+ \partial_\phi F^{a\phi} + \Gamma_{\phi a}^\theta F^{a\phi} + \Gamma_{\phi a}^\phi F^{\phi a}
\end{aligned} \tag{17}$$

where all six Γ are

$$\begin{aligned}
\Gamma_{ta}^\theta &= \frac{g^{\theta\theta}}{2} [\partial_t g_{\theta a} + \partial_a g_{\theta t} - \partial_\theta g_{ta}] = 0 \\
\Gamma_{ta}^t &= \Gamma_{tr}^t = \frac{1}{2} \left(\frac{1}{g} \partial_r g + \frac{1}{f} \partial_r f \right) \\
\Gamma_{ra}^\theta &= \frac{g^{\theta\theta}}{2} [\partial_r g_{\theta a} + \partial_a g_{\theta r} - \partial_\theta g_{ra}] = \Gamma_{r\theta}^\theta = \frac{1}{2g} \partial_r g + \frac{1}{r} \\
\Gamma_{ra}^r &= \Gamma_{rr}^r = \frac{1}{2g} \partial_r g - \frac{1}{2f} \partial_r f \\
\Gamma_{\phi a}^\theta &= \frac{g^{\theta\theta}}{2} [\partial_\phi g_{\theta a} + \partial_a g_{\theta\phi} - \partial_\theta g_{\phi a}] = \Gamma_{\phi\phi}^\theta = -\sin\theta \cos\theta \\
\Gamma_{\phi r}^\phi &= \frac{1}{2g} \partial_r g + \frac{1}{r}, \quad \Gamma_{\phi\theta}^\phi = \cot\theta
\end{aligned}$$

Equation (17) becomes

$$\begin{aligned}
0 &= -\frac{1}{r^2 f g^2} \partial_t F_{\theta t} + 0 + \frac{1}{2} \left(\frac{1}{g} \partial_r g + \frac{1}{f} \partial_r f \right) g^{\theta\theta} g^{rr} F_{\theta r} \\
&+ \partial_r \frac{f}{r^2 g^2} F_{\theta r} + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{\theta\theta} g^{rr} F_{\theta r} + \frac{1}{2} \left(\frac{1}{g} \partial_r g - \frac{1}{f} \partial_r f \right) g^{\theta\theta} g^{rr} F_{\theta r} \\
&+ \frac{1}{r^4 g^2 \sin^2 \theta} \partial_\phi F_{\theta\phi} + 0 + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{\theta\theta} g^{rr} F_{\theta r} \\
&= -\frac{1}{r^2 f g^2} [\partial_t \partial_\theta A_t - \partial_t^2 A_\theta] + \partial_r \frac{f}{r^2 g^2} [\partial_\theta A_r - \partial_r A_\theta] + \frac{1}{r^4 g^2 \sin^2 \theta} [\partial_\phi \partial_\theta A_\phi - \partial_\phi^2 A_\theta] \\
&+ 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} [\partial_\theta A_r - \partial_r A_\theta] \\
&= -\frac{1}{r^2 f g^2} \left\{ \partial_t f^{lm} \partial_\theta Y_{lm} - \left[\partial_t^2 a^{lm} \frac{1}{\sin\theta} \partial_\phi Y_{lm} + \partial_t^2 k^{lm} \partial_\theta Y_{lm} \right] \right\} \\
&+ \partial_r \frac{f}{r^2 g^2} \left\{ h^{lm} \partial_\theta Y_{lm} - \left[\partial_r a^{lm} \frac{1}{\sin\theta} \partial_\phi Y_{lm} + \partial_r k^{lm} \partial_\theta Y_{lm} \right] \right\} \\
&+ \frac{1}{r^4 g^2 \sin^2 \theta} \left\{ [-a^{lm} \partial_\theta \sin\theta \partial_\phi \partial_\phi Y_{lm} + k^{lm} \partial_\theta \partial_\phi^2 Y_{lm}] - \left[a^{lm} \partial_\phi^2 \frac{1}{\sin\theta} \partial_\phi Y_{lm} + k^{lm} \partial_\theta \partial_\phi^2 Y_{lm} \right] \right\} \\
&+ 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} \left\{ h^{lm} \partial_\theta Y_{lm} - \left[\partial_r a^{lm} \frac{1}{\sin\theta} \partial_\phi Y_{lm} + \partial_r k^{lm} \partial_\theta Y_{lm} \right] \right\}
\end{aligned} \tag{18}$$

The coefficients of k^{lm} are canceled each other and the coefficients of a^{lm} are combined to be

$$-\frac{1}{r^4 g^2 \sin^2 \theta} \left\{ \partial_\theta \sin\theta \partial_\phi \partial_\phi Y_{lm} - \frac{1}{\sin\theta} \partial_\phi^3 Y_{lm} \right\} = \frac{1}{r^4 g^2} \frac{l(l+1)}{\sin\theta} \partial_\phi Y_{lm}$$

Then equation (17) becomes

$$\begin{aligned}
0 = & -\frac{1}{r^2 f g^2} \left\{ \partial_t f^{lm} \partial_\theta Y_{lm} - \left[\partial_t^2 a^{lm} \frac{1}{\sin \theta} \partial_\phi Y_{lm} + \partial_t^2 k^{lm} \partial_\theta Y_{lm} \right] \right\} \\
& + \partial_r \frac{f}{r^2 g^2} h^{lm} \partial_\theta Y_{lm} - \partial_r \frac{f}{r^2 g^2} \partial_r a^{lm} \frac{1}{\sin \theta} \partial_\phi Y_{lm} - \partial_r \frac{f}{r^2 g^2} \partial_r k^{lm} \partial_\theta Y_{lm} \\
& + \frac{1}{r^4 g^2} a^{lm} \frac{l(l+1)}{\sin \theta} \partial_\phi Y_{lm} \\
& + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} \left\{ h^{lm} \partial_\theta Y_{lm} - \partial_r a^{lm} \frac{1}{\sin \theta} \partial_\phi Y_{lm} - \partial_r k^{lm} \partial_\theta Y_{lm} \right\}
\end{aligned} \tag{19}$$

We'll keep equation (19) and subtract it from the result from equation (8) later

4 The fourth Maxwell equation

The Maxwell equation (8) is

$$\begin{aligned}
0 = & \partial_t F^{\phi t} + \Gamma_{ta}^\phi F^{at} + \Gamma_{ta}^t F^{\phi a} \\
& + \partial_r F^{\phi r} + \Gamma_{ra}^\phi F^{ar} + \Gamma_{ra}^r F^{\phi a} \\
& + \partial_\theta F^{\phi \theta} + \Gamma_{\theta a}^\phi F^{a\theta} + \Gamma_{\theta a}^\theta F^{\phi a}
\end{aligned} \tag{20}$$

where all six Γ are

$$\begin{aligned}
\Gamma_{ta}^\phi &= \frac{g^{\phi\phi}}{2} [\partial_t g_{\phi a} + \partial_a g_{\phi t} - \partial_\phi g_{ta}] = 0 \\
\Gamma_{ta}^t &= \Gamma_{tr}^t = \frac{1}{2} \left(\frac{1}{g} \partial_r g + \frac{1}{f} \partial_r f \right) \\
\Gamma_{ra}^\phi &= \frac{g^{\phi\phi}}{2} [\partial_r g_{\phi a} + \partial_a g_{\phi r} - \partial_\phi g_{ra}] = \Gamma_{r\phi}^\phi = \frac{1}{2g} \partial_r g + \frac{1}{r} \\
\Gamma_{ra}^r &= \Gamma_{rr}^r = \frac{1}{2g} \partial_r g - \frac{1}{2f} \partial_r f \\
\Gamma_{\theta a}^\phi &= \frac{g^{\phi\phi}}{2} [\partial_\theta g_{\phi a} + \partial_a g_{\phi\theta} - \partial_\phi g_{\theta a}] = \Gamma_{\theta\phi}^\phi = \cot \theta \\
\Gamma_{\theta r}^\theta &= \frac{1}{2g} \partial_r g + \frac{1}{r},
\end{aligned}$$

Equation (20) becomes

$$\begin{aligned}
0 = & -\frac{1}{r^2 f g^2 \sin^2 \theta} \partial_t F_{\phi t} + 0 + \frac{1}{2} \left(\frac{1}{g} \partial_r g + \frac{1}{f} \partial_r f \right) g^{\phi\phi} g^{rr} F_{\phi r} \\
& + \partial_r \frac{f}{r^2 g^2 \sin^2 \theta} F_{\phi r} + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{\phi\phi} g^{rr} F_{\phi r} + \frac{1}{2} \left(\frac{1}{g} \partial_r g - \frac{1}{f} \partial_r f \right) g^{\phi\phi} g^{rr} F_{\phi r} \\
& + \partial_\theta \frac{1}{r^4 g^2 \sin^2 \theta} F_{\phi\theta} + \cot \theta g^{\phi\phi} g^{\theta\theta} F_{\phi\theta} + \left(\frac{1}{2g} \partial_r g + \frac{1}{r} \right) g^{\phi\phi} g^{rr} F_{\phi r} \\
= & -\frac{1}{r^2 f g^2 \sin^2 \theta} [\partial_t \partial_\theta A_t - \partial_t^2 A_\phi] + \partial_r \frac{f}{r^2 g^2 \sin^2 \theta} [\partial_\phi A_r - \partial_r A_\phi] + \partial_\theta \frac{1}{r^4 g^2 \sin^2 \theta} [\partial_\phi A_\theta - \partial_\theta A_\phi] \\
& + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2 \sin^2 \theta} [\partial_\phi A_r - \partial_r A_\phi]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\cos \theta}{r^4 g^2 \sin^3 \theta} [\partial_\phi A_\theta - \partial_\theta A_\phi] \\
& = -\frac{1}{r^2 f g^2 \sin^2 \theta} \{ \partial_t f^{lm} \partial_\phi Y_{lm} - [-\partial_t^2 a^{lm} \sin \theta \partial_\theta Y_{lm} + \partial_t^2 k^{lm} \partial_\phi Y_{lm}] \} \\
& + \partial_r \frac{f}{r^2 g^2 \sin^2 \theta} \{ h^{lm} \partial_\phi Y_{lm} - [-\partial_r a^{lm} \sin \theta \partial_\theta Y_{lm} + \partial_r k^{lm} \partial_\phi Y_{lm}] \} \\
& + \frac{1}{r^4 g^2 \sin^2 \theta} \left\{ \left[a^{lm} \frac{1}{\sin \theta} \partial_\phi^2 Y_{lm} + k^{lm} \partial_\theta \partial_\phi Y_{lm} \right] - [-a^{lm} \partial_\theta \sin \theta \partial_\theta Y_{lm} + k^{lm} \partial_\theta \partial_\phi Y_{lm}] \right\} \\
& + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2 \sin^2 \theta} \{ h^{lm} \partial_\phi Y_{lm} - [-\partial_r a^{lm} \sin \theta \partial_\theta Y_{lm} + \partial_r k^{lm} \partial_\phi Y_{lm}] \} \\
& + \frac{\cos \theta}{r^4 g^2 \sin^3 \theta} \left\{ a^{lm} \frac{1}{\sin \theta} \partial_\phi^2 Y_{lm} + k^{lm} \partial_\theta \partial_\phi Y_{lm} - [-a^{lm} \partial_\theta \sin \theta \partial_\theta Y_{lm} + k^{lm} \partial_\theta \partial_\phi Y_{lm}] \right\} \quad (21)
\end{aligned}$$

The coefficients of k^{lm} are canceled among themselves. The coefficients of a^{lm} are combined to be $-\frac{l(l+1)}{r^4 g^2} \partial_\theta \frac{1}{\sin \theta} Y_{lm}$ and $-\frac{\cos \theta}{r^4 g^2 \sin^2 \theta} l(l+1) Y_{lm}$. These two terms are $-\frac{l(l+1)}{r^4 g^2} \partial_\theta \frac{1}{\sin \theta} Y_{lm} - \frac{\cos \theta}{r^4 g^2 \sin^2 \theta} l(l+1) Y_{lm} = -\frac{1}{r^4 g^2} \frac{l(l+1)}{\sin \theta} \partial_\theta Y_{lm}$

Then equation (21) becomes

$$\begin{aligned}
0 & = -\frac{1}{r^2 f g^2 \sin^2 \theta} \{ \partial_t f^{lm} \partial_\phi Y_{lm} + \partial_t^2 a^{lm} \sin \theta \partial_\theta Y_{lm} - \partial_t^2 k^{lm} \partial_\phi Y_{lm} \} \\
& + \frac{1}{\sin^2 \theta} \left\{ \partial_r \frac{f}{r^2 g^2} h^{lm} \partial_\phi Y_{lm} + \partial_r \frac{f}{r^2 g^2} \partial_r a^{lm} \sin \theta \partial_\theta Y_{lm} - \partial_r \frac{f}{r^2 g^2} \partial_r k^{lm} \partial_\phi Y_{lm} \right\} \\
& - \frac{1}{r^4 g^2} a^{lm} \frac{l(l+1)}{\sin \theta} \partial_\theta Y_{lm} \\
& + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2 \sin^2 \theta} \{ h^{lm} \partial_\phi Y_{lm} + \partial_r a^{lm} \sin \theta \partial_\theta Y_{lm} - \partial_r k^{lm} \partial_\phi Y_{lm} \} \quad (22)
\end{aligned}$$

5 Rearranging All Four equations

Equation (19) and (22) can be reduced together by combining them as $-\frac{1}{\sin \theta} \partial_\phi (19) + \frac{1}{\sin \theta} \partial_\theta \sin^2 \theta (22)$. That is

$$\begin{aligned}
0 & = \frac{1}{r^2 f g^2} \partial_t^2 a^{lm} - \partial_r \frac{f}{r^2 g^2} \partial_r a^{lm} + \frac{l(l+1)}{r^4 g^2} a^{lm} - 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} \partial_r a^{lm} \\
0 & = \frac{1}{r^2 f g^2} \partial_t^2 a^{lm} - \frac{1}{r^2 g^2} \partial_r f \partial_r a^{lm} + \frac{l(l+1)}{r^4 g^2} a^{lm}, \quad \text{or} \\
0 & = \partial_t^2 a^{lm} - f \partial_r (f \partial_r a^{lm}) + \frac{l(l+1)}{r^2} f a^{lm} \quad (23)
\end{aligned}$$

where I took out the summation of Y_{lm} and g is decoupled out.

If we rearrange and regroup equation (19) and (22) in terms of coefficients of the polynomials $\partial_\theta Y_{lm}$ and $\partial_\phi Y_{lm}$. Equation (19) gives

$$\begin{aligned}
0 & = \left\{ -\frac{1}{r^2 f g^2} [\partial_t f^{lm} - \partial_t^2 k^{lm}] + \partial_r \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] \right\} \partial_\theta Y_{lm} \\
& + \left\{ \frac{1}{r^2 f g^2} \partial_t^2 a^{lm} - \frac{1}{r^2 g^2} \partial_r (f \partial_r a^{lm}) + \frac{l(l+1)}{r^4 g^2} a^{lm} \right\} \frac{1}{\sin \theta} \partial_\phi Y_{lm} \quad (24)
\end{aligned}$$

But the coefficients of $\frac{1}{\sin \theta} \partial_\phi Y_{lm}$ is equation (23) which is zero, that means the coefficients of $\partial_\theta Y_{lm}$ has to be zero

too, in which can be reduced to

$$\begin{aligned} 0 &= -\frac{1}{r^2 f g^2} [\partial_t f^{lm} - \partial_t^2 k^{lm}] + \partial_r \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] \\ 0 &= -\partial_t [f^{lm} - \partial_t k^{lm}] + f \partial_r f [h^{lm} - \partial_r k^{lm}] \end{aligned} \quad (25)$$

Also equation (22) can be rewritten similar to (19)

$$\begin{aligned} 0 &= \left\{ -\frac{1}{r^2 f g^2} \partial_t^2 a^{lm} + \frac{1}{r^2 g^2} \partial_r (f \partial_r a^{lm}) - \frac{l(l+1)}{r^4 g^2} a^{lm} \right\} \frac{1}{\sin \theta} \partial_\theta Y_{lm} \\ &+ \left\{ -\frac{1}{r^2 f g^2} [\partial_t f^{lm} - \partial_t^2 k^{lm}] + \partial_r \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] + 2 \left(\frac{1}{g} \partial_r g + \frac{1}{r} \right) \frac{f}{r^2 g^2} [h^{lm} - \partial_r k^{lm}] \right\} \frac{1}{\sin^2 \theta} \partial_\phi Y_{lm} \end{aligned} \quad (26)$$

which is the same (24).

Recall equation (12) and (16) and write them together with (25) and (23)

$$\begin{aligned} (12) \quad 0 &= -f \partial_r r^2 [\partial_t h^{lm} - \partial_r f^{lm}] + l(l+1) [\partial_t k^{lm} - f^{lm}] \\ (16) \quad 0 &= \partial_t [-\partial_r f^{lm} + \partial_t h^{lm}] - l(l+1) \frac{f}{r^2} [\partial_r k^{lm} - h^{lm}] \\ (25) \quad 0 &= -\partial_t [f^{lm} - \partial_t k^{lm}] + f \partial_r f [h^{lm} - \partial_r k^{lm}] \\ (23) \quad 0 &= \partial_t^2 a^{lm} - f \partial_r (f \partial_r a^{lm}) + \frac{l(l+1)}{r^2} f a^{lm} \end{aligned}$$

If we define

$$A^{lm} \equiv \partial_t k^{lm} - f^{lm} \quad (27)$$

$$\frac{1}{f} B^{lm} \equiv \partial_r k^{lm} - h^{lm} \quad (28)$$

then equations (12), (16) and (25) can be rewritten as

$$0 = f \partial_r r^2 \left[\partial_r A^{lm} - \frac{1}{f} B^{lm} \right] - l(l+1) A^{lm} \quad (29)$$

$$0 = \partial_t \left[\partial_r A^{lm} - \frac{1}{f} B^{lm} \right] - \frac{l(l+1)}{r^2} B^{lm} \quad (30)$$

$$0 = \partial_t A^{lm} - f \partial_r B^{lm} \quad (31)$$

We can new relations between these three equation.

First let $\partial_t(29)$ and substitute $\partial_t \left[\partial_r A^{lm} - \frac{1}{f} B^{lm} \right]$ from (30) and $\partial_t A^{lm}$ from (31) into $\partial_t(29)$, i.e. we get zero both sides of the equation.

Second, let $\partial_r(31)$ and substitute $\partial_r \partial_t A^{lm}$ from $\partial_r(31)$ into (30), then one get

$$f \partial_r f \partial_r B^{lm} - \partial_t^2 B^{lm} - \frac{l(l+1)}{r^2} f B^{lm} = 0 \quad (32)$$

where B^{lm} satisfies the same equation as a^{lm} does in (23).

Third, let define

$$C^{lm} \equiv \partial_r A^{lm} - \frac{1}{f} B^{lm} = -\partial_r f^{lm} + \partial_t h^{lm} \quad (33)$$

Then the equations (12) and (16) can be written as

$$0 = f \partial_r r^2 C^{lm} - l(l+1) A^{lm} \quad (34)$$

$$0 = \partial_t C^{lm} - \frac{l(l+1)}{r^2} B^{lm} \quad (35)$$

then combine them as $\partial_r(34) - \frac{r^2}{f} \partial_t(35)$, i.e.

$$\begin{aligned} 0 &= \partial_r f \partial_r r^2 C^{lm} - \frac{1}{f} \partial_t^2 r^2 C^{lm} - \frac{l(l+1)}{r^2} r^2 \left[\partial_r A^{lm} - \frac{1}{f} B^{lm} \right] \\ 0 &= f \partial_r f \partial_r (r^2 C^{lm}) - \partial_t^2 r^2 C^{lm} - \frac{l(l+1)}{r^2} f (r^2 C^{lm}) \end{aligned} \quad (36)$$

$r^2 C^{lm} = r^2 [-\partial_r f^{lm} + \partial_t h^{lm}]$ satisfies the same equation as B^{lm} and a^{lm} do

We can conclude the result as the following

$$\frac{\partial^2 \Psi}{\partial r_*^2} - \partial_t^2 \Psi - f(r) \frac{\ell(\ell+1)}{r^2} \Psi = 0 \quad (37)$$

where $\frac{\partial r}{\partial r_*} = f$ and factor $g(r)$ in the metric are decouple out from the equations. Ψ is satisfied by the following three cases,

$$\begin{aligned} \Psi &= a^{lm}(t, r) \\ \Psi &= B^{lm} = f(r) [\partial_r k^{lm} - h^{lm}] \\ \Psi &= r^2 C^{lm} = r^2 [-\partial_r f^{lm} + \partial_t h^{lm}] \end{aligned}$$

And the last equation is (25)

$$\partial_t [\partial_t k^{lm} - f^{lm}] - f \partial_r f [\partial_r k^{lm} - h^{lm}] = 0$$

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- Eleftherios Papantonopoulos and George Siopsis, *Journal of High Energy Physics*, 10, (2006) 006, hep-th/0606096
- ii) Perturbative Calculation of Quasi-normal Modes of AdS Schwarzschild Black Holes, S. Musiri, S.Ness and G. Siopsis, *Physical Review D* **73**, (2006) 064001 hep-th/0511113
 - iii). Perturbative Calculation of Quasi-normal Modes of Schwarzschild Black Holes, Suphot Musiri and George Siopsis, *Classical and Quantum Gravity*, **20** No. 24 (21 December 2003) L285-L291, hep-th/0308168
 - iv). Asymptotic Form of Quasi-normal Modes of Large AdS Black Holes, Suphot Musiri and George Siopsis, *Physics Letters*, **B 576** (2003) 309-313, hep-th/0308196
 - v). On Quasi-normal Modes of Kerr Black Holes, Suphot Musiri and George Siopsis, *Physics Letters*, **B 579** (2004) 25-30, hep-th/0309227
 - vi). Quasinormal modes of large AdS black holes, Suphot Musiri and George Siopsis, *Physics Letters*, **B 563** (2003) 102-106, hep-th/0301081
 - vii) Quantization of a self-interacting maximally charged string, Suphot Musiri and George Siopsis, *Physics Letters*, **B 524** (2002) 192-198, hep-th/0109190.
 - viii) Temperature of D3-branes off extremality, , Suphot Musiri and George Siopsis, *Physics Letters*, **B 504** (2001) 314-322, hep-th/000384.

