EQUIVALENCES OF BRUN'S CRITERION AND APPLICATIONS

A THESIS
BY
KANTAPHON KUHAPATANAKUL

Presented in Partial Fulfillment of the Requirements for the
Doctor of Philosophy in Mathematics
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สมมูลของเกณฑ์ความเป็นอตรรกยะของบรุนและการประยุกต์

บทคัดย่อ
ของ
กันตภณ คูหาพัฒนกุล

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา
ตามหลักสูตรปริญญาปรัชญาดุษฎีบัณฑิต สาขาวิชาคณิตศาสตร์
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การวิจัยมีวัตถุประสงค์เพื่อแสดงความสมมูลของเกณฑ์การตรวจสอบความเป็นอตรรกยะของลำดับจำนวนตรรกยะโดยทฤษฎีบทของบรุน(Brun) กับเกณฑ์การตรวจสอบความเป็นอตรรกยะของอนุกรมซึ่งแต่ละพจน์เป็นจำนวนตรรกยะโดยทฤษฎีบทของบาเดีย(Badea) และยังเทียบเท่ากับเกณฑ์การตรวจสอบความเป็นอตรรกยะของผลคูณซึ่งแต่ละพจน์เป็นจำนวนตรรกยะ ซึ่งเราต้องปรับทฤษฎีบทของบรุนไปในทางเดียวกันกับทฤษฎีบทของบาเดียในปี 1993 ส่วนทฤษฎีบทของบาเดียต้องเปลี่ยนเงื่อนไขของลำดับจากจำนวนเต็มเป็นลำดับของจำนวนตรรกยะ

จากการเทียบเท่ากันข้างต้น นำไปสู่การพิสูจน์เอกลักษณ์ที่เกี่ยวข้องกับเศษส่วนต่อเนื่องอีกแบบหนึ่ง และศึกษาความเชื่อมโยงของเกณฑ์ความเป็นอตรรกยะที่ได้กับเกณฑ์ความเป็นอตรรกยะที่เกี่ยวข้องอื่นๆ

EQUIVALENCES OF BRUN'S CRITERION AND APPLICATIONS

AN ABSTRACT
BY
KANTAPHON KUHAPATANAKUL

Presented in Partial Fulfillment of the Requirements for the
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Kantaphon Kuhapatanakul. (2009). *Equivalences of Brun's Irrationality Criterion and Applications.* Dissertation, Ph.D. (Mathematics). Bangkok: Graduate School, Srinakharinwirot University. Advisor Committee: Prof. Dr. Narong Punnim, Prof. Dr. Vichian Laohakosol, Associate Prof. Dr. Nittiya Pabhapote.

An old irrationality criterion, due to Brun since 1910, for a sequence of rational numbers is shown to be essentially equivalent to the one for a series of rational terms proved by Badea in 1993, and to one for an infinite product of rational numbers. A similar criterion with certain inequality conditions of the classical Brun's criterion being reversed is also shown to be equivalent to a reversed form of Badea's 1993 irrationality criterion for series of rational terms. This is accomplished on one hand by suitably modifying Brun's criterion, and on the other hand by extending Badea's criterion to series of quotients whose numerators and denominators are rational numbers.

Applying these equivalences, interesting new proofs of identities involving convergents of continued fractions are obtained and related irrationality criteria are derived. Connections with other irrationality criteria are investigated with more irrationality criteria for infinite products of rational numbers as applications.

The dissertation titled
“Equivalences of Brun’s Irrationality Criterion and Applications”
by
Kantaphon Kuhapatanakul
has been approved by the Graduate School as partial fulfillment of the requirements for
the Doctor of Philosophy in Mathematics of Srinakharinwirot University.

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Kantaphon Kuhapatanakul

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CHAPTER 1

INTRODUCTION

A real number is said to be *rational* if it can be written as a quotient of two integers with nonzero denominator, and is called *irrational* otherwise. To determine whether a real number is rational is in general not a trivial task. For example, that $\sqrt{2}$ is irrational can be concluded from the unique factorization theorem. For if $\sqrt{2}$ could be represented in the form a/b with $\gcd(a,b) = 1$, it would follow that $a^2 = 2b^2$. Hence, a^2 is even, so is a. If $a = 2c$, then $4c^2 = 2b^2, 2c^2 = b^2$, and b is also even, contrary to the hypothesis that $\gcd(a,b) = 1$. Other examples include $e, \pi, \sqrt[3]{5}$, etc. The irrational of e was proven by Euler in 1737 and the irrationality of π was proven by Lambert in 1760.

There have been a number of rationality characterizations, e.g. [16], [25]. In 1910, Viggo Brun, [7], established the following classical irrationality criterion which nowadays bears his name.

Brun's criterion. Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be two sequences of positive integers satisfying $X_n < X_{n+1}$ for all $n \geq 1$. Assume that $L = \lim_{n\to\infty} Y_n/X_n \in \mathbb{R}$, and

$$\frac{Y_n}{X_n} < \frac{Y_{n+1}}{X_{n+1}} \quad (n \geq 1).$$

If

$$\frac{Y_{n+2} - Y_{n+1}}{X_{n+2} - X_{n+1}} < \frac{Y_{n+1} - Y_n}{X_{n+1} - X_n}$$

for all large n, then L is irrational.

Brun, [7], using his criterion, showed the exponential series $e = \sum_n 1/n!$ to be irrational by observing that the sequence of its n^{th} partial sums $\left\{\frac{2}{1}, \frac{5}{2}, \frac{16}{6}, \frac{65}{24}, \frac{326}{120}, \ldots\right\}$ is increasing, while the sequence of the quotients of numerator-differences by denominator-differences $\left\{\frac{3}{1}, \frac{11}{4}, \frac{49}{18}, \frac{261}{96}, \ldots\right\}$ is decreasing. This criterion was later improved by Froda, [13], in 1963. Using Brun's criterion, Badea, [1], in 1987 deduced the following test for the irrationality of certain infinite series of rational numbers.

Badea's criterion. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers. If

$$a_{n+1} > \frac{b_{n+1}}{b_n}\, a_n^2 - \frac{b_{n+1}}{b_n}\, a_n + 1$$

holds for every large n, then the sum $\sum_{n=1}^{\infty} b_n/a_n$, whenever convergent, is irrational.

Badea, [1], used this criterion to answer an old conjecture of Erdös and Graham, [11], about the irrationality of a series of reciprocals of Fibonacci and Lucas numbers. Badea's criterion has indeed been mentioned as an application in the 1910 paper of Brun, [7]. Later in 1993, Badea, [2], established an extension of his criterion *without using Brun's criterion* by modifying the inequality condition from two consecutive terms to that of a block of terms.

Badea's criterion II. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers. For an increasing sequence $\mathcal{N} = \{n(k)\}_{k\geq 1}$ of positive integers, define

$$S_k := S_k(\mathcal{N}) = a_{n(k)+1} \cdots a_{n(k+1)}, \qquad R_k := R_k(\mathcal{N}) = \sum_{j=1}^{d(k)} S_k \frac{b_{n(k)+j}}{a_{n(k)+j}},$$

where $d(k) = n(k+1) - n(k)$. Assume that $\sum_{n=1}^{\infty} b_n/a_n$ converges to a rational number. If

$$S_{k+1} \geq \frac{R_{k+1}}{R_k} S_k(S_k - 1) + 1 \tag{1}$$

for all large k, then in (1) we have equality from some k onwards.

As another application of Brun's criterion, Brun, [7], proved that the infinite product of positive rational elements $\prod_{n=1}^{\infty} b_n/a_n$ converges to an irrational number when $b_n > a_n$ and

$$\frac{a_{n+1} - 1}{a_n - 1} > \frac{b_n}{a_n} \cdot \frac{b_{n+1} - 1}{b_n - 1}.$$

In particular, if $\{a_n\}_{n\geq 1}$ is a sequence of positive integers satisfying $a_{n+1} > a_n^2$ for all $n \geq 1$, then the infinite product $\prod_{n=1}^{\infty} (1 + 1/a_n)$ converges to an irrational number. This is contained in a more general result about Cantor products, cf. [29].

There are several known conditions for an infinite convergent series of positive rational numbers to have an irrational sum. There are also results about the irrationality of particular constants expressed as series of positive rationals (see [1], [2], [9], [11], [12], [15], [21], [32] and the references cited therein). In general, it is difficult to obtain general irrationality criteria which are both sufficient and necessary and to find one is an interesting problem. Surprisingly, in comparison, very little attention has been paid to finding such sufficiency conditions in the case of infinite products. One such sufficiency condition is attributed to Cantor (see [29]), and some generalizations of this condition have also been obtained in [32].

There are other related results such as Tachiya, [34], proved that the infinite product $\prod_{k=0}^{\infty}(1 + m_k/R_k)$, and Becker-Töpfer, [4], proved that the infinite series $\sum_{k=1}^{\infty} m_k/R_k$, where $\{m_k\}$ is a sequence of algebraic numbers and $\{R_k\}$ is a binary recurrence, are both transcendental except for a few cases where they represent algebraic numbers.

The objectives of this thesis are:

1. to gather general criteria for irrationality of sequences of rational terms, including sequences of partial sums and partial products;
2. to search for possible relations among various irrationality criteria, including those of Brun and Badea;
3. to find new irrationality criteria for infinite sums and infinite products deducible from the relations found in the second objective;
4. to give applications of the criteria and relations so obtained including the identification of the kind of algebraic numbers arising from the work of Tachiya and Becker-Töpfer.

CHAPTER 2

REVIEW OF THE LITERATURE

The classical Brun's irrationality criterion since 1910, [7], states that:

Theorem 2.1 Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of positive integers satisfying $X_n < X_{n+1}$ for all $n \geq 1$. Assume that $L = \lim_{n\to\infty} Y_n/X_n \in \mathbb{R}$, and

$$\frac{Y_n}{X_n} < \frac{Y_{n+1}}{X_{n+1}} \quad (n \geq 1).$$

If

$$\frac{Y_{n+2} - Y_{n+1}}{X_{n+2} - X_{n+1}} < \frac{Y_{n+1} - Y_n}{X_{n+1} - X_n}$$

for all large n, then L is irrational.

Regarding the irrationality of infinite series, in 1957 Erdös, [10], posed the following problem: let $n_1 < n_2 < \ldots$ be a sequence of integers such that

$$\lim_{k\to\infty} \frac{n_k}{n_1 n_2 \cdots n_{k-1}} = \infty.$$

Show that $\sum_{k=1}^{\infty} 1/n_k$ is irrational. The problem was solved elementarily by many people. In 1963, Erdös and Straus, [12], gave the following irrationality criterion.

Theorem 2.2 Let $\{n_k\}_{k\geq 1}$ be an increasing sequence of positive integers such that

$$\lim_{k\to\infty} \frac{n_k^2}{n_{k+1}} \leq 1 \quad \text{and} \quad \left\{ \frac{N_k}{n_{k+1}} \right\} \quad \text{is unbounded,}$$

where N_k denotes the least common multiple of $n_1, ..., n_k$. Then the Ahmes series $\sum_{k=1}^{\infty} 1/n_k$ is rational if and only if $n_{k+1} = n_k^2 - n_k + 1$ for all $k \geq k_0$, in which case

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = \frac{1}{n_1} + \cdots + \frac{1}{n_{k_0-1}} + \frac{1}{n_{k_0} - 1}.$$

In 1984, Sandor, [32], extended the result of Erdös, [10], to:

Theorem 2.3 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers with

$$\limsup_{n\to\infty} \frac{a_n}{a_1 a_2 \cdots a_{n-1} b_n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{b_{n-1} a_n}{a_{n-1} b_n} > 1.$$

If $\xi = \sum_{n=1}^{\infty} b_n/a_n < \infty$, then ξ is an irrational number.

It is generally believed that series of rational terms with rapid convergence should give rise to irrational sums. In 1987, Badea, [1], deduced the following test for irrationality of certain infinite series of rational numbers by using Brun's criterion.

Theorem 2.4 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers. If

$$a_{n+1} > \frac{b_{n+1}}{b_n}\, a_n^2 - \frac{b_{n+1}}{b_n}\, a_n + 1 \tag{2}$$

holds for every large n, then $\sum_{n=1}^{\infty} b_n/a_n$, whenever convergent, is irrational.

The converse of Theorem 2.4 is false in general, [6], as seen e.g. from the exponential series $\sum 1/n! = e$. Theorem 2.4 is best possible in the sense that should the inequality condition (2) be replaced by an equality, the result fails as the following example shows: for a given sequence $\{b_n\}$ of positive integers and a given a positive integer t, define the sequence of positive integers $\{w_n\}, w_n = w_n(b_n, t)$ by $w_1 = 1 + tb_1$ and $w_{n+1} = 1 + tb_{n+1}w_1 \cdots w_n$ for $n \geq 1$. Then

$$w_{n+1} = \frac{b_{n+1}}{b_n}\, w_n^2 - \frac{b_{n+1}}{b_n}\, w_n + 1,$$

and

$$\frac{b_1}{w_1} + \frac{b_2}{w_2} + \cdots + \frac{b_n}{w_n} = \frac{1}{t} - \frac{b_{n+1}}{w_{n+1} - 1} \longrightarrow \frac{1}{t} \in \mathbb{Q}.$$

Moreover, there is a problem, [24], which asks for conclusion when the inequality (2) becomes equality from certain point onward. This settled by noting that (2) is equivalent to

$$\frac{b_n}{a_n - 1} - \frac{b_{n+1}}{a_{n+1} - 1} > \frac{b_n}{a_n}. \tag{3}$$

If equality holds in (3) for all $n \geq m$, then summing from $n = m$ onwards and nothing that $b_n/(a_n - 1) \to \infty$ as $n \to \infty$, we have

$$\sum_{n=m}^{\infty} \frac{b_n}{a_n} = \frac{b_m}{a_m - 1},$$

rendering the infinite sum to be rational.

The result of Badea's criterion (Theorem 2.4) has indeed been mentioned as an application in the 1910 paper of Brun, [7].

Define the sequence of Fibonacci numbers $\{F_n\}$ by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0)$$

and the sequence of Lucas numbers $\{L_n\}$ by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).$$

Badea, [1], used Theorem 2.4 to answer an old conjecture of Erdös and Graham (see [11] pp. 64-65) that the sums $\sum_{n=1}^{\infty} 1/F_{2^n+1}$ and $\sum_{n=1}^{\infty} 1/L_{2^n+1}$ are irrational. Later in 1993, Badea [2] established an extension of his criterion (Theorem 2.4) *without using Brun's criterion* by modifying the inequality condition from two consecutive terms to that of a block of terms.

Theorem 2.5 (Badea's criterion II.) Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers and $\mathcal{N} = \{n(k)\}_{k\geq 1}$ an increasing sequence of positive integers. Define

$$S_k := S_k(\mathcal{N}) = a_{n(k)+1} \cdots a_{n(k+1)}, \qquad R_k := R_k(\mathcal{N}) = \sum_{j=1}^{d(k)} S_k \frac{b_{n(k)+j}}{a_{n(k)+j}},$$

where $d(k) = n(k+1) - n(k)$. Assume that $\sum_{n=1}^{\infty} b_n/a_n$ converges to a rational number. If

$$S_{k+1} \geq \frac{R_{k+1}}{R_k} S_k(S_k - 1) + 1 \tag{4}$$

for all large k, then in (4) we have equality from some k onwards.

Specializing $n(k) = k$ in Theorem 2.5, we have

Corollary 2.6 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers. Assume that $\sum_{n=1}^{\infty} b_n/a_n$ converges to a rational number. If

$$a_{n+1} \geq \frac{b_{n+1}}{b_n}\, a_n^2 - \frac{b_{n+1}}{b_n}\, a_n + 1 \tag{5}$$

for every large n, then in (5) we have equality from some n onwards.

Badea, [2], used Corollary 2.6 to answer another problem of Erdös and Graham, [11], by showing that $\sum_{k=1}^{\infty} 1/F_{n(k)}$ is irrational when $\{n(k)\}$ is a sequence of positive integers satisfying $n(k+1) \geq 2n(k)$. Badea, [2], gave another corollary of his main result which yields an irrationality criterion for alternating series as:

Corollary 2.7 Let $\{b_n/a_n\}$ be a decreasing sequence of positive rationals such that

$$a_{2k+1}a_{2k+2} \geq 1 + a_{2k-1}a_{2k}\frac{(a_{2k-1}a_{2k}-1)(b_{2k+1}a_{2k+2}-b_{2k+2}a_{2k+1})}{b_{2k-1}a_{2k}-b_{2k}a_{2k-1}} \tag{6}$$

for every k. If $\sum_{n=1}^{\infty}(-1)^{n+1}b_n/a_n$ is rational, then in (6) we have equality for all sufficiently large k.

Corollary 2.7 provides an improvement of an old result of Sierpiński which says that: if $a_{n+1} \geq a_n(a_n+1)$, then $\sum_{n=1}^{\infty}(-1)^{n+1}/a_n$ is irrational. There is another extension of this result due to Sándor, [33], who gave an irrationality assertion for alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n/a_n$. Specializing $b_n = 1$ for all n in Corollary 2.7, we have

Corollary 2.8 If $\{a_n\}_{n\geq 1}$ is a sequence of integers satisfying

$$a_{2k+1} \geq a_{2k}a_{2k-1}\frac{a_{2k}a_{2k-1}-1}{a_{2k}-a_{2k-1}}$$

for all sufficiently large k, then $\sum_{n=1}^{\infty}(-1)^{n+1}/a_n$ is irrational.

In 1996 Hančl, [15], uses Brun's criterion (Theorem 2.1) to prove an irrationality result under inequality conditions opposite to (2). A particular case reads:

Theorem 2.9 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive integers. If

(i) $a_n > 2$ and $a_{n+1} < \frac{b_{n+1}}{b_n}\, a_n^2 - \frac{b_{n+1}}{b_n}\, a_n + 1$

(ii) $a_{n+1}(-a_nb_{n-1} + 2a_{n-1}^2b_n - a_{n-1}b_n) >$

$$b_{n+1}a_na_{n-1}(a_na_{n-1} - 2a_{n-1} + 1) + 3b_na_{n-1}^2 - 2a_nb_{n-1} - 2a_{n-1}b_n + b_{n-1}$$

for every large n, then the sum $\sum_{n=1}^{\infty} b_n/a_n$ is irrational.

In the direction of infinite products of rational terms, Brun gave the following result.

Theorem 2.10 Let $a_n, b_n \in \mathbb{N}$ when $b_n > a_n > 1$ for $n \in \mathbb{N}$. If

$$\frac{a_{n+1}-1}{a_n-1} > \frac{b_n}{a_n}\cdot\frac{b_{n+1}-1}{b_n-1},$$

then the infinite product $\prod_{n=1}^{\infty} b_n/a_n$, whenever convergent, is an irrational number.

In particular, if $\{a_n\}_{n\geq 1}$ is a sequence of positive integers satisfying $a_{n+1} > a_n^2$ for all $n \geq 1$, then the infinite product $\prod_{n=1}^{\infty}(1+1/a_n)$ converges to an irrational number. This is contained in a more general result about Cantor products which we quote from [29].

Theorem 2.11 Each real number $x > 1$ can be represented uniquely as an infinite product of the form

$$x = \prod_{n=0}^{\infty}\left(1+\frac{1}{a_n}\right), \tag{7}$$

where a_n's are positive integers not all equal to 1 and satisfy

$$a_{n+1} \geq a_n^2. \tag{8}$$

Moreover, the value in (7) is rational if and only if in (8) we have equality from some n onwards.

In 1968, Oppenheim, [28], proved the following:

Theorem 2.12 Suppose that the positive integers a_n satisfy the conditions

$$\limsup_{n\to\infty}\frac{a_n^2}{a_{n+1}} < \infty \qquad \text{and} \qquad \limsup_{n\to\infty} a_n = \infty.$$

Then if $\prod_{n=1}^{\infty}(1+1/a_n)$, necessarily convergent, is rational, then there exists a sequence of positive integers $\{c_n\}$ with the following properties

$$c_n a_{n+1} - c_{n+1} = (a_n+1)(c_{n-1}a_n - c_n) > 0, \quad (n \geq n_0), \tag{9}$$

$$c_n = o(a_n), \quad 0 < c_n < a_n, \tag{10}$$

$$\limsup_{n\to\infty}\frac{c_n}{c_{n-1}} = \limsup_{n\to\infty}\frac{a_n^2}{a_{n+1}}.$$

Conversely if (9) and (10) are satisfied, then the infinite product is rational since

$$\prod_{n\geq n_0}\left(1+\frac{1}{a_n}\right) = 1 + \frac{c_{n_0-1}}{c_{n_0-1}a_{n_0} - c_{n_0}}.$$

There are also other irrationality results for infinite products such as the one in Lynch and Mycielski, [22], which states that $\prod_{n=1}^{\infty}(1-1/a^n)$ is irrational for $a \in \mathbb{Z}$ with $|a| > 1$ and another irrationality criterion due to Sándor, Proposition 2 of [32], which states that:

Theorem 2.13 Let $\{b_n\}$ be a sequence of primes with $\lim_{n\to\infty} b_n = \infty$ and let $\{a_n\}$ be a sequence of positive integers satisfying

$$a_{n+k} \geq b_{n+k}\, a_n^{2^k} \qquad (n,k \in \mathbb{N})\,. \tag{11}$$

Then the infinite product $\prod_{n=1}^{\infty}(1 + b_n/a_n)$ is irrational.

In 1986, Badea, [3], found some conditions for the value of the infinite product to be irrational.

Theorem 2.14 Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive integers suct that

$$a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 + \frac{b_{n+1}(b_n - 1)}{b_n} a_n + 1 - b_{n+1} \tag{12}$$

holds for all large n. Then the value of the product $\prod_{n=1}^{\infty}(1 + b_n/a_n)$ is an irrational number.

More recently, A. Knopfmacher and J. Knopfmacher, [19], discovered the following product representation of rational numbers as infinite product of the form considered above.

Theorem 2.15 Let $c_1 \leq c_2 \leq \cdots \leq c_r$ be positive integers with $c_r \geq 2$. If $c_n = b^{2^{n-r-1}}$ for $n \geq r+1$, where $b = c_1 c_2 \cdots c_r$, then

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{c_1 c_2 \cdots c_n}\right) = \frac{b}{b-1}\prod_{n=1}^{r-1}\left(1 + \frac{1}{c_1 c_2 \cdots c_n}\right). \tag{13}$$

As evidenced earlier, a number of infinite series and infinite products mentioned above have terms which are elements belonging to sequences which satisfy linear recurrence relations. There have appeared a great deal of irrationality results for series and products of such elements. We mention here only two recent ones which will be further investigated in later chapters.

Let $\{R_n\}_{n\geq 0}$ be a sequence of integers satisfying a binary linear recurrence defined by

$$R_{n+2} = E_1 R_{n+1} + E_2 R_n, \qquad (E_1, E_2 \in \mathbb{Z} \setminus \{0\}) \tag{14}$$

where the initial values R_0, R_1 are not all zero and $\Delta = E_1^2 + 4E_2 > 0$. Let ρ_1, ρ_2 with $|\rho_1| \geq |\rho_2|$ be the distint real roots of the equation $x^2 - E_1 x + E_2 = 0$. Then

R_n can be written as

$$R_n = g_1\rho_1^n + g_2\rho_2^n, \tag{15}$$

where $g_1, g_2 \in \mathbb{Q}(\rho_1)$.

In 2007 Tachiya, Theorem 3 of [34], proved the following transcendence results about infinite products whose elements are related to a binary sequence.

Theorem 2.16 Let $r \geq 2, c \geq 1$ and d be integers and $\{R_n\}$ a binary recurrence given by (14) and (15). Let K be algebraic number field and $\{m_k\}_{k\geq 0}$ a sequence in K satisfying $m_k \neq R_{cr^k+d}$ $(k > 0)$ and $\log ||m_k|| = o(r^k)$. Then

$$\prod_{k=0}^{\infty}(1 + \frac{m_k}{R_{cr^k+d}})$$

is algebraic if and only if at least one of the following conditions is satisfied:
(i) $m_n = 0$ for every large n.
(ii) $r = 2$ and $m_n = R_d$ for every large n.
(iii) $r = 2, g_1\rho_1^d = g_2\rho_2^d$, and there exists a root of unity ω such that $m_n = g_1\rho_1^d(\omega^{2^n} + \omega^{-2^n})$ for every large n.

In 1994 Becker and Töpfer, [4], characterized the algebraicity of infinite series which are reciprocal sums of elements in a binary sequence.

Theorem 2.17 Let c, d, r be integers with $r \geq 2, c \geq 1$ and $\{R_n\}$ a binary recurrence sequence defined above. Let $\{m_n\}$ be a periodic sequence of algebraic numbers which is not identically zero. Then

$$\sum_{n=0}^{\infty}\frac{m_n}{R_{c\cdot r^n+d}}$$

is algebraic if and only if $\{m_n\}$ is a constant sequence, $r = 2, |A_2| = 1$ and $R_d = 0$.

CHAPTER 3

EQUIVALENCES AND APPLICATIONS

In this chapter, we first show that Brun's and Badea's criteria are essentially equivalent. To do so, we need to modify the original Brun's criterion in the direction of the conclusion of Badea's 1993 criterion, and in Badea's 1993 criterion, we raise the complexity of rational terms b_n/a_n one level up to β_n/α_n with α_n, β_n being positive rational numbers. Applying this equivalences, interesting new proofs of identities invilving convergents of continued fractions are obtained. Then we deduce connections with some other irrationality criteria for infinite series and identify the result of Becker-Töpfersome of these exceptional cases which are algebraic but not rational. In the second, Brun's criterion for sequences of rational numbers is shown to be equivalent to a useful irrationality criterion for infinite product of rational numbers. There are several applications of this equivalence and connections with other irrationality results about infinite products. The final section is investigated what conclusion can be obtained should only one of the two inequality of Brun's criterion be reversed.

3.1 Equivalences of Brun and Badea

3.1.1 Modifying Brun's Criterion

Our modification of the original Brun's criterion in the direction of the conclusion of Badea's 1993 criterion reads:

Theorem 3.1 (Brun's criterion II) Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be two sequences of positive integers with $X_n < X_{n+1}$ for all $n \geq 1$. Assume that $L = \lim_{n\to\infty} Y_n/X_n$ is rational and

$$\frac{Y_n}{X_n} < \frac{Y_{n+1}}{X_{n+1}} \qquad (n \geq 1). \tag{16}$$

If

$$\frac{Y_{n+1} - Y_n}{X_{n+1} - X_n} \geq \frac{Y_{n+2} - Y_{n+1}}{X_{n+2} - X_{n+1}} \tag{17}$$

for all large n, then in (17) we have equality from some n onwards.

Proof. Suppose that $L = a/b$ where $a, b \in \mathbb{N}$. From (16), we deduce that

$$\frac{Y_{n+1} - Y_n}{X_{n+1} - X_n} > \frac{Y_n}{X_n}. \tag{18}$$

Since $\left\{ \frac{Y_{n+1}-Y_n}{X_{n+1}-X_n} \right\}$ is a non-increasing sequence of positive rationals, it must converge to a limit. Invoking upon (18) we get

$$\lim_{n\to\infty} \frac{Y_{n+1} - Y_n}{X_{n+1} - X_n} \geq \lim_{n\to\infty} \frac{Y_n}{X_n} = L = \frac{a}{b},$$

which implies, for all large n, that

$$aX_n - bY_n \geq aX_{n+1} - bY_{n+1} > 0;$$

the last inequality being followed from $\frac{Y_n}{X_n} \leq L$. This gives us an infinite non-increasing sequence of nonnegative integers. Thus, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we must have

$$aX_n - bY_n = aX_{n+1} - bY_{n+1}, \quad \text{i.e.,} \quad \frac{Y_{n+1} - Y_n}{X_{n+1} - X_n} = \frac{a}{b},$$

and the result follows. □

There is also a criterion called here the ***reverse Brun's criterion*** that reversing the inequalities in (16) and (17) by changing " $<$ " to " $>$ " and " $\geq$ " to " $\leq$ ", respectively. Its proof is analogous and then omitted.

Theorem 3.2 Let $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ be two sequences of positive integers such that $U_n < U_{n+1}$ for all $n \geq 1$. Assume that $L' = \lim_{n\to\infty} V_n/U_n$ is rational and

$$\frac{V_n}{U_n} > \frac{V_{n+1}}{U_{n+1}} \qquad (n \geq 1).$$

If

$$\frac{V_{n+1} - V_n}{U_{n+1} - U_n} \leq \frac{V_{n+2} - V_{n+1}}{U_{n+2} - U_{n+1}} \tag{19}$$

for all large n, then in (19) we have equality from some n onwards.

3.1.2 An Extension of Badea's Criterion

We start to re-state Badea's criterion, [2], of 1993 and give a simplified proof. To facilitate the writing, the following terminology will be kept fixed for the rest of this chapter.

- $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ three sequences of positive rational numbers;
- $\{\nu_n\}_{n\geq 1}$ a sequence of rationals with $\nu_1 > 0$ and $\nu_n < 0$ for all $n \geq 2$;
- $P_n := \alpha_1\alpha_2\cdots\alpha_n$ and $Q_n := \mu_1\mu_2\cdots\mu_n$ for all $n \geq 1$;
- $A_n := P_n \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j}$ and $B_n := Q_n \sum_{j=1}^{n} \frac{\nu_j}{\mu_j}$ for all $n \geq 1$;
- $\mathcal{N} := \{n(k)\}_{k\geq 1}$ an increasing sequence of positive integers;
- $d(k) := n(k+1) - n(k) \quad$ for all $k \geq 1$;
- $s_k := s_k(\mathcal{N}) = \alpha_{n(k)+1}\cdots\alpha_{n(k+1)} \quad (k \geq 1), \quad s_0 := \alpha_1\cdots\alpha_{n(1)}$;
- $r_k := r_k(\mathcal{N}) = s_k \sum_{j=1}^{d(k)} \frac{\beta_{n(k)+j}}{\alpha_{n(k)+j}} \quad (k \geq 1), \quad r_0 := s_0\left(\frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_{n(1)}}{\alpha_{n(1)}}\right)$.

Theorem 3.3 Assume that P_n and A_n are both integral for all large n. If $\sum_{n=1}^{\infty}\beta_n/\alpha_n$ is a rational number, there is a sequence $\mathcal{N}$ such that $s_k > 1$ and

$$\frac{s_{k+1}-1}{r_{k+1}} \geq \frac{s_k(s_k-1)}{r_k} \tag{20}$$

for all sufficiently large k, then in (20) we have equality from some k onwards.

Proof. We assume that the conditions of the theorem are satisfied but the inequality in (20) is strict for infinitely many values of k. Let $\sum_{n=1}^{\infty}\beta_n/\alpha_n = p/q$, where $p,\ q \in \mathbb{N},\ \gcd(p,q) = 1$. Thus,

$$\frac{p}{q} = \lim_{n\to\infty}\frac{A_n}{P_n} = \lim_{k\to\infty}\frac{A_{n(k)}}{P_{n(k)}}.$$

Now

$$A_{n(k+1)} = P_{n(k+1)}\left(\sum_{j=1}^{n(k)}\frac{\beta_j}{\alpha_j} + \sum_{j=1}^{d(k)}\frac{\beta_{n(k)+j}}{\alpha_{n(k)+j}}\right) = s_k A_{n(k)} + r_k P_{n(k)}. \tag{21}$$

Defining $v_k = \frac{A_{n(k+1)}-A_{n(k)}}{P_{n(k+1)}-P_{n(k)}}$ and using (21), we get that

$$v_k = \frac{(s_k-1)\,A_{n(k)} + r_k P_{n(k)}}{P_{n(k)}(s_k-1)} = \frac{A_{n(k)}}{P_{n(k)}} + \frac{r_k}{s_k-1} \tag{22}$$

and

$$\frac{A_{n(k+1)}}{P_{n(k+1)}} - \frac{A_{n(k)}}{P_{n(k)}} = \frac{A_{n(k+1)} - s_k A_{n(k)}}{P_{n(k+1)}} = \frac{r_k}{s_k},$$

so

$$v_{k+1} - v_k = \frac{r_{k+1}}{s_{k+1}-1} - \frac{r_k}{s_k(s_k-1)}.$$

Together with (20) and our assumptions, we have $v_{k+1} \le v_k$ for all sufficiently large k and $v_{k+1} < v_k$ for infinitely many k. We express this by saying that $\{v_k\}$ is *almost strictly decreasing*. Note that $\{A_n/P_n\}$ is an increasing sequence and thus $\frac{A_n}{P_n} < \frac{p}{q}$, i.e., for all large n, we have

$$pP_n - qA_n > 0. \tag{23}$$

Since $s_k > 1$ when k is large enough, this together with (22) give $v_k > \frac{A_{n(k)}}{P_{n(k)}}$. Thus, the almost strictly decreasing of $\{v_k\}$ and the increasing of $\{A_n/P_n\}$ to p/q imply that $v_k \ge p/q$ for all large k. Using this last relation and (23) we have

$$0 < pP_{n(k+1)} - qA_{n(k+1)} \le pP_{n(k)} - qA_{n(k)} \tag{24}$$

for all large k. The relation (24) gives us an infinite non-increasing sequence of positive integers and so $pP_{n(k+1)} - qA_{n(k+1)} = pP_{n(k)} - qA_{n(k)}$, which gives

$$\frac{p}{q} = \frac{A_{n(k+1)} - A_{n(k)}}{P_{n(k+1)} - P_{n(k)}} = v_k$$

for all large k, contradicting the almost strictly decreasing of v_k. □

Remark. The main idea of the proof given above, which does not make use of Brun's criterion, is similar to that of Badea [2], yet the steps around (22)-(23), differ and substantially simplify those of Badea. Theorem 2.5 is a special case where $\beta_n = b_n \in \mathbb{N}, \alpha_n = a_n \in \mathbb{N}$; in this situation the integrality of A_n and P_n are automatic, while the requirement that $s_k > 1$ is immediate from the fact that $a_n \to \infty$ because of the series convergence.

Specializing $n(k) = k$ in Theorem 3.3, we have:

Corollary 3.4 Assume that P_n and A_n are both integral for all large n. If $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ converges to a rational number, $\alpha_n > 1$ and

$$\frac{\alpha_{n+1}-1}{\beta_{n+1}} \ge \frac{\alpha_n(\alpha_n-1)}{\beta_n} \tag{25}$$

for all large n, then equality occurs in (25) from some n onwards.

As in the same case of the reverse Brun's criterion, we also have a version of the reverse Badea's criterion as:

Theorem 3.5 Let $\sum_{n=1}^{\infty} \nu_n/\mu_n$ converge to a positive rational number. Assume that Q_n and B_n are both integral and $\mu_n > 1$ for all large n. If

$$\frac{\nu_{n+1}}{\mu_{n+1}-1} \geq \frac{\nu_n}{\mu_n(\mu_n - 1)} \tag{26}$$

for all large n, then in (26) we have equality from some n onwards.

3.1.3 Brun ≡ Badea

Now, we are ready to show that Brun's criterion II is essentially equivalent to Badea's criterion and the reverse Brun's criterion is essentially equivalent to the reverse Badea's criterion. We do so by proving that

- Theorem 3.1 implies Theorem 3.3
- Corollary 3.4 implies Theorem 3.1
- Theorem 3.2 and Theorem 3.5 are equivalent.

Theorem 3.6 Theorem 3.1 implies Theorem 3.3.

Proof. From the two given sequences $\{\alpha_n\}$, $\{\beta_n\}$ of positive rational numbers satisfying all the properties as stated in Theorem 3.3, we construct another two sequences of positive integers $\{X_n\}$, $\{Y_n\}$ as follows:

$$X_1 = \alpha_1\alpha_2\cdots\alpha_{n(1)}, \quad X_2 = \alpha_1\alpha_2\cdots\alpha_{n(2)}, \quad \cdots, \quad X_k = \alpha_1\alpha_2\cdots\alpha_{n(k)},$$

$$Y_1 = X_1\sum_{j=1}^{n(1)}\frac{\beta_j}{\alpha_j}, \quad Y_2 = X_2\sum_{j=1}^{n(2)}\frac{\beta_j}{\alpha_j}, \quad \cdots, \quad Y_k = X_k\sum_{j=1}^{n(k)}\frac{\beta_j}{\alpha_j}.$$

Clearly,

$$Y_{k+1} = s_kY_k + r_kX_k \quad \text{and} \quad \frac{Y_{k+1}-Y_k}{X_{k+1}-X_k} = \frac{Y_k}{X_k} + \frac{r_k}{s_k - 1}.$$

That Theorem 3.1 implies Theorem 3.3 follows immediately from the following set of equivalences, for k sufficiently large:

$$\begin{aligned}\frac{X_k}{X_{k-1}} > 1 \quad &\Leftrightarrow \quad s_{k-1} = \alpha_{n(k-1)+1} \cdots \alpha_{n(k)} > 1;\\ \frac{Y_k}{X_k} > \frac{Y_{k-1}}{X_{k-1}} \quad &\Leftrightarrow \quad \frac{r_{k-1}}{s_{k-1}} = \frac{\beta_{n(k-1)+1}}{\alpha_{n(k-1)+1}} + \cdots + \frac{\beta_{n(k)}}{\alpha_{n(k)}} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{Y_{k+2}-Y_{k+1}}{X_{k+2}-X_{k+1}} \le \frac{Y_{k+1}-Y_k}{X_{k+1}-X_k} \quad &\Leftrightarrow \quad \frac{Y_{k+1}}{X_{k+1}} + \frac{r_{k+1}}{s_{k+1}-1} \le \frac{Y_k}{X_k} + \frac{r_k}{s_k-1}\\ &\Leftrightarrow \quad \frac{Y_{k+1}}{X_{k+1}} - \frac{Y_k}{X_k} \le \frac{r_k}{s_k-1} - \frac{r_{k+1}}{s_{k+1}-1}\\ &\Leftrightarrow \quad \frac{r_k}{s_k} \le \frac{r_k}{s_k-1} - \frac{r_{k+1}}{s_{k+1}-1}\\ &\Leftrightarrow \quad \frac{r_{k+1}}{s_{k+1}-1} \le \frac{r_k}{s_k\left(s_k-1\right)}\end{aligned}$$

which complete. □

Theorem 3.7 Corollary 3.4 implies Theorem 3.1.

Proof. From the two given sequences of positive integers $\{X_n\}$, $\{Y_n\}$ satisfying all the properties as stated in Theorem 3.1, note that the construction in the last theorem is reversible, i.e., two corresponding sequences of positive rational numbers $\{\alpha_n\}$ and $\{\beta_n\}$ can be constructed, namely,

$$\begin{aligned}&\alpha_1 = X_1, \qquad \alpha_2 = \frac{X_2}{X_1}, \quad \ldots, \quad \alpha_n = \frac{X_n}{X_{n-1}};\\ &\beta_1 = Y_1, \qquad \beta_2 = \frac{X_2}{X_1}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad \ldots, \quad \beta_n = \frac{X_n}{X_{n-1}}\left(\frac{Y_n}{X_n} - \frac{Y_{n-1}}{X_{n-1}}\right).\end{aligned}$$

The two sequences $\{\alpha_n\}, \{\beta_n\}$ are merely counter-parts of the previously defined sequences $\{s_n\}, \{r_n\}$, respectively. Using this observation, that Corollary 3.4 implies Theorem 3.1 follows immediately through the same set of equivalences as in the last theorem. □

Theorem 3.8 The reverse Brun's criterion (Theorem 3.2) and the reverse Badea's criterion (Theorem 3.5) are equivalent.

Proof. Given two sequences $\{\mu_n\}, \{\nu_n\}$ satisfying all the properties as stated in Theorem 3.5. By hypothesis, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have

$Q_n, B_n \in \mathbb{Z}$, $\mu_n > 1$ and the inequality (20) holds. Throwing away finitely many terms at the beginning if necessary, without loss of generality, we may take $N = 1$. We construct two sequences of positive integers $\{U_n\}$, $\{V_n\}$ as follows:

$$U_1 = Q_1 = \mu_1, \quad V_1 = B_1 = Q_1 \sum_{j=1}^{1} \frac{\nu_j}{\mu_j},$$

$$U_2 = Q_2 = \mu_1\mu_2, \quad V_2 = B_2 = Q_2 \sum_{j=1}^{2} \frac{\nu_j}{\mu_j},$$

and in general,

$$U_n = Q_n \ , \qquad V_n = B_n = Q_n \sum_{j=1}^{n} \frac{\nu_j}{\mu_j}$$

so that

$$\frac{V_n}{U_n} = \sum_{j=1}^{n} \frac{\nu_j}{\mu_j}.$$

Since $\nu_n < 0$ for $n > 2$, the sequence $\{V_n/U_n\}$ is strictly decreasing and since $\mu_n > 1$, we have $U_n < U_{n+1}$. Clearly,

$$V_{n+1} = \mu_{n+1}V_n + \nu_{n+1}U_n \ \text{ and } \ \frac{V_{n+1} - V_n}{U_{n+1} - U_n} = \frac{V_n}{U_n} + \frac{\nu_{n+1}}{\mu_{n+1} - 1}.$$

Thus,

$$\frac{V_{n+2} - V_{n+1}}{U_{n+2} - U_{n+1}} \geq \frac{V_{n+1} - V_n}{U_{n+1} - U_n} \Leftrightarrow \frac{\nu_{n+2}}{\mu_{n+2} - 1} \geq \frac{\nu_{n+1}}{\mu_{n+1}\left(\mu_{n+1} - 1\right)},$$

which, by the reverse Brun's criterion (Theorem 3.2), implies the result of the reverse Badea's criterion (Theorem 3.5).

The other implication follows from the facts that the above construction is reversible, i.e., given two sequences of positive integers $\{U_n\}$, $\{V_n\}$, two corresponding sequences of rational numbers $\{\mu_n\}$ and $\{\nu_n\}$ can be constructed, namely,

$$\mu_1 = U_1 > 0 \quad , \quad \nu_1 = V_1 > 0$$

and for $n \geq 1$,

$$\mu_{n+1} = \frac{U_{n+1}}{U_n} > 0 \ , \quad \nu_{n+1} = \frac{U_{n+1}}{U_n}\left(\frac{V_{n+1}}{U_{n+1}} - \frac{V_n}{U_n}\right) < 0$$

and the above implications are all reversible. □

3.1.4 Applications

Apart from the two equivalence proofs above, it is natural to ask whether the consideration of series whose terms are quotients of rational numbers over rational numbers (rather that integers over integers) is noteworthy and nontrivial. An affirmative answer to this query is illustrated using some knowledge of continued fractions. Take a positive irrational number γ and write it as an infinite simple continued fraction

$$\gamma = [e_0; e_1, e_2, \ldots], \quad e_0 \geq 0, \ e_i \in \mathbb{N} \ \ (i \geq 1).$$

Let $\{h_n/k_n\}$ be its corresponding sequence of convergents. It is well-known, see e.g. Chapter X of [16], that h_n and k_n satisfy the recurrences

$$h_{-1} = 1, \ h_0 = e_0, \ h_{n+2} = e_{n+2}h_{n+1} + h_n \quad (n \geq -1), \tag{27}$$

$$k_{-1} = 0, \ k_0 = 1, \ \ k_{n+2} = e_{n+2}k_{n+1} + k_n \quad (n \geq -1). \tag{28}$$

We first prove some auxiliary results about numerators and denominators of the convergents.

Lemma 1. *Let $n \in \mathbb{N}$ be fixed. Define the sequence of positive integers $\{C_{n,i}\}_{i\geq 1}$ via*

$$C_{n,1} = 1, \ C_{n,2} = e_{n+2}, \ C_{n,i} = e_{n+i}C_{n,i-1} + C_{n,i-2} \quad (i \geq 3).$$

Then $\quad h_{n+i}k_n - h_nk_{n+i} = (-1)^n C_{n,i} \quad$ *for all $i \in \mathbb{N}$.*

Proof. From Theorems 150 and 151 of [16], we know that

$$h_{n+1}k_n - h_nk_{n+1} = (-1)^n, \quad h_{n+2}k_n - h_nk_{n+2} = (-1)^n e_{n+2},$$

showing that the assertion holds for $i = 1, 2$. Assume it holds up to i. Using (27), (28) and the induction hypothesis, we get

$$\begin{aligned} h_{n+i+1}k_n - h_nk_{n+i+1} &= e_{n+i+1}(h_{n+i}k_n - h_nk_{n+i}) + (h_{n+i-1}k_n - h_nk_{n+i-1}) \\ &= (-1)^n e_{n+i+1}C_{n,i} + (-1)^n C_{n,i-1} \\ &= (-1)^n C_{n,i+1}. \end{aligned}$$

and complete the result. □

Lemma 2. *Let* $n, t \in \mathbb{N}$*. Then* $h_{n+2t} - h_n = \sum_{i=1}^{t} e_{n+2i} h_{n2i-1}$.

Proof. This follows immediately from applying (27) successively. □

Lemma 3. *Let* $l, m, n,$ *be positive integers such that* $l > m > n$.
1) *If* l, m, n *are all even, then* $\frac{h_l - h_m}{k_l - k_m} < \frac{h_m - h_n}{k_m - k_n}$.
2) *If* l, m, n *are all odd, then* $\frac{h_l - h_m}{k_l - k_m} > \frac{h_m - h_n}{k_m - k_n}$.

Proof. Consider the case when l, m, n are all even. Then $l = 2r, m = 2s, n = 2t$, where $r, s, t \in \mathbb{N}$. By Lemma 2, it suffices to show

$$\frac{e_{2r} h_{2r-1} + \cdots + e_{2s+2} h_{2s+1}}{e_{2r} k_{2r-1} + \cdots + e_{2s+2} k_{2s+1}} < \frac{e_{2s} h_{2s-1} + \cdots + e_{2t+2} h_{2t+1}}{e_{2s} k_{2s-1} + \ldots + e_{2t+2} k_{2t+1}}.$$

This is equivalent to

$$0 > \sum_{i=s+1}^{r} \sum_{j=t+1}^{s} e_{2i} e_{2j} (h_{2i-1} k_{2j-1} - h_{2j-1} k_{2i-1}) = \sum_{i=s+1}^{r} \sum_{j=t+1}^{s} e_{2i} e_{2j} (-1)^{2j-1} C_{2j-1, 2(i-j)}.$$

The result now follows from Lemma 1. The other assertion is proved similarly. □

Remark. There is another proof of Lemma 3 using ideas from Farey fractions. For $a, b, c, d \in \mathbb{N}$, observe that if $0 < \frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}. \tag{29}$$

By Theorem 152 of [16], the sequence $\left\{ \frac{h_{2n+1}}{k_{2n+1}} \right\}$ is strictly decreasing, so the recurrences (27) and (28) yield

$$\frac{h_{2n+4} - h_{2n+2}}{k_{2n+4} - k_{2n+2}} = \frac{e_{2n+4} h_{2n+3}}{e_{2n+3} k_{2n+1}} < \frac{e_{2n+2} h_{2n+1}}{e_{2n+2} k_{2n+1}} = \frac{h_{2n+2} - h_{2n}}{k_{2n+2} - k_{2n}}.$$

Using this relation repeatedly for $r, s, t \in \mathbb{N}$ when $r > s > t$ we get that

$$\frac{h_{2r} - h_{2r-2}}{k_{2r} - k_{2r-2}} < \frac{h_{2r-2} - h_{2r-4}}{k_{2r-2} - k_{2r-4}} < \cdots < \frac{h_{2s} - h_{2s-2}}{k_{2s} - k_{2s-2}} < \\ \cdots < \frac{h_{2t+4} - h_{2t+2}}{k_{2t+4} - k_{2t+2}} < \frac{h_{2t+2} - h_{2t}}{k_{2t+2} - k_{2t}}.$$

Replacing the leftmost two quotients and the rightmost two quotients by their respective middle quotient as in (29), while keeping all the remaining quotients as before, we obtain

$$\frac{h_{2r} - h_{2r-4}}{k_{2r} - k_{2r-4}} < \frac{h_{2r-4} - h_{2r-6}}{k_{2r-4} - k_{2r-6}} < \cdots < \frac{h_{2t+6} - h_{2t+4}}{k_{2t+6} - k_{2t+4}} < \frac{h_{2t+4} - h_{2t}}{k_{2t+4} - k_{2t}}.$$

Now repeat the step of replacing the leftmost and rightmost quotients by their middle quotients and using (29) for $\frac{r-s-2}{2}$ times and $\frac{s-t-2}{2}$ times, respectively, to get Lemma 3 part 1). The other part is similarly proved.

Theorem 3.9 Let γ be an irrational number whose simple continued fraction is

$$\gamma = [e_0; e_1, e_2, \ldots], \quad e_0 \geq 0, \ e_n \in \mathbb{N} \ \ (n \geq 1).$$

Let $\{h_n/k_n\}$ be its n^{th} convergent and $f(n)$ a strictly increasing function of $n \in \mathbb{N}$. If either $f(n) \in 2\mathbb{N}$ or $f(n) \in 2\mathbb{N}+1$ for all $n \in \mathbb{N}$, then

$$\gamma = \frac{h_{f(1)}}{k_{f(1)}} + \sum_{n=1}^{\infty} \frac{h_{f(n+1)}\, k_{f(n)} - h_{f(n)}\, k_{f(n+1)}}{k_{f(n)}\, k_{f(n+1)}}.$$

Proof. If $f(n) \in 2\mathbb{N}$ for all $n \in \mathbb{N}$, we have to use the equivalence of the Brun's and Badea's criteria (Theorem 3.6 and Theorem 3.7) take $Y_n = h_{f(n)}$ and $k_n = B_{f(n)}$. Since the sequence $\{X_n\}$ and $\{Y_n/X_n\}$ are strictly increasing and Lemma 3 part 1), the sequence $\{(Y_{n+1} - Y_n)/(X_{n+1} - X_n)\}$ is strictly decreasing. The two sequences $\{Y_n\}$ and $\{X_n\}$ thus satisfy all the conditions of the Brun's criterion II (Theorem 3.1). The two corresponding sequences of rationals $\{\alpha_n\}$ and $\{\beta_n\}$, constructed as in the proof of Theorem 3.6 and 3.7, are $\alpha_1 = k_{f(1)}$, $\beta_1 = h_{f(1)} > 0$, and for $n \geq 1$,

$$\alpha_{n+1} = \frac{k_{f(n+1)}}{k_{f(n)}} \qquad \text{and} \qquad \beta_{n+1} = \frac{k_{f(n+1)}}{k_{f(n)}} \left(\frac{h_{f(n+1)}}{k_{f(n+1)}} - \frac{h_{f(n)}}{k_{f(n)}} \right) > 0.$$

The equivalence of the Brun's criterion II and Badea's criterion yields the desired result.

If $f(n) \in 2\mathbb{N}+1$ for all $n \in \mathbb{N}$, we have to use the equivalence of the reverse Brun's and Badea's criteria (Theorem 3.8) instead, take $V_n = h_{f(n)}$ and $U_n = k_{f(n)}$. Since the sequence $\{U_n\}$ is strictly increasing, while the sequence $\{V_n/U_n\}$ is strictly decreasing and by Lemma 3 part 2), the sequence $\{(V_{n+1} - V_n)/(U_{n+1} - U_n)\}$ is strictly increasing. The two sequences $\{V_n\}$ and $\{U_n\}$ thus satisfy all the conditions of reverse Brun's criterion (Theorem 3.2). The two corresponding sequences of rationals μ_n and ν_n, constructed as in the proof of Theorem 3.8, are $\mu_1 = k_{f(1)}, \nu_1 = h_{f(1)} > 0$, and for $n \geq 1$,

$$\mu_{n+1} = \frac{k_{f(n+1)}}{k_{f(n)}} \qquad \text{and} \qquad \nu_{n+1} = \frac{k_{f(n+1)}}{k_{f(n)}} \left(\frac{h_{f(n+1)}}{k_{f(n+1)}} - \frac{h_{f(n)}}{k_{f(n)}} \right) < 0,$$

where the last inequality follows from Lemma 1. The equivalence of the reverse Brun's criterion and reverse Badea's criteria yields the desired result. □

Theorem 3.9 is a host of a good deal of identities about reciprocal sums. In particular, it can be further applied to give interesting new proofs of identities about the Fibonacci numbers. The following identities are easily verified

$$F_{2n} = F_n L_n \tag{30}$$

$$2F_{n+1} = F_n + L_n \tag{31}$$

$$(-1)^m 2F_n = L_m F_{m+n} - L_{m+n} F_m \tag{32}$$

$$L_{2n} + 2 = \begin{cases} 5F_n^2 & \text{if } n \text{ is odd} \\ L_n^2 & \text{otherwise.} \end{cases} \tag{33}$$

The elements of the corresponding two sequences $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are generally rational. As an example, consider

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots].$$

Its sequences of partial numerators and partial denominators are merely two shifted Fibonacci sequences:

$$\{h_n\}_{n\geq 1} = \{2, 3, 5, 8, 13, 21, \ldots\}, \quad \{k_n\}_{n\geq 1} = \{1, 2, 3, 5, 8, 13, \ldots\}.$$

Given two sequences of positive integers $\{Y_n\} = \{h_{2n}\}$ and $\{X_n\} = \{k_{2n}\}$, the corresponding two sequences of numerators and denominators of the terms in the infinite sum are:

$$\{\alpha_n\}_{n\geq 1} = \left\{2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \ldots\right\}, \quad \{\beta_n\}_{n\geq 1} = \left\{3, \frac{1}{4}, \frac{1}{25}, \frac{1}{169}, \ldots\right\}.$$

The α_n's are generally rational but not integral because

$$\alpha_n = \frac{k_{2n}}{k_{2n-2}} = \frac{F_{2n+1}}{F_{2n-1}} = 2 + \frac{F_{2n-2}}{F_{2n-1}}$$

and the last fraction is positive, strictly less than 1. The same holds for

$$\beta_n = \left(\frac{h_{2n}}{k_{2n}} - \frac{h_{2n-2}}{k_{2n-2}}\right) \frac{k_{2n}}{k_{2n-2}} = \frac{1}{k_{2n-2}^2} = \frac{1}{F_{2n-1}^2}.$$

It is exciting to note incidentally that we have discovered the following result.

Corollary 3.10 Let $f(n)$ be a strictly increasing function of $n \in \mathbb{N}$. If either $f(n) \in 2\mathbb{N}$ or $f(n) \in 2\mathbb{N}+1$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{(-1)^{f(n)} F_{f(n+1)-f(n)}}{F_{f(n)+1} F_{f(n+1)+1}} = \frac{1+\sqrt{5}}{2} - \frac{F_{f(1)+2}}{F_{f(1)+1}}.$$

Proof. Take the golden number

$$\gamma = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, ...].$$

In its continued fraction expansion, the sequences of partial numerators and partial denominators of the convergents h_n/k_n, are merely two shifted Fibonacci sequences: $h_n = F_{n+2}, k_n = F_{n+1}$. From the well-known identity, see p.87 of [20],

$$F_{n+1}F_{k-1} - F_n F_k = (-1)^k F_{n-k+1} \quad (1 \le k \le n),$$

we have

$$\begin{aligned} h_{f(n+1)}\, k_{f(n)} - h_{f(n)}\, k_{f(n+1)} &= F_{f(n+1)+2}\, F_{f(n)+1} - F_{f(n)+2}\, F_{f(n+1)+1} \\ &= (-1)^{f(n)} F_{f(n+1)-f(n)}. \end{aligned}$$

Substituting into Theorem 3.9 gives

$$\frac{1+\sqrt{5}}{2} = \frac{F_{f(1)+2}}{F_{f(1)+1}} + \sum_{n=1}^{\infty} \frac{(-1)^{f(n)} F_{f(n+1)-f(n)}}{F_{f(n)+1} F_{f(n+1)+1}},$$

which is the result. □

Remarks.

1. In Corollary 3.10, putting $f(n) = 2n-2, 2n-1$, we respectively get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}\, F_{2n+1}} = \frac{\sqrt{5}-1}{2}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n}\, F_{2n+2}} = \frac{3-\sqrt{5}}{2}.$$

Summing and subtracting these two equations yield

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} = 2 - \sqrt{5};$$

the former identity is equation (4) in [5], see also Identity 35 on page 442 of [20], while the latter identity is a problem posed in [8].

2. In Corollary 3.10, putting $f(n) = 4n - j$ for $j \in \{4, 3, 2, 1\}$, we respectively get

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-3}F_{4n+1}} = \frac{\sqrt{5}}{6} - \frac{1}{6} \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}F_{4n+2}} = \frac{1}{2} - \frac{\sqrt{5}}{6} \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-1}F_{4n+3}} = \frac{\sqrt{5}}{6} - \frac{1}{3} \tag{36}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}F_{4n+4}} = \frac{7}{18} - \frac{\sqrt{5}}{6}. \tag{37}$$

Adding (34) and (36) yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+3}} = \frac{2\sqrt{5} - 3}{6}. \tag{38}$$

Adding (35) and (37) yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}F_{2n+4}} = \frac{8 - 3\sqrt{5}}{9}. \tag{39}$$

Subtracting (34) and (36) yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n-1}F_{2n+3}} = \frac{1}{6}.$$

Subtracting (35) and (37) yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{2n}F_{2n+4}} = \frac{1}{9}.$$

Adding (38) and (39) yields

$$\sum_{n=1}^{\infty} \frac{1}{F_nF_{n+4}} = \frac{7}{18},$$

which is equation (5) in [5], see also Identity 36 on page 442 of [20]. Subtracting (39) and (38) yields

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_nF_{n+4}} = \frac{25}{18} - \frac{2\sqrt{5}}{3}.$$

3. In Corollary 3.10, putting $f(n) = 2tn + j$ for $t \in \mathbb{N}$ and $j \geq -2t$, we have

$$\sum_{n=1}^{\infty} \frac{1}{F_{2tn+j+1}F_{2t(n+1)+j+1}} = \frac{(-1)^j}{F_{2t}} \left(\frac{1 + \sqrt{5}}{2} - \frac{F_{2t+j+2}}{F_{2t+j+1}} \right).$$

Taking $j = 2m-1 \in \{-2t, \ldots, 2t-1\}$ and adding zero term, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{F_{2tn+2m}F_{2t(n+1)+2m}} &= \frac{-1}{F_{2t}}\left(\frac{1+\sqrt{5}}{2} - \frac{F_{2t+2m+1}}{F_{2t+2m}}\right) + \frac{1}{F_{2m}F_{2t+2m}} \\
&= \frac{1}{F_{2t}}\left(\frac{F_{2t+2m+1}}{F_{2t+2m}} + \frac{F_{2t}}{F_{2m}F_{2t+2m}} - \frac{1+\sqrt{5}}{2}\right) \\
&= \frac{1}{F_{2t}}\left(\frac{L_{2t+2m}F_{2m} + 2F_{2t}}{2F_{2m}F_{2t+2m}} - \frac{\sqrt{5}}{2}\right) \qquad \text{(using (31))} \\
&= \frac{1}{F_{2t}}\left(\frac{L_{2m}}{2F_{2m}} - \frac{\sqrt{5}}{2}\right) \qquad \text{(using (32))},
\end{aligned}$$

which is the last equation on page 263 of [30].

4. In Corollary 3.10, putting $f(n) = at^n + i$ for $a, t \in \mathbb{N}$ and $i \geq -at$, we get

$$\sum_{n=1}^{\infty} \frac{F_{(t-1)at^n}}{F_{at^n+i+1}F_{at^{n+1}+i+1}} = (-1)^{at+i}\left(\frac{1+\sqrt{5}}{2} - \frac{F_{at+i+2}}{F_{at+i+1}}\right).$$

Taking $t = 2, i = -1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{a2^{n+1}}} = \frac{F_{2a+1}}{F_{2a}} - \frac{1+\sqrt{5}}{2}. \tag{40}$$

We can rewrite this as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{F_{a2^n}} &= \frac{F_{2a+1}}{F_{2a}} + \frac{1}{F_a} + \frac{1}{F_{2a}} - \frac{1+\sqrt{5}}{2} \\
&= \frac{2F_{2a+1} + 2L_a + 2 - F_{2a} - F_{2a}\sqrt{5}}{2F_{2a}} \qquad \text{(using(30))} \\
&= \frac{L_{2a} + 2 + 2L_a - F_{2a}\sqrt{5}}{2F_{2a}} \qquad \text{(using(31))} \\
&= \begin{cases} \dfrac{5F_a^2 + 2L_a - F_{2a}\sqrt{5}}{2F_{2a}} & \text{if } a \text{ is odd} \\ \dfrac{L_a + 2 - F_a\sqrt{5}}{2F_a} & \text{otherwise} \end{cases} \qquad \text{(using(33))},
\end{aligned}$$

which is a result of Hoggatt and Bicknell [18]. In equation (40), taking $a = 1$ we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^{n+1}}} = \frac{3-\sqrt{5}}{2},$$

which was proposed as a problem by Millin [23] and solved by Good in [14], see also many proofs in [17].

We determine the nature of certain numbers in the exceptional case of the result of Becker-Töpfer, using Badea's criterion.

Proposition 3.11 Let $c \geq 1, d$ be integers and $\{R_k\}$ defined by (14) and (15) with $|A_2| = 1$, $R_d = 0$. Let $\{m_k\}_{k\geq 1}$ be a sequence of positive integers. Assume that m_k is constant for all large k, $g_1 \in \mathbb{R}^+$ and $\rho_1^d \in \mathbb{R}$. If there is an $\epsilon > 0$ such that $g_1 - g_1^2\rho_1^d \geq \epsilon$, then $\sum_{k=1}^{\infty} m_k/R_{c\cdot 2^k+d}$ represents a non-rational algebraic number.

Proof. Using Badea's criterion, it suffices to show that

$$R_{c\cdot 2^{k+1}+d} > R_{c\cdot 2^k+d}^2 - R_{c\cdot 2^k+d} + 1, \tag{41}$$

for all large k. The equation (41) is equivalent to

$$g_1\rho_1^{c\cdot 2^{k+1}+d} + g_2\rho_2^{c\cdot 2^{k+1}+d} > \left(g_1\rho_1^{c\cdot 2^k+d} + g_2\rho_2^{c\cdot 2^k+d}\right)^2 - \left(g_1\rho_1^{c\cdot 2^k+d} + g_2\rho_2^{c\cdot 2^k+d}\right) + 1,$$

i.e.,

$$g_1 - g_1^2\rho_1^d > \frac{2g_1g_2(\rho_1\rho_2)^{c\cdot 2^k+d} + g_2\rho_2^{c\cdot 2^{k+1}+d}(g_2\rho_2^d - 1) - (g_1\rho_1^{c\cdot 2^k+d} + g_2\rho_2^{c\cdot 2^k+d}) + 1}{\rho_1^{c\cdot 2^{k+1}+d}}.$$

Since the right-hand expression $\to 0$ as $k \to \infty$, this requirement is fulfilled under the stated hypothesis. $\square$

Example 1. Let $b \in \mathbb{N}$ and $\{T_k\}_{k\geq 0}$ be a sequence of integers satisfying the bilinear recurrence relation

$$T_0 = 0, \quad T_1 = 1, \qquad T_{k+2} = bT_{k+1} + T_k.$$

Then $T_2 = b$, $T_k \geq 2$ $(k \geq 3)$ and

$$T_k = \frac{1}{\sqrt{b^2+4}}\left(\frac{b+\sqrt{b^2+4}}{2}\right)^k + \frac{-1}{\sqrt{b^2+4}}\left(\frac{b-\sqrt{b^2+4}}{2}\right)^k \qquad (k \in \mathbb{N}).$$

Taking $A_1 = b$, $A_2 = 1$, $d = 0$ and assuming a_k is constant for all large k in Proposition 3.11, and noting $g_1 = 1/\sqrt{b^2+4}$, $\rho_1 = (b+\sqrt{b^2+4})/2$, $1 - g_1\rho_1 > 0$, we deduce $\sum_{k\geq 1} a_k/R_{c\cdot 2^k}$ are non-rational algebraic numbers for any $c \in \mathbb{N}$.

A particular case involving the sequence of Fibonacci numbers is the well-known result that $\sum_{k\geq 1} 1/F_{c\cdot 2^k}$ is a non-rational algebraic number for any $c \in \mathbb{N}$, see in equation (40) or e.g. [18].

3.1.5 Connections with Other Irrational Results

Immediate from Corollary 3.4 are the following irrationality criteria, the first of which is an extension of Theorem 2.4.

Theorem 3.12 Assume that $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ converges, each $\alpha_n > 1$ and $P_n,\ A_n$ are both integral for all large n. If

$$\alpha_{n+1} > \frac{\beta_{n+1}}{\beta_n}\alpha_n^2 - \frac{\beta_{n+1}}{\beta_n}\alpha_n + 1$$

for all sufficiently large n, then the sum $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is irrational.

Proof. Note first that

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n} = \lim_{n\to\infty} (\frac{\beta_1}{\alpha_1} + ... + \frac{\beta_n}{\alpha_n}) = \lim_{n\to\infty} \frac{A_n}{P_n}.$$

Since $\alpha_n > 1$ and all terms are positive, we see that $P_n < P_{n+1}$, and $\{A_n/P_n\}$ is an increasing sequence. In the Brun's criterion taking $y_n = A_n, x_n = P_n$, we need only check that

$$\frac{A_{n+2} - A_{n+1}}{P_{n+2} - P_{n+1}} < \frac{A_{n+1} - A_n}{P_{n+1} - P_n} \tag{42}$$

for all large n. Now the inequality (42) is equivalent to

$$\left(\frac{A_{n+2}}{P_{n+2}} - \frac{A_{n+1}}{P_{n+1}}\right) P_{n+1}P_{n+2} + \left(\frac{A_{n+1}}{P_{n+1}} - \frac{A_n}{P_n}\right) P_nP_{n+1} < \left(\frac{A_{n+2}}{P_{n+2}} - \frac{A_n}{P_n}\right) P_nP_{n+2},$$

and so

$$\frac{\beta_{n+2}}{\beta_{n+1}}\alpha_{n+1}^2 - \frac{\beta_{n+2}}{\beta_{n+1}}\alpha_{n+1} + 1 < \alpha_{n+2}.$$

This conclusion follows immediately from Brun's criterion. □

Theorem 3.13 Assume that P_n and A_n are both integral for all large n. If $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ converges and there exists $\lambda > 0$ such that $s_k > 1 + \lambda$ and

$$\frac{s_{k+1} - 1}{r_{k+1}} = \frac{s_k(s_k - 1)}{r_k} \tag{43}$$

for all sufficiently large k, then $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational.

Proof. Since $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ converges, for every $\epsilon > 0$ there is a $N_\epsilon \in \mathbb{N}$ such that for sufficiently large k, we have $n(k+1) > n(k) > N_\epsilon$ and Cauchy's criterion yields

$$\epsilon > \sum_{n=1}^{n(k+1)} \frac{\beta_n}{\alpha_n} - \sum_{n=1}^{n(k)} \frac{\beta_n}{\alpha_n} = \sum_{n=n(k)+1}^{n(k+1)} \frac{\beta_n}{\alpha_n} = \frac{r_k}{s_k} \geq 0, \tag{44}$$

i.e.,

$$\lim_{k\to\infty} \frac{r_k}{s_k} = 0. \tag{45}$$

Suppose that (43) holds for $j \geq k_0$ where k_0 is sufficiently large. Then

$$\frac{r_j}{s_j} = \frac{r_j}{s_j - 1} - \frac{r_{j+1}}{s_{j+1} - 1}.$$

Summing from k_0 to k, we get

$$\sum_{j=k_0}^{k} \frac{r_j}{s_j} = \frac{r_{k_0}}{s_{k_0} - 1} - \frac{r_{k+1}}{s_{k+1} - 1}.$$

By the condition $s_k > 1 + \lambda$ and (45), we have as $k \to \infty$

$$0 < \frac{r_{k+1}}{s_{k+1} - 1} = \frac{r_{k+1}}{s_{k+1}} \cdot \frac{1}{1 - \frac{1}{s_{k+1}}} < \frac{r_{k+1}}{s_{k+1}} \cdot \frac{\lambda + 1}{\lambda} \to 0.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n} = \sum_{j=0}^{k_0 - 1} \frac{r_j}{s_j} + \frac{r_{k_0}}{s_{k_0} - 1}$$

is rational. □

Remark. The extra requirement, $s_k > 1 + \lambda$, can be dropped if $n(k) = k$ and $\alpha_n, \beta_n \in \mathbb{N}$ for then $s_k = \alpha_{k+1} \geq 2$ by the convergence of the series.

For alternating series, using the technique of proof of Corollary 3.5 in [2], we have the next two results.

Theorem 3.14 Let $\{\beta_n/\alpha_n\}_{n\geq 1}$ be a decreasing sequence of positive rational numbers whose numerators and denominators are also positive rational numbers. For an increasing sequence of positive integers $\mathcal{N} = \{n(k)\}_{k\geq 1}$, define

$$\begin{aligned} s_2(k) &:= s_2(k, \mathcal{N}) = \alpha_{2n(k)+1} \cdots \alpha_{2n(k+1)}, \\ G_o(k) &:= G_o(k, \mathcal{N}) = \frac{\beta_{2n(k)+1}}{\alpha_{2n(k)+1}} + \frac{\beta_{2n(k)+3}}{\alpha_{2n(k)+3}} + \cdots + \frac{\beta_{2n(k+1)-1}}{\alpha_{2n(k+1)-1}}, \\ G_e(k) &:= G_e(k, \mathcal{N}) = \frac{\beta_{2n(k)+2}}{\alpha_{2n(k)+2}} + \frac{\beta_{2n(k)+4}}{\alpha_{2n(k)+4}} + \cdots + \frac{\beta_{2n(k+1)}}{\alpha_{2n(k+1)}}. \end{aligned}$$

Assume that $s_2(k) > 1$, $P_{2k} := \alpha_1\alpha_2\cdots\alpha_{2k} \in \mathbb{N}$, $P_{2k}\sum_{j=1}^{2k}(-1)^{j+1}\beta_j/\alpha_j \in \mathbb{N}$ and

$$\frac{s_2(k+1)-1}{s_2(k+1)\,(s_2(k)-1)} \geq \frac{G_o(k+1)-G_e(k+1)}{G_o(k)-G_e(k)}, \tag{46}$$

for all sufficiently large k. If $\sum_{n=1}^{\infty}(-1)^{n+1}\beta_n/\alpha_n$ is rational, then in (46) we have equality for all sufficiently large n.

Proof. Let

$$\frac{D_n}{C_n} := \frac{\beta_{2n-1}}{\alpha_{2n-1}} - \frac{\beta_{2n}}{\alpha_{2n}} \quad \text{and} \quad C_n := \alpha_{2n}\alpha_{2n-1}$$

so that $\{D_n/C_n\}$ is a sequence of positive rationals whose numerators and denominators are also positive rationals. Observe that

$$\begin{aligned} G_o(k)-G_e(k) &= \left(\frac{\beta_{2n(k)+1}}{\alpha_{2n(k)+1}} - \frac{\beta_{2n(k)+2}}{\alpha_{2n(k)+2}}\right) + \cdots + \left(\frac{\beta_{2n(k+1)-1}}{\alpha_{2n(k+1)-1}} - \frac{\beta_{2n(k+1)}}{\alpha_{2n(k+1)}}\right) \\ &= \frac{D_{n(k)+1}}{C_{n(k)+1}} + \cdots + \frac{D_{n(k+1)}}{C_{n(k+1)}}, \\ s_2(k) &= \alpha_{2n(k)+1}\cdots\alpha_{2n(k+1)} = C_{n(k)+1}\cdots C_{n(k+1)}. \end{aligned}$$

The condition (46) becomes

$$\frac{C_{n(k+1)+1}\cdots C_{n(k+2)}-1}{C_{n(k+1)+1}\cdots C_{n(k+2)}\left(C_{n(k)+1}\cdots C_{n(k+1)}-1\right)} \geq \frac{\frac{D_{n(k+1)+1}}{C_{n(k+1)+1}} + \cdots + \frac{D_{n(k+2)}}{C_{n(k+2)}}}{\frac{D_{n(k)+1}}{C_{n(k)+1}} + \cdots + \frac{D_{n(k+1)}}{C_{n(k+1)}}},$$

the result follows directly from Theorem 3.3 by considering the series $\sum D_n/C_n$. □

Theorem 3.15 Let $\{|\nu_n/\mu_n|\}$ be a strictly decreasing sequence and B_{2n}, Q_{2n} be both integral for all large n. Assume that $\sum_{n=1}^{\infty}(-1)^{n+1}\nu_n/\mu_n$ is a positive rational number. If $\mu_{2n-1}\mu_{2n} > 1$ and

$$\frac{\mu_{2n+1}\mu_{2n+2}-1}{\nu_{2n+1}\mu_{2n+2}-\nu_{2n+2}\mu_{2n+1}} \leq \frac{\mu_{2n-1}\mu_{2n}(\mu_{2n-1}\mu_{2n}-1)}{\nu_{2n-1}\mu_{2n}-\nu_{2n}\mu_{2n-1}} \tag{47}$$

for all large n, then in (47) we have equality from certain n onwards.

Proof. Since

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\nu_n}{\mu_n} = \sum_{k=1}^{\infty}\left(\frac{\nu_{2k-1}}{\mu_{2k-1}} - \frac{\nu_{2k}}{\mu_{2k}}\right) = \sum_{k=1}^{\infty}\frac{\nu_{2k-1}\mu_{2k}-\nu_{2k}\mu_{2k-1}}{\mu_{2k-1}\mu_{2k}}$$

and the sequence $\{|\nu_n/\mu_n|\}_{n\geq 2}$ is strictly decreasing, the last series has negative terms except the first term. The result follows from the reverse Badea's criterion. □

Related to Theorem 4.3 of Tijdeman and Yuan, [35], we have:

Theorem 3.16 Let P_n and A_n be both integral for all large n. Assume that the sequence $\{\frac{s_k-1}{P_{n(k)}r_k}\}$ is non-decreasing and $s_k > 1$ for all sufficiently large k. If $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational, then $\frac{s_k-1}{P_{n(k)}r_k}$ is constant from some k onwards.

On the other hand, if $\frac{s_k-1}{P_{n(k)}r_k}$ is constant and there exists $\lambda > 0$ such that $s_k > 1 + \lambda$ from some k onwards, then $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational.

Proof. For k sufficiently large, since the sequence $\{\frac{s_k-1}{P_{n(k)}r_k}\}$ is non-decreasing, from

$$\frac{s_{k+1}-1}{r_{k+1}} \geq \frac{s_k(s_k-1)}{r_k} \quad \Leftrightarrow \quad \frac{s_{k+1}-1}{P_{n(k+1)}r_{k+1}} \geq \frac{s_k-1}{P_{n(k)}r_k},$$

we deduce from Theorem 3.3 that if $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational, then $\frac{s_k-1}{P_{n(k)}r_k}$ is constant from some k onwards. The converse is immediate from Theorem 3.13. □

Related to Theorem 4.1 and Corollary 4.1 (i) of Tijdeman-Yuan, [35], we have:

Theorem 3.17 Let P_n and A_n be both integral for all large n. Assume that there is a constant $c > 0$ such that $r_k > c, s_k > 1$ for all large k, and

$$\limsup_{k\to\infty} P_{n(k)} \left(\frac{r_{k+1}s_k}{s_{k+1}} - \frac{r_k}{s_k} \right) \leq 0.$$

If $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational, then

$$\frac{s_{k+1}-1}{r_{k+1}} = \frac{s_k(s_k-1)}{r_k} \tag{48}$$

for all sufficiently large k.

Proof. Suppose $S = \sum_{n=1}^{\infty} \beta_n/\alpha_n = p/q$ with $p, q \in \mathbb{N}$. Assume that there exists K_0 such that $s_k > 1$ and $P_{n(k)}, A_{n(k)} \in \mathbb{Z}$ for all $k \geq K_0$. Put $t_k = \sum_{j=k+1}^{\infty} r_j/s_j$, we get $qP_{n(k+1)}t_k \in \mathbb{Z}$. By the limsup condition and the series convergence, given $\epsilon > 0$, there is K_ϵ such that

$$\frac{r_{k+1}s_k}{s_{k+1}} - \frac{r_k}{s_k} \leq \frac{\epsilon}{P_{n(k)}} \quad \text{and} \quad \frac{r_k}{s_k} \leq \epsilon,$$

which implies $s_k < \epsilon s_{k+1}$, for $k \geq K_\epsilon$. For $k \geq K = max\{K_0, K_\epsilon\}$, we have, assuming that $\epsilon \leq 1/2$,

$$\begin{aligned} s_k t_k - t_{k-1} &= \left(\frac{r_{k+1}s_k}{s_{k+1}} - \frac{r_k}{s_k}\right) + \left(\frac{r_{k+2}s_k}{s_{k+2}} - \frac{r_{k+1}}{s_{k+1}}\right) + \left(\frac{r_{k+3}s_k}{s_{k+3}} - \frac{r_{k+2}}{s_{k+2}}\right) + \cdots \\ &\le \left(\frac{r_{k+1}s_k}{s_{k+1}} - \frac{r_k}{s_k}\right) + \frac{s_k}{s_{k+1}}\left(\frac{r_{k+2}s_{k+1}}{s_{k+2}} - \frac{r_{k+1}}{s_{k+1}}\right) \\ &\quad + \frac{s_k}{s_{k+1}}\frac{s_{k+1}}{s_{k+2}}\left(\frac{r_{k+3}s_{k+2}}{s_{k+3}} - \frac{r_{k+2}}{s_{k+2}}\right) + \cdots \\ &< \frac{\epsilon}{P_{n(k)}} + \frac{\epsilon^2}{P_{n(k+1)}} + \frac{\epsilon^3}{P_{n(k+2)}} + \cdots \le \frac{2\epsilon}{P_{n(k)}}. \end{aligned}$$

Choose $\epsilon = 1/2q$. We get that the integer $qP_{n(k+1)}t_k - qP_{n(k)}t_{k-1} < 1$, which implies that the sequence $qP_{n(k+1)}t_k$ is non-increasing of positive integers for $k \ge K$. Then the sequence is ultimately constant, so $s_k t_k = t_{k-1}$ for $k \ge K$. Observe that $t_k = \frac{r_k}{s_k(s_k-1)}$, hence $\frac{r_{k+1}}{s_{k+1}-1} = s_{k+1}t_{k+1} = t_k = \frac{r_k}{s_k(s_k-1)}$ for $k \ge K$. □

Theorem 3.18 Let P_n, A_n be both integral for all large n and the sequence $\{s_k\}$ a non-decreasing. Assume that there is a constant $c > 0$ such that $r_k > c$ and $s_k > 1$ for all large k and

$$\limsup_{k\to\infty} P_{n(k)}\left(\frac{r_{k+1}s_k - r_k}{s_k}\right) \le 0.$$

Then $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ is rational if and only if $\frac{s_k-1}{P_{n(k)}r_k}$ is constant from some k onwards.

Proof. For all large k, since $s_{k+1} \ge s_k > 1$, we have $\frac{r_{k+1}s_k}{s_{k+1}} - \frac{r_k}{s_k} \le \frac{r_{k+1}s_k - r_k}{s_k}$, and the necessity part follows Theorem 3.17. The sufficiency part follows from the equivalence in Theorem 3.16 and the proof of Theorem 3.13 noting that the presence of λ can be eliminated using the non-decreasing of $\{s_k\}$. □

3.2 Equivalence of Brun's Criterion and Infinite Products

3.2.1 Brun ≡ Infinite Products

We begin with a modified form of Theorem 2.10:

Theorem 3.19 Let $\{\alpha_n\}_{n\ge1}$ and $\{\mu_n\}_{n\ge1}$ be two sequences of positive rational numbers and set

$$P_n := \alpha_1\alpha_2\cdots\alpha_n, \quad Q_n := \mu_1\mu_2\cdots\mu_n.$$

Assume that P_n, Q_n are both integral for all large n and $\lim_{n\to\infty} P_n/Q_n$ is rational.

1) If $\alpha_{n+1} > \mu_{n+1} > 1$ for all $n \geq 1$ and

$$\frac{\alpha_n(\alpha_{n+1}-1)}{\mu_n(\mu_{n+1}-1)} \leq \frac{\alpha_n - 1}{\mu_n - 1} \tag{49}$$

for all large n, then in (49) we have equality from some n onwards.

2) If $\mu_{n+1} > \alpha_{n+1} > 1$ for all $n \geq 1$ and

$$\frac{\alpha_n(\alpha_{n+1}-1)}{\mu_n(\mu_{n+1}-1)} \geq \frac{\alpha_n - 1}{\mu_n - 1} \tag{50}$$

for all large n, then in (50) we have equality from some n onwards.

Proof. We only prove part 1), for that of part 2) is analogous. Since $\alpha_{n+1} > \mu_{n+1} > 1$ for all $n \geq 1$, we have

$$Q_n < Q_{n+1}, \qquad \frac{P_n}{Q_n} < \frac{P_{n+1}}{Q_{n+1}} \qquad (n \geq 1).$$

Let $N \in \mathbb{N}$ be such that $P_n, Q_n \in \mathbb{N}$ and the inequality (49) holds for all $n \geq N$. Multiplying (49) by $\alpha_1\alpha_2\cdots\alpha_{n-1}/\mu_1\mu_2\cdots\mu_{n-1}$, we get

$$\frac{P_{n+1}-P_n}{Q_{n+1}-Q_n} \leq \frac{P_n - P_{n-1}}{Q_n - Q_{n-1}} \quad (n \geq N). \tag{51}$$

The conclusion follows from Theorem 3.1 by taking $X_n = Q_n,\ Y_n = P_n$ and shifting indices beyond N. □

We are now ready to establish the promised equivalence.

Theoem 3.20 Theorem 3.1 is equivalent to Theorem 3.19 part 1), and Theorem 3.2 is equivalent to Theorem 3.19 part 2).

Proof. We only give a proof for the first part as that of the second is similar. The proof of Theorem 3.19 shows that Theorem 3.1 implies Theorem 3.19 part 1). We proceed to show the other implication. Given two sequences $\{X_n\}, \{Y_n\}$ satisfying all the properties as stated in Theorem 3.1, we construct two sequences $\{\mu_n\}$ and $\{\alpha_n\}$ of positive rational numbers as follows: $\mu_1 = X_1,\quad \alpha_1 = Y_1$ and

$$\mu_{n+1} = \frac{X_{n+1}}{X_n}, \quad \alpha_{n+1} = \frac{Y_{n+1}}{Y_n} \quad (n \geq 1).$$

Then $P_n = Y_n$ and $Q_n = X_n$, we see that both are integral for each $n \in \mathbb{N}$. Since

$$X_n < X_{n+1}, \ \frac{Y_n}{X_n} < \frac{Y_{n+1}}{X_{n+1}},$$

we get $\alpha_{n+1} > \mu_{n+1} > 1 \quad (n \geq 1)$. To deduce Theorem 3.1 from Theorem 3.19 part 1), it suffices to note the following equivalences

$$\begin{aligned} \frac{Y_{n+2}-Y_{n+1}}{X_{n+2}-X_{n+1}} \leq \frac{Y_{n+1}-Y_n}{X_{n+1}-X_n} \quad &\Longleftrightarrow \quad \frac{P_{n+2}-P_{n+1}}{Q_{n+2}-Q_{n+1}} \leq \frac{P_{n+1}-P_n}{Q_{n+1}-Q_n} \\ &\Longleftrightarrow \quad \frac{\alpha_{n+1}(\alpha_{n+2}-1)}{\mu_{n+1}(\mu_{n+2}-1)} \leq \frac{\alpha_{n+1}-1}{\mu_{n+1}-1} \end{aligned}$$

which complete. □

3.2.2 Applications

Corollary 3.21 Let $\{b_n\}_{n\geq 1}$ and $\{a_n\}_{n\geq 1}$ be two sequences of positive integers.

1) If $a_1 > 1$, $\prod_{n=1}^{\infty}(1+b_n/a_n)$ is rational and

$$\frac{a_{n+1}-1}{b_{n+1}} \geq (a_n+b_n)\left(\frac{a_n-1}{b_n}\right) \quad (n \geq 1), \tag{52}$$

then in (52) we have equality from some n onwards.

2) If $a_n > b_n + 1 \quad (n \geq 1)$, $\prod_{n=1}^{\infty}(1-b_n/a_n)$ is rational and

$$\frac{a_{n+1}-1}{b_{n+1}} \geq (a_n-b_n)\left(\frac{a_n-1}{b_n}\right) \quad (n \geq 1), \tag{53}$$

then in (53) we have equality from some n onwards.

3) If $a_1 > 1$ and

$$\frac{a_{n+1}-1}{b_{n+1}} > (a_n+b_n)\left(\frac{a_n-1}{b_n}\right) \tag{54}$$

for infinitely many n, then $\prod_{n=1}^{\infty}(1+b_n/a_n)$ converges to an irrational number.

4) If $a_n > b_n + 1 \quad (n \geq 1)$ and

$$\frac{a_{n+1}-1}{b_{n+1}} > (a_n-b_n)\left(\frac{a_n-1}{b_n}\right) \tag{55}$$

for infinitely many n, then $\prod_{n=1}^{\infty}(1-b_n/a_n)$ converges to an irrational number.

Proof. To prove part 1), first note that the inequalities $a_1 > 1$ and (52) show that all $a_n > 1$. Let $M_n = (a_n-1)/b_n > 0$. Using the condition (52) recursively yields $M_{n+1} \geq 3^n M_1$ which shows that

$$\frac{b_{n+1}}{a_{n+1}} < \frac{1}{3^n M_1}.$$

It implies by induction that the infinite product is convergent. The result thus follows from Theorem 3.19 part 1) by taking $\alpha_n = a_n + b_n,\ \mu_n = a_n$.

To prove part 2), using the inequalities $a_n > b_n + 1$ and (53), we deduce as before that

$$\frac{b_{n+1}}{a_{n+1}} < \frac{1}{2^n M_1}$$

implying the product convergence and the result again follows from Theorem 3.19 part 2) by taking $\alpha_n = a_n - b_n,\ \mu_n = a_n$.

For parts 3) and 4) are immediate consequences of Parts 1) and 2), respectively. □

Note that the second half of Theorem 2.11 follows from Corollary 3.21 part 1) by setting $b_n = 1, a_n = q_n$. We see that the inequality (54) is equivalent to the inequality (12) of Badea [3].

The next application may be regarded as a partial answer to the final remarks on page 44 of [19] which ask for simple cases where the digits satisfy explicit formulae of products representing rational numbers in the open interval $(0, 1)$. This is in fact an analogue of Corollary 3.21 providing an irrationality criterion for products with values in the open interval $(0, 1)$.

Corollary 3.22 Let $\{a_n\}_{n\geq 1}$ be a sequence of the positive integers with $a_1 > 1$. If

$$a_{n+1} - 1 \geq (a_n - 1)^2 \quad (n \geq 1), \tag{56}$$

then the infinite product $\prod_{n=1}^{\infty}(1 - 1/a_n)$ represents a rational number if and only if we have equality from some n onwards in (56).

Proof. The product convergence and the necessity part follow immediately from Corollary 3.21 part 2) by taking $b_n = 1$ and shifting index n if necessary. As to the sufficiency, we observe that (56) is equivalent to the condition (8) in Theorem 2.11. Since

$$1 - \frac{1}{a_n} = \frac{1}{1 + \frac{1}{a_n - 1}},$$

$\prod_{n=1}^{\infty}(1 + (a_n - 1)^{-1})$ converges to a rational number and so does $\prod_{n=1}^{\infty}(1 - 1/a_n)$. □

There are also other irrationality results for infinite products of the shape considered in Corollary 3.22, such as the one in [22] which states that $\prod_{n=1}^{\infty}(1-a^{-n})$ is irrational for $a \in \mathbb{Z}$ with $|a| > 1$.

Finally, in connection with Tachiya's result mentioned in the introduction, we derive a sufficient condition for irrationality, which will be used to determine the algebraicity of an exceptional case in Theorem 3 of [34].

Proposition 3.23 Let $c \geq 1$ and d be integers and $\{R_k\}$ a binary recurrence sequence of positive integers defined by (14) and (15). Let $\{m_k\}_{k\geq 1}$ be a sequence of positive integers with $\log m_k = o(2^k)$. Assume there is an $\epsilon > 0$ such that

$$\epsilon \leq g_1 - \frac{m_{k+1}}{m_k} g_1^2 \rho_1^d \in \mathbb{R} \qquad (k \geq 1) \tag{57}$$

for infinite many k.

I. If $R_{2c+d} > 1$, then $\prod_{k=1}^{\infty}\left(1 + m_k/R_{c\cdot 2^k+d}\right)$ converges to irrational.

II. If $R_{c\cdot 2^k+d} > m_k + 1$, then $\prod_{k=1}^{\infty}\left(1 - m_k/R_{c\cdot 2^k+d}\right)$ converges to irrational.

Proof. I. Put $b_k = m_k,\ a_k = R_{c\cdot 2^k+d}$ in Corollary 3.21 part 3). Then $a_1 = R_{2c+d} > 1$ and the assertion follows at once if we can show that

$$R_{c\cdot 2^{k+1}+d} > \frac{m_{k+1}}{m_k}\left(R_{c\cdot 2^k+d} - 1\right)\left(R_{c\cdot 2^k+d} + m_k\right) + 1 \tag{58}$$

for infinite many k. Since $R_k = g_1\rho_1^k + g_2\rho_2^k$, we get that the equation (58), after rearanging, is equivalent to

$$g_1 - \frac{m_{k+1}}{m_k} g_1^2\rho_1^d > \frac{1 - g_2\rho_2^{c\cdot 2^{k+1}+d}}{\rho_1^{c\cdot 2^{k+1}+d}} + \frac{m_{k+1}h(k)}{m_k\rho_1^{c\cdot 2^{k+1}+d}}$$

for infinite many k, where

$$h(k) = g_1\rho_1^{c\cdot 2^k+d}\left(g_2\rho_2^{c\cdot 2^k+d} - 1\right) + \left(g_2\rho_2^{c\cdot 2^k+d} + m_k\right)\left(g_1\rho_1^{c\cdot 2^k+d} + g_2\rho_2^{c\cdot 2^k+d} - 1\right).$$

Since $|\rho_2/\rho_1| < 1,\ |\rho_1|^2 > |\rho_1\rho_2| = |A_2| \geq 1$ and $\log m_k = o(2^k)$, the right-hand expression $\to 0$ as $k \to \infty$. By the hypothesis (57), this last condition holds.

The proof of II is similar. $\square$

Immediate from Proposition 3.23 is the next corollary which show that products of two particular forms in the exceptional case of Tachiya's result are non-rational algebraic numbers.

Corollary 3.24 Let $c \geq 1$ and d be integers. Let $\{R_k\}$ be a binary recurrence sequence of positive integers defined by (14) and (15) with $|A_2| = 1, g_1 \in \mathbb{R}^+$ and $\rho_1^d \in \mathbb{R}$. Let $\{m_k\}_{k\geq 1}$ be a sequence of positive integers such that $m_k = R_d$ for all large k.
I. Assume that $R_{2c+d} > 1$. If

$$1 - g_1\rho_1^d > 0, \tag{59}$$

then the infinite product $\prod_{k=1}^{\infty}(1 + m_k/R_{c\cdot 2^k+d})$ is a non-rational algebraic number.
II. Assume that $R_{c\cdot 2^k+d} > m_k + 1$ and $m_k \neq R_{c\cdot 2^k+d}$ $(k \geq 1)$. If (59) holds, then the infinite product $\prod_{k=1}^{\infty}(1 - m_k/R_{c\cdot 2^k+d})$ is a non-rational algebraic number.

A typical example of Corollary 3.24 is the sequence of generalized Fibonacci numbers.

Example 2. Taking $\{T_k\}$ as in Example 1, $c \in \mathbb{N}$, $A_1 = b$, $A_2 = 1$, $d = 1$ and $m_k = 1$ for all k in Corollary 3.24, we deduce $\prod_{k=1}^{\infty}(1 \pm 1/T_{c\cdot 2^k+1})$ is non-rational algebraic numbers.

In particular $\prod_{k=1}^{\infty}(1 \pm 1/F_{c\cdot 2^k+1})$ and $\prod_{k=1}^{\infty}(1 \pm 1/F_{c\cdot 2^k+2})$ are non-rational algebraic numbers.

3.2.3 Connections with Other Results

We give connections of our irrationality criterion for infinite products established in the previous section with two other results of Sándor,[32], and A. Knopfmacher and J. Knopfmacher,[19], respectively.

As another application of Corollary 3.21, we derive two extensions of an irrationality criterion due to Sándor, see Theorem 2.13 in chapter 2.

Corollary 3.25 Let $\{b_n\}$ and $\{a_n\}$ be two sequences of positive integers such that

$$a_{n+1} \geq b_{n+1}\, a_n^2 \qquad (n \in \mathbb{N}). \tag{60}$$

1) If all $b_n > 1$, then the infinite product $\prod_{n=1}^{\infty}(1 + b_n/a_n)$ converges to an irrational number.
2) If $a_1 > 1$, then the infinite product $\prod_{n=1}^{\infty}(1 - b_n/a_n)$ converges to an irrational number.

Proof. Without loss of generality, in 1) we may assume that $a_1 > 1$ for (60) and $b_n > 1$ show that $a_n > 1$ for all $n \geq 2$, and we need merely shift the index n. Now $a_1 > 1$ and (60) show that $a_{n+1} \geq 2^{2^n}$. In addition, the condition (60) yields

$$\frac{b_{n+1}}{a_{n+1}} \leq \frac{1}{a_n^2}$$

which implies that the two infinite products in 1) and 2) both converge. To prove part 1), from (60), we have

$$\begin{aligned} a_{n+1} &\geq b_{n+1}\ a_n^2 > b_{n+1}\left(a_n^2 - 1\right) + 1 = b_{n+1}(a_n - 1)(a_n + 1) + 1 \\ &> b_{n+1}(a_n - 1)\left(\frac{a_n}{b_n} + 1\right) + 1. \end{aligned}$$

This last relation is equivalent to (54) and the conclusion follows from Corollary 3.21 part 3).

To prove part 2), note first that (60) gives $a_n > b_n + 1$ for all $n \geq 2$ and that

$$a_{n+1} \geq b_{n+1}a_n^2 > b_{n+1}(a_n - 1)^2 + b_{n+1} \geq \frac{b_{n+1}}{b_n}(a_n - b_n)(a_n - 1) + 1.$$

This last relation is equivalent to (54) and the result follows at once from Corollary 3.21 part 4). □

Remark. Sándor's conditions in [32] are $\lim_{n\to\infty} b_n = \infty$ and $a_{n+k} \geq b_{n+k}\ a_n^{2^k}$, where $n, k \in \mathbb{N}$, which are much too strong compared to our Corollary 3.25 part 1).

Next,we give a converse, with an extra condition, of Proposition 4.1 in [19], see Theorem 2.15 in chapter 2.

Corollary 3.26 Let $\{c_n\}_{n\geq 1}$ be a sequence of positive integers. Assume that

$$c_1 \geq 2, \quad c_{n+1} \geq c_1 c_2 \cdots c_n \quad (n \in \mathbb{N}). \tag{61}$$

If $\prod_{n=1}^{\infty}(1 + 1/c_1 c_2 \cdots c_n)$ is rational, then there exists $k \in \mathbb{N}$ such that

$$c_n = d^{2^{n-k-1}} \quad (n \geq k + 1),$$

where $d = c_1 c_2 \cdots c_k$.

Proof. The product clearly converges due to the conditions in (61). Taking $b_n = 1, a_n = c_1c_2\cdots c_n$ in Corollary 3.21 part 1), the condition (52) is equivalent to the condition (61) and we deduce that there exists $k \in \mathbb{N}$ such that

$$c_{n+1} = c_1c_2\cdots c_n \quad (n \geq k).$$

Repeated use of this relation yields the desired result. □

A counterpart of Corollary 3.26 for an infinite product with terms smaller than 1 is:

Corollary 3.27 Let $\{c_n\}_{n\geq 1}$ be a sequence of positive integers. Assume that

$$c_1 \geq 2, \quad c_{n+1} \geq c_1c_2\cdots c_n \quad (n \in \mathbb{N}). \tag{62}$$

Then $\prod_{n=1}^{\infty}(1 - (c_1c_2\cdots c_n + 1)^{-1})$ converges to a rational number if and only if there exists $k \in \mathbb{N}$ such that

$$c_n = d^{2^{n-k-1}} \quad (n \geq k+1),$$

where $d = c_1c_2\cdots c_k$.

Proof. The necessity proof proceeds much like the one in Corollary 3.26 but we take here $b_n = 1,\ a_n = c_1c_2\cdots c_n + 1$ and appeal instead to the results of Corollary 3.21 part 2). The sufficiency follows from (13). □

3.3 Remarks of Brun's Criterion

We allude to a natural question of what conclusion can be obtained should only one of the two inequalities from (16) and (17) be reversed. We begin with the case where the inequality (17) is reversed while the inequality (16) remains the same.

Theorem 3.28 Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences of positive integers satisfying $x_n < x_{n+1}$ and

$$\frac{y_n}{x_n} < \frac{y_{n+1}}{x_{n+1}} \quad (n \geq 1). \tag{63}$$

Assume that $S = \lim_{n\to\infty} y_n/x_n \in \mathbb{Q}$. If

$$\frac{y_{n+2} - y_{n+1}}{x_{n+2} - x_{n+1}} \geq \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \tag{64}$$

for all large n, then $\lim_{n\to\infty} \frac{y_{n+1}-y_n}{x_{n+1}-x_n}$ exists and is equal to S.

Proof. Let $S = a/b$, where $a, b \in \mathbb{N}$. From (63) we have

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} > \frac{y_n}{x_n}. \tag{65}$$

We first claim that the sequence $\{\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\}$ is bounded above. Suppose not. Then using (64) and (65), there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\frac{y_{n+2} - y_{n+1}}{x_{n+2} - x_{n+1}} \geq \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \geq \frac{a}{b} = S. \tag{66}$$

This together with (63) yield

$$ax_n - by_n \geq ax_{n+1} - by_{n+1} \geq ax_{n+2} - by_{n+2} > 0 \qquad (n \geq N_1)$$

showing that $\{ax_n - by_n\}_{n\geq 1}$ is non-increasing sequence of positive integers and so is ultimately constant. Thus, there is $N_2 \in \mathbb{N}$ for which

$$ax_n - by_n = ax_{n+1} - by_{n+1} = ax_{n+2} - by_{n+2} \qquad (n \geq N_2 \geq N_1),$$

i.e.,

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{a}{b} \qquad (n \geq N_2)$$

contradicting the fact that the sequence $\{\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\}$ is not bounded above. The sequence $\left\{\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\right\}$, being non-decreasing and bounded above, converges to a limit $S' > 0$. Taking the limit as $n \to \infty$ in (65), we deduce that $S' \geq S$. There are then two possible cases to consider.

Case I: There is $N \in \mathbb{N}$ such that $\frac{y_{N+1}-y_N}{x_{N+1}-x_N} \geq S$.
Embarking upon (64), we deduce that (66) holds for all $n \geq N$. Following the steps in the proof of the preceding claim, we deduce that $\frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \frac{a}{b}$ for all large n, which yields $S' = S$.

Case II: The relation $\frac{y_{n+1}-y_n}{x_{n+1}-x_n} < S$ holds for all n.
In this case, we have $S \leq S' = \lim_{n\to\infty} \frac{y_{n+1}-y_n}{x_{n+1}-x_n} \leq S$, as desired. □

Using a similar proof, the result with the inequality (16) reveresd and the inequality (17) remaining the same reads:

Theorem 3.29 Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences of positive integers satisfying $x_n < x_{n+1}$ and

$$\frac{y_n}{x_n} > \frac{y_{n+1}}{x_{n+1}} \quad (n \geq 1).$$

Assume that $T = \lim_{n\to\infty} y_n/x_n \in \mathbb{Q}$. If

$$\frac{y_{n+2} - y_{n+1}}{x_{n+2} - x_{n+1}} \leq \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$$

for all large n, then $\lim_{n\to\infty} \frac{y_{n+1}-y_n}{x_{n+1}-x_n}$ exists and is equal to T.

The next two examples show that the two possible cases in the proof of Theorem 3.28 do actually exist.

Example 3. Let $x_n = 2^n,\ y_n = 2^n - 1\ \ (n \geq 1)$. Then

$$x_n = 2^n < 2^{n+1} = x_{n+1}, \quad \frac{y_n}{x_n} = \frac{2^n - 1}{2^n} < \frac{2^{n+1} - 1}{2^{n+1}} = \frac{y_{n+1}}{x_{n+1}}, \ S = \lim_{n\to\infty} \frac{y_n}{x_n} = 1$$

and

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = 1 \ \ (n \geq 1).$$

Example 4. Let $x_n = 3^n,\ y_n = 3^n - 2^n\ \ (n \geq 1)$. Then

$$x_n = 3^n < 3^{n+1} = x_{n+1}, \quad \frac{y_n}{x_n} = 1 - \frac{2^n}{3^n} < 1 - \frac{2^{n+1}}{3^{n+1}} = \frac{y_{n+1}}{x_{n+1}}, \ S = \lim_{n\to\infty} \frac{y_n}{x_n} = 1$$

and

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{(3^{n+1} - 2^{n+1}) - (3^n - 2^n)}{3^{n+1} - 3^n} < 1 \quad (n \geq 1).$$

Note also that unlike the Brun's and reverse Brun's criteria, in Theorems 3.28 and 3.29 only the convergence not the being ultimately constant of the sequence of quotients of differences can be drawn which in turn indicates that no useful irrationality criteria can be extracted.

CHAPTER 4

SUMMARY AND OPEN PROBLEMS

The research work contained in this dissertation deals with results arising from the Brun's irrationality criterion. In Section 4.1, a summary of our main results are gathered. A list of some open problems related to the findings is presented in Section 4.2.

4.1 Summary

The results obtained are separated into three parts. In the first part, the classical irrationality criterion for a sequence of rational numbers due to Brun in 1910 is shown to be essentially equivalent to the one for a series of rational terms proved by Badea in 1993. There are also versions of Brun's and Badea's criteria with main inequalities pointing in the opposite direction, referred to as reverse Brun's and Badea's criteria whose proof are analogous and both results are shown to be equivalent. Applying this equivalence, several applications, namely, to continued fractions, to determine the algebraicity of infinite sums are given. Main results in this part are:

1. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of positive integers with $X_n < X_{n+1}$. Assume that $L = \lim_{n\to\infty} Y_n/X_n$ is rational. Then $\dfrac{Y_{n+1}-Y_n}{X_{n+1}-X_n}$ is constant for all large n if at least one of the following conditions is satisfied:

$$\text{(i)}\ \frac{Y_n}{X_n} < \frac{Y_{n+1}}{X_{n+1}} \text{ and } \frac{Y_{n+1}-Y_n}{X_{n+1}-X_n} \geq \frac{Y_{n+2}-Y_{n+1}}{X_{n+2}-X_{n+1}} \text{ for all } n \geq 1.$$

$$\text{(ii)}\ \frac{Y_n}{X_n} > \frac{Y_{n+1}}{X_{n+1}} \text{ and } \frac{Y_{n+1}-Y_n}{X_{n+1}-X_n} \leq \frac{Y_{n+2}-Y_{n+1}}{X_{n+2}-X_{n+1}} \text{ for all } n \geq 1.$$

2. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\mu_n\}$ be three sequences of positive rational numbers and $\{\nu_n\}$ a sequence of rationals with $\nu_1 > 0$ and $\nu_n < 0$ for all $n \geq 2$. Assume that

$$P_n = \alpha_1\alpha_2\cdot\cdots\cdot\alpha_n, \quad Q_n = \beta_1\beta_2\cdots\beta_n, \quad A_n = P_n\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j}, \quad B_n = Q_n\sum_{j=1}^{n}\frac{\nu_j}{\mu_j}$$

are both integral for all large n.

(i) If $\sum_{n=1}^{\infty} \beta_n/\alpha_n$ converges to a rational number and

$$\frac{\alpha_{n+1}-1}{\beta_{n+1}} \geq \frac{\alpha_n(\alpha_n-1)}{\beta_n} \tag{$*$}$$

for all large n, then in $(*)$ we have equality from some n onwards.

(ii) If $\sum_{n=1}^{\infty} \nu_n/\mu_n$ converges to a positive rational number and

$$\frac{\mu_{n+1}-1}{\nu_{n+1}} \leq \frac{\mu_n(\mu_n-1)}{\nu_n} \tag{$**$}$$

for all large n, then in $(**)$ we have equality from some n onwards.

3. Let γ be a positive irrational number whose simple continued fraction is

$$\gamma = [e_0; e_1, e_2, \ldots], \quad e_0 \geq 0, \ e_n \in \mathbb{N} \ \ (n \geq 1).$$

Let $\{h_n/k_n\}$ be its n^{th} convergents and $f(n)$ a strictly increasing function of $n \in \mathbb{N}$. If either $f(n) \in 2\mathbb{N}$ or $f(n) \in 2\mathbb{N}+1$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{h_{f(n+1)}\,k_{f(n)} - h_{f(n)}\,k_{f(n+1)}}{k_{f(n)}\,k_{f(n+1)}} = \gamma - \frac{h_{f(1)}}{k_{f(1)}}.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{(-1)^{f(n)} F_{f(n+1)-f(n)}}{F_{f(n)+1}F_{f(n+1)+1}} = \frac{1+\sqrt{5}}{2} - \frac{F_{f(1)+2}}{F_{f(1)+1}},$$

where F_n is n^{th} the Fibonacci number.

In the second part, we show that an irrational criterion of Brun for sequences of rational numbers is equivalent to one for product of rational numbers. Moreover, there are also equivalent criterion with the main inequality conditions being reversed. Using this equivalence, more irrationality criteria for infinite products of rational numbers are derived.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive integers.

(i) If $\prod_{n=1}^{\infty}(1 \pm b_n/a_n)$ is rational and

$$\frac{a_{n+1}-1}{b_{n+1}} > (a_n \pm b_n)\frac{a_n-1}{b_n} \qquad (n \geq 1), \tag{$***$}$$

then in $(***)$ we have equality from some n onwards.

(ii) If

$$\frac{a_{n+1}-1}{b_{n+1}} > (a_n \pm b_n)\frac{a_n-1}{b_n}$$

for infinitely many n, then $\prod_{n=1}^{\infty}(1 \pm b_n/a_n)$ converges to an irrational number.

From above result, we derived some extensions of irrationality criteria of infinite products due to Sándor [32] and the Knopfmacher [19], and analyzed the infinite products in the exceptional case of the result of Tachiya, [34], which are algebraic but not rational.

The final part, a natural question that should only one of two main inequalities (16) and (17) of Brun's criterion II be kept, what kind of conclusion can be extracted. It is shown that the sequence of the quotients of differences converges but not necessarily ultimately constant and so no useful irrationality criteria can be derived.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of positive integers satisfying $x_n < x_{n+1}$. Assume that $S = \lim_{n\to\infty} y_n/x_n$ is rational. Then $\lim_{n\to\infty} \frac{y_{n+1}-y_n}{x_{n+1}-x_n} = S$ if at least one of the following conditions is satisfied:

(i) If the sequence $\{y_n/x_n\}$ is a strictly increasing for all $n \geq 1$ and the sequence $\{(y_{n+1}-y_n)/(x_{n+1}-x_n)\}$ is non-decreasing for all large n.

(ii) If the sequence $\{y_n/x_n\}$ is a strictly decreasing for all $n \geq 1$ and the sequence $\{(y_{n+1}-y_n)/(x_{n+1}-x_n)\}$ is non-increasing for all large n.

4.2 Open Problems

Two open problems which may be taken as further studies are:

1. In the Brun's irrationality criterion, there are two main conditions. The first one is the monotonicity of the sequence of rational numbers involved and the second is the monotonicity of the quotients of the numerator-differences by those of the denominator-differences. The first condition is quite natural and is generally compulsory. A natural open question is whether there are other variations for the second condition, such as being the monotonicity of the quotients of numerator-sums by denominator-sums.

2. Referring to the two results of Becker-Töpfer and Tachiya, what we have done in this thesis is to determine the irrationality of certain exceptional cases where the numbers are algebraic. There are some other cases that similar questions can be asked.

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