THE CHARACTERIZATION OF GAMMA-LABELINGS OF GRAPHS

A DISSERTATION

BY

SUPAPORN SADUAKDEE

Presented in Partial Fulfillment of the Requirements for the

Doctor of Philosophy in Mathematics

at Srinakharinwirot University

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การศึกษาลักษณะเฉพาะของแกมมา-เลเบิลลิงของกราฟ

บทคัดย่อ

ของ

สุภาภรณ์ สดวกดี

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา

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สำหรับกราฟ G ที่มีอันดับ n และขนาด m ให้ γ - *เลเบิลลิงของ* G คือฟังก์ชันหนึ่งต่อหนึ่ง $f : V(G) \to \{0,\ 1,\ 2, \ldots,\ m\}$ ซึ่งก่อให้เกิด *เลเบิลลิงเส้นเชื่อม* $f' : E(G) \to \{1,\ 2, \ldots,\ m\}$ บน G นิยามโดย

$$f'(e) = \left|f(u) - f(v)\right|, \text{ สำหรับแต่ละเส้นเชื่อม } e = uv \text{ ใน } E(G)$$

ค่าของ f นิยามโดย $\operatorname{val}(f) = \sum_{e \in E(G)} f'(e)$

ค่าสูงสุดของ γ - เลเบิลลิงของ G นิยามโดย

$$\operatorname{val}_{\max}(G) = \max\{\operatorname{val}(f) \mid f \text{ เป็น } \gamma\text{-เลเบิลลิงของ } G\}$$

ขณะที่*ค่าต่ำสุด*ของ γ -เลเบิลลิงของ G คือ

$$\operatorname{val}_{\min}(G) = \min\{\operatorname{val}(f) \mid f \text{ เป็น } \gamma\text{-เลเบิลลิงของ } G\}$$

γ - เลเบิลลิง g ของ G เป็น γ - *เลเบิลลิงสูงสุด* ถ้า $\operatorname{val}(g) = \operatorname{val}_{\max}(G)$ และ γ - เลเบิลลิง h ของ G เป็น γ - *เลเบิลลิงต่ำสุด* ถ้า $\operatorname{val}(h) = \operatorname{val}_{\min}(G)$

สำหรับ γ - เลเบิลลิง f ของ G ที่มีขนาด m ให้ *เลเบิลลิงเติมเต็ม* $\overline{f} : V(G) \to \{0,\ 1,\ 2, \ldots,\ m\}$ ของ f นิยามโดย

$$\overline{f}(v) = m - f(v) \text{ สำหรับ } v \in V(G)$$

ให้ f เป็น γ - เลเบิลลิงต่ำสุดของ G แล้ว G มี γ - *เลเบิลลิงต่ำสุดแบบหนึ่งเดียว* ถ้า f และ $\overline{f}$ เป็นเพียงสอง γ - เลเบิลลิงต่ำสุดของ G

ในปริญญานิพนธ์นี้ เราได้ศึกษาลักษณะเฉพาะของค่าสุดขีดของ γ -เลเบิลลิงของกราฟวงที่มีหนึ่งคอร์ดและกราฟที่มีจุดยอดหลักภายนอก และยังได้นำเสนอการพิสูจน์อีกแบบหนึ่งโดยอุปนัยเชิงคณิตศาสตร์เพื่อให้ได้ค่าสูงสุดของ γ - เลเบิลลิงของกราฟสองส่วนบริบูรณ์และกราฟแบบบริบูรณ์ นอกจากนั้น เราได้ศึกษากราฟที่มี γ - เลเบิลลิงต่ำสุดแบบหนึ่งเดียว

THE CHARACTERIZATION OF GAMMA-LABELINGS OF GRAPHS

AN ABSTRACT

BY

SUPAPORN SADUAKDEE

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Asst. Prof. Dr. Varanoot Khemmani.

For a graph G of order n and size m, let a γ-*labeling* of G be a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces an *edge-labeling* $f' : E(G) \to \{1, 2, \ldots, m\}$ on G defined by

$$f'(e) = |f(u) - f(v)|, \text{ for each edge } e = uv \text{ in } E(G)\,.$$

The *value* of f is defined as $\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$

The *maximum value* of a γ-labeling of G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\};$$

while the *minimum value* of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if $\mathrm{val}(g) = \mathrm{val}_{\max}(G)$ and a γ-labeling h of G is a γ-min *labeling* if $\mathrm{val}(h) = \mathrm{val}_{\min}(G)$.

For a γ-labeling f of G of size m, let the *complementary labeling* $\bar{f}: V(G) \to \{0, 1, \ldots, m\}$ of f be defined by

$$\bar{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Let f be a γ-min labeling of G. Then G has a *unique* γ-min *labeling* if f and $\bar{f}$ are only two γ-min labelings of G.

In this dissertation, we characterize the extremal values of γ-labelings of the cycles with one chord and the graphs with exterior major vertices. There is also an alternative proof by mathematical induction to achieve the maximum values of γ-labelings of the complete bipartite graphs and the complete graphs. Moreover, graphs are studied for the property of unique γ-min labeling.

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Supaporn Saduakdee

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CHAPTER 1

INTRODUCTION

1. Prologue

In the mathematical field of graph theory, one of the particular graph labelings is the γ-*labeling* of a graph which is defined as follows.

Let G be a graph of order n and size m. A γ-*labeling* of G is a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m\}$ that induces an edge-labeling $f'\colon E(G) \to \{1, \ldots, m\}$ on G defined by

$$f'(e) = |f(u) - f(v)|, \quad \text{for each edge } e = uv \text{ of } G.$$

The *value* of f is defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

This particular graph labeling was first introduced by Chartrand et al. [4], in 2005, and it was motivated by Rosa's influential paper [18], where a variety of types of "valuations" were introduced. (We remark that this notion should not be confused with the γ-labeling recently introduced by Blinco et al. in [1].)

If the induced edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling*. Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a paper of Rosa [18], who used the term β-valuations. A few years later, Golomb [12] called these labelings "graceful" and this is the terminology that has been used since then. A graph that has a graceful labeling is called a *graceful graph*. One of the major conjectures in graph theory concerns graceful graphs and is due to Kotzig [15].

The Graceful Tree Conjecture Every tree is graceful.

The conjecture has been very widely studied. However, it has not been possible to devise a single algorithm to stand out against the conjecture for any arbitrary tree. The graceful labeling has been computed for all larger trees up to a fixed order. Computationally, it has been shown so far that all trees with up to 35 vertices are graceful in [7].

Moreover, a more general vertex labeling of a graph was introduced by Hegde in [14]. A vertex function f of a graph G is defined from $V(G)$ to the set of nonnegative integers that induces an edge function f' defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Such a function is called a *geodetic function* of G. A one-to-one geodetic function is a *geodetic labeling* of G if the induced edge function f' is also one-to-one. The following result was established by Hegde which provides an upper bound for $\mathrm{val}_{\max}(G)$.

Theorem 1.1.1 ([14])**.** *For any geodetic labeling f of a graph G of order n,*

$$\sum_{e\in E(G)} f'(e) \le \sum_{i=0}^{n-1}(2i - n + 1)f(v_i)\,.$$

Gallian [11] has written an extensive survey on labelings of graphs. The subject of γ-labelings of graphs has been studied since 2005. The extremal values of γ-labelings of some well-known classes and some special classes of connected graphs have been determined in [3]-[5] and [8]-[9]. A characterization of γ-labelings of some classes of connected graphs has been established in [3], [16] and [10].

Our particular interest is a study of the extremal values of γ-labelings of some special classes of connected graphs such as the cycles with one chord and the graphs with exterior major vertices. We also characterize γ-max labelings of some well-known classes of graphs such as the complete bipartite graphs and the complete graphs, providing alternative and improved proof. Moreover, we characterize the graphs having the unique γ-min labeling.

2. Background

We begin with some of the fundamental results to graphs in general. As there are considerable variations in graph theory notation and terminology used in the literature, we present, in this section, basic notation and terminology that will be used. For most part, our graph theoretic notation and terminology can be found in the textbook of Chartrand and Zhang [6]. In particular, a *graph* G consists of a finite nonempty set V of objects called *vertices* (the singular is *vertex*) and a set E of 2-element subsets of V called *edges*. The sets V and E are the *vertex set* and *edge set* of G, respectively. So a graph G is a pair (actually an ordered pair) of two sets V and E. For this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than V and E to emphasize that these are the vertex and edge sets of a particular graph G. Although G is the common symbol to use for a graph, we also use F and H, as well as G', G'' and G_1, G_2, etc. Vertices are sometimes called *points* or *nodes* and edges are sometimes called *lines*. Indeed, there are some who use the term *simple graph* for what we call a graph. Two graphs G and H are *equal* if $V(G) = V(H)$ and $E(G) = E(H)$, in which case we write $G = H$.

When dealing with graphs, it is customary and simpler to represent an edge $\{u, v\}$ by uv (or vu). If uv is an edge of G, then u and v are said to be *adjacent* in G. The number of vertices in G is often called the *order* of G, while the number of edges is its *size*. Since the vertex set of every graph is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a *trivial graph*, implying that the order of a nontrivial graph is at least 2.

If $e = uv$ is an edge of G, then the adjacent vertices u and v are said to be *joined* by the edge e. The vertices u and v are referred to as *neighbors* of each other. In this case, the vertex u and the edge e (as well as v and e) are said to be *incident* with each other. Distinct edges incident with a common vertex are *adjacent edges*.

Two graphs G_1 and G_2 are *isomorphic* if there exists a one-to-one correspondence ϕ from $V(G_1)$ to $V(G_2)$ such that $u_1v_1 \in E(G_1)$ if and only if $\phi(u_1)\phi(v_1) \in E(G_2)$. In this case, ϕ is called an *isomorphism* from G_1 to G_2. Thus, if G_1 and G_2 are isomorphic graphs, then we say that G_1 is *isomorphic* to G_2 and we write $G_1 \cong G_2$. If two graphs G and H are not isomorphic, then they are called *non-isomorphic graphs* and we write $G \ncong H$.

A graph H is called a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say that G contains H as a subgraph. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G. If a subgraph of a graph G has the same vertex set as G, then it is a *spanning subgraph* of G.

A subgraph F of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. If S is a nonempty set of vertices of a graph G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S. This induced subgraph is denoted by $\langle S\rangle$ or $\langle S\rangle_G$.

Any proper subgraph of a graph G can be obtained by removing vertices and edges from G. For an edge e of G, we write $G-e$ for the spanning subgraph of G whose edge set consists of all edges of G except e. More generally, if X is a set of edges of G, then $G-X$ is the spanning subgraph of G with $E(G-X)=E(G)-X$. If $X=\{e_1,e_2,\ldots,e_k\}$, then we also write $G-X$ as $G-e_1-e_2-\cdots-e_k$.

For a vertex v of a nontrivial graph G, the subgraph $G-v$ consists of all vertices of G except v and all edges of G except those incident with v. For a proper subset U of $V(G)$, the subgraph $G-U$ has vertex set $V(G)-U$ and its edges set consists of all edges of G joining two vertices in $V(G)-U$. Necessarily, $G-U$ is an induced subgraph of G.

If u and v are nonadjacent vertices of a graph G, then $e=uv\notin E(G)$. By $G+e$, we mean the graph with vertex set $V(G)$ and edge set $E(G)\cup\{e\}$. Thus G is a spanning subgraph of $G+e$.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices K_n is $\binom{n}{2}$ and so

$$\text{the size of } K_n \text{ is } \binom{n}{2}=\frac{n(n-1)}{2}.$$

The *complement* $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u,v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2}-m$. The graph $\overline{K_n}$ then has n vertices and no edges; it is called the *empty graph* of order n. Therefore, empty graphs have empty edge sets.

A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W. We call G a *complete bipartite graph* if every vertex of U is adjacent to every vertex of W. A complete bipartite graph with $|U|=s$ and $|W|=t$ is denoted by $K_{s,t}$ or $K_{t,s}$. If either $s=1$ or $t=1$, then $K_{s,t}$ is a *star*.

We now define a number of concept arising from the adjacency and incidence relations in a graph, leading to the concept of a connected graph.

A $u-v$ *walk* W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as

$$W : u = v_0, v_1, \ldots, v_k = v,$$

where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, \ldots, k-1$. Each vertex v_i $(0 \leq i \leq k)$ and each edge v_iv_{i+1} $(0 \leq i \leq k-1)$ is said to lie on or belong to W. If $u = v$ then the walk W is *closed*; while if $u \neq v$, then W is *open*. The number of edges encountered in walk (including multiple occurrences of an edge) is called the *length* of the walk.

A $u-v$ *trail* in a graph G is a $u-v$ walk in which no edge is traversed more than once. A $u-v$ walk in a graph in which no vertices are repeated is a $u-v$ *path*. A *circuit* in a graph G is a closed trail of length 3 or more. A circuit that repeats no vertex, except for the first and last, is a *cycle*. A cycle of odd length is called an *odd cycle*; while, not surprisingly, a cycle of even length is called an *even cycle*. A path of order n is called an n-*path* and is denoted by P_n. We can express P_n as

$$P_n : v_1, v_2 \ldots, v_n$$

where $n \geq 1$ and v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, n-1$. A cycle of order n is called an n-*cycle* and is denoted by C_n. We can express C_n as

$$C_n : v_1, v_2 \ldots, v_n, v_1$$

where $n \geq 3$, v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, n-1$ and also v_1 and v_n are adjacent.

If G contains a $u-v$ path, then u and v are said to be *connected* and u *is connected to* v (and v is connected to u). So, saying that u and v are connected only means that there is some $u-v$ path in G; it doesn't say that u and v are joined by an edge. Of course, if u is joined to v, then u is connected to v as well. A graph G is *connected* if every two vertices of G are connected, that is, if G contains a $u-v$ path for every pair u, v of vertices of G. Since every vertex is connected to itself, the trivial graph is connected. A graph G that is not connected is called *disconnected*. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a *component* of G.

A graph G is called *acyclic* if it has no cycles. A *tree* is an acyclic connected graph. A spanning subgraph H of a connected graph G such that H is a tree is called a *spanning tree* of G.

Let G be a connected graph of order n, and let u and v be two vertices of G. The *distance* between u and v is the smallest length of any $u - v$ path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if the graph G under consideration is clear. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by $\text{diam}(G)$.

The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $\deg_G v$ or simply by $\deg v$ if the graph G is clear from the context. Also, $\deg v$ is the number of vertices adjacent to v. Recall that two adjacent vertices are referred to as *neighbors* of each other. The set $N(v)$ of neighbors of a vertex v is called the *neighborhood* of v. Thus $\deg v = |N(v)|$. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end vertex* (or a *leaf*). The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the *maximum degree* of G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$. So if G is a graph of order n and v is any vertex of G, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1\,.$$

If $\delta(G) = \Delta(G)$, then the vertices of G have the same degree and G is called *regular*. If $\deg v = r$ for every vertex v of G, where $0 \leq r \leq n - 1$, then G is *r-regular* or *regular of degree r*.

A vertex of odd degree is called an *odd vertex* and a vertex of even degree is called an *even vertex*. The following theorem is known as *The First Theorem of Graph Theory*.

Theorem 1.2.1 (The First Theorem of Graph Theory). *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Corollary 1.2.2. *Every graph has an even number of odd vertices.*

3. γ-labelings of graphs

Let G be a graph of order n and size m. A γ-*labeling* of G is a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m\}$ that induces an edge-labeling $f'\colon E(G) \to \{1, \ldots, m\}$ on G defined by

$$f'(e) = |f(u) - f(v)|, \quad \text{for each edge } e = uv \text{ of } G.$$

The *value* of f is defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e)\,.$$

Obviously, since f is one-to-one, it follows that $f'(e) \geq 1$, for any edge e, and therefore, $\text{val}(f) \geq m$. Moreover, G has a γ-labeling if and only if $m \geq n-1$ and every connected graph has a γ-labeling.

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

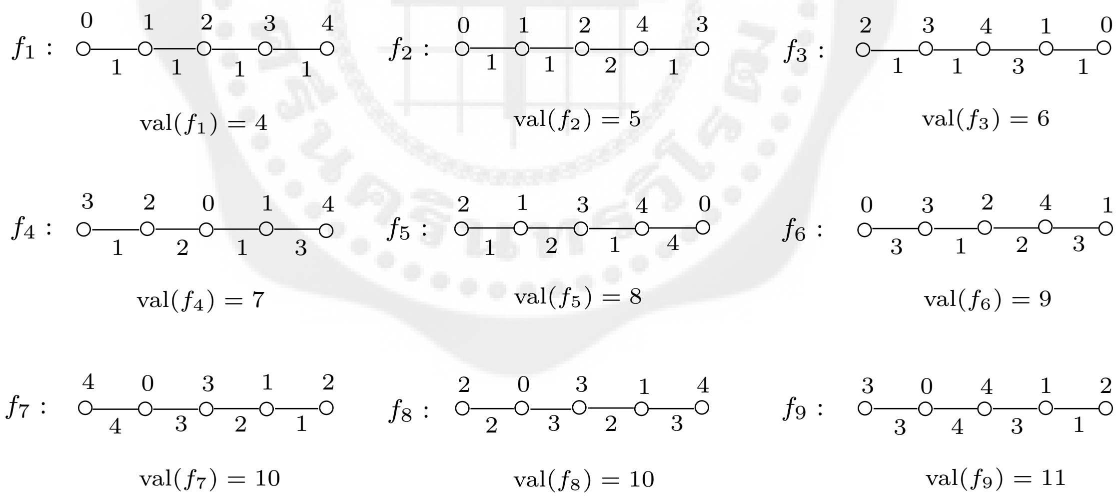

Figure 1 Some γ-labelings of P_5

The value of a graceful labeling of a graph G of order n and size m is necessarily $\binom{m+1}{2}$. For example, the γ-labeling f_7 of P_5 shown in Figure 1 is graceful and consequently $\text{val}(f_7) = \binom{5}{2} = 10$. However, the labeling f_8 shows that it is not necessary for a γ-labeling to be graceful in order to have a value of $\binom{m+1}{2}$.

The *maximum value* and the *minimum value* of a γ-labeling of G are defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\}$$

and

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\}\,,$$

respectively.

A γ-labeling g of G is a γ-max *labeling* if

$$\mathrm{val}(g) = \mathrm{val}_{\max}(G)$$

and a γ-labeling h of G is a γ-min *labeling* if

$$\mathrm{val}(h) = \mathrm{val}_{\min}(G).$$

Since $\mathrm{val}(f_1) = 4$ for the γ-labeling f_1 of P_5 shown in Figure 1 and the size of P_5 is 4, it follows that f_1 is a γ-min labeling of P_5 and $\mathrm{val}_{\min}(P_5) = \mathrm{val}(f_1) = 4$. By a case-by-case analysis, the γ-labeling f_9 shown in Figure 1 is a γ-max labeling of P_5 and $\mathrm{val}_{\max}(P_5) = \mathrm{val}(f_9) = 11$.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\bar{f}\colon V(G) \to \{0, 1, \ldots, m\}$ of f is defined by

$$\bar{f}(v) = m - f(v) \ \text{ for } \ v \in V(G).$$

Not only is $\bar{f}$ a γ-labeling of G as well but $\mathrm{val}(\bar{f}) = \mathrm{val}(f)$. The following observation appears in [4].

Observation 1.3.1 ([4]). *Let f be a γ-labeling of a graph G. Then f is a γ-max labeling (γ-min labeling) of G if and only if $\bar{f}$ is a γ-max labeling (γ-min labeling) of G.*

CHAPTER 2

REVIEW OF THE LITERATURE

The purpose of this chapter is to review some relevant works on the γ-labelings of graphs. As we have already mentioned in Section 1.1 that this particular graph labeling was first introduced by Chartrand et al. [4], in 2005, and it was motivated by Rosa's influential paper [18], where a variety of types of "valuations" were introduced. Gallian [11] has written an extensive survey on labelings of graphs. The subject of γ-labelings of graphs was studied in [3]-[5], [8]-[10] and [16].

1. γ-labelings of subgraphs

In [4] and [5], a simple and useful connection between minimum and maximum values of a connected graph and that of a proper connected subgraph is found.

Proposition 2.1.1 ([4]). *If H is a proper connected subgraph of a connected graph G, then*

$$\text{val}_{\min}(H) < \text{val}_{\min}(G) \textit{ and } \text{val}_{\max}(H) < \text{val}_{\max}(G).$$

The *subdivision graph* of a graph G is the graph obtained from G by replacing each edge uv of G by a new vertex w and the two new edges uw and vw.

Proposition 2.1.2 ([4]). *If H is a subdivition of a connected graph G, then*

$$\text{val}_{\min}(G) < \text{val}_{\min}(H) \quad \textit{and} \quad \text{val}_{\max}(G) < \text{val}_{\max}(H).$$

Theorem 2.1.3 ([5]). *If G is a connected graph of size m and G' is a connected subgraph of G having size m', then*

$$\text{val}_{\min}(G) \geq (m - m') + \text{val}_{\min}(G').$$

The extended result of Theorem 2.1.3 is shown as follows.

Theorem 2.1.4 ([5]). *If G is a connected graph of size m containing pairwise edge-disjoint connected subgraphs $G_1, G_2, \ldots, G_k$, where G_i has size m_i for $1 \leq i \leq k$, then*

$$\text{val}_{\min}(G) \geq \left(m - \sum_{i=1}^{k} m_i\right) + \sum_{i=1}^{k} \text{val}_{\min}(G_i).$$

Theorem 2.1.5 ([5]). *Let f be a γ-labeling of a connected graph G. If P is a $u-v$ path in G, then*

$$\sum_{e\in E(P)} f'(e) \geq |f(u) - f(v)|.$$

2. γ-labelings of some well-known classes of graphs

In [3]-[5], the maximum and minimum values of a γ-labeling of some well-known classes of graphs, namely path P_n, cycle C_n, complete graph K_n, double star $S_{p,q}$ and complete bipartite graph $K_{r,s}$ are determined.

Theorem 2.2.1 ([4]). *For every integer $n \geq 3$, there exists a γ-max labeling f of $P_n : v_1, v_2, \ldots, v_n$ having the property that for every integer i with $1 \leq i \leq n-2$, the 3-term sequence*

$$s_i(f) = (f(v_i), f(v_{i+1}), f(v_{i+2}))$$

is not monotone.

Theorem 2.2.2 ([4]). *For each integer $n \geq 2$,*

$$\mathrm{val}_{\min}(P_n) = n-1$$

and

$$\mathrm{val}_{\max}(P_n) = \left\lfloor \frac{n^2-2}{2} \right\rfloor.$$

Theorem 2.2.3 ([4]). *Let G be a connected graph of order n and size m. Then*

$$\mathrm{val}_{\min}(G) = m \text{ if and only if } G \cong P_n\,.$$

Proposition 2.2.4 ([4]). *If G is a connected r-regular bipartite graph of order n and size m, where $r \geq 2$, then*

$$\mathrm{val}_{\max}(G) = \frac{rn(2m-n+2)}{4}\,.$$

Theorem 2.2.5 ([4]). *For each integer $n \geq 3$,*

$$\mathrm{val}_{\min}(C_n) = 2(n-1)$$

and

$$\mathrm{val}_{\max}(C_n) = \begin{cases} \frac{(n-1)(n+3)}{2} & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{2} & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2.2.6 ([4]). *For every positive integer n,*

$$\mathrm{val}_{\min}(K_n) = \binom{n+1}{3}$$

and

$$\mathrm{val}_{\max}(K_n) = \begin{cases} \frac{(n^2-1)(3n^2-5n+6)}{24} & \text{if } n \text{ is odd} \\ \frac{n(3n^3-5n^2+6n-4)}{24} & \text{if } n \text{ is even.} \end{cases}$$

Last, we turn to the double star $S_{p,q}$ containing central vertices u and v such that $\deg u = p$ and $\deg v = q$ and the complete bipartite graph $K_{r,s}$.

Theorem 2.2.7 ([5]). *For every pair p, q of positive integers,*

$$\mathrm{val}_{\min}(S_{p,q}) = \left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right)^2 + \left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right)^2 - \left(n_p \left\lfloor \frac{p+2}{2} \right\rfloor + n_q \left\lfloor \frac{q+2}{2} \right\rfloor + 1\right),$$

where

$$n_p = \begin{cases} 1 & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases} \quad \text{and} \quad n_q = \begin{cases} 1 & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases},$$

and

$$\mathrm{val}_{\max}(S_{p,q}) = \frac{1}{2}\left[p^2 + q^2 + 4pq - 3p - 3q + 2\right].$$

Theorem 2.2.8 ([3]). *For any two positive integers $r \geq s$,*

$$\mathrm{val}_{\min}(K_{r,s}) = \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor$$

and

$$\mathrm{val}_{\max}(K_{r,s}) = rs\left(rs - \frac{1}{2}(r+s) + 1\right).$$

3. γ^δ-labelings of connected graphs

A slightly extension of the main notion of γ-labeling of a graph is shown as follows.

For a nonnegative integer δ and a γ-labeling of a connected graph G of order n and size m, the *extension γ^δ-labeling* of G is defined in [8] as a one-to-one function $f : V(G) \to \{0, 1, \ldots, m+\delta-1, m+\delta\}$ that induces an edge-labeling $f' \colon E(G) \to \{1, \ldots, m+\delta\}$ on G defined by

$$f'(uv) = |f(u) - f(v)|, \quad \text{for each edge } uv \text{ of } G.$$

The *value* of the extension γ^δ-labeling f is defined as $\mathrm{val}(f) = \sum_{e \in E(G)} f'(e)$. The other definitions are similar to the γ-labeling case.

C_8 :

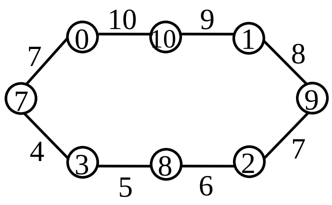

Figure 2 The extension γ^2-labeling of the cycle C_8

For example, the cycle C_8 of Figure 2 has eight edges. We consider $\delta = 2$ and let $f\colon V(C_8) \to \{0, 1, \ldots, 8+2 = 10\}$ be a one-to-one function that induces an edge-labeling $f'\colon E(C_8) \to \{1, 2, \ldots, 8+2 = 10\}$ defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of C_8 (where the vertex labels are shown within the circle representing each vertex and the induced edge labels are shown close to each edge). Then, a one-to-one function f is the extension γ^2-labeling of C_8.

Moreover, Fonseca, Saenpholphat, and Zhang [8] determined the minimum value of the extension γ^δ-labeling of a connected graph and the maximum value of the extension γ^δ-labeling of a cycle.

Theorem 2.3.1 ([8]). *For any connected graph G of order n,*

$$\mathrm{val}^{\delta}_{\min}(G) = \mathrm{val}_{\min}(G).$$

Theorem 2.3.2 ([8]). *For every pair δ, n of nonnegative integers with $n \geq 4$,*

$$\mathrm{val}^{\delta}_{\max}(C_n) = \begin{cases} \mathrm{val}_{\max}(C_n) + n\delta = \frac{n(n+2\delta+2)}{2} & \text{if } n \text{ is even} \\ \mathrm{val}_{\max}(C_n) + (n-1)\delta = \frac{(n-1)(n+2\delta+3)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

4. γ-spectra of some well-known classes of graphs

The *γ-spectrum* of a graph G which is defined in [4] as

$$\mathrm{spec}(G) = \{\mathrm{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\}.$$

Observe that $\mathrm{val}_{\min}(G), \mathrm{val}_{\max}(G) \in \mathrm{spec}(G)$ for every graph G.

For integers a and b with $a < b$, let

$$[a, b] = \{a, a+1, \ldots, b\}$$

be a *consecutive set* of integers between a and b.

Thus for every graph G,

$$\operatorname{spec}(G) \subseteq [\operatorname{val}_{\min}(G),\ \operatorname{val}_{\max}(G)].$$

The γ-spectra of some well-known classes of graphs, namely, stars, paths, cycles and complete graphs are determined in [4] and [9].

Proposition 2.4.1 ([4]). *For each integer $t \geq 2$,*

$$\operatorname{spec}(K_{1,t}) = \left\{ \binom{t+1-k}{2} + \binom{k+1}{2} \mid 0 \leq k \leq t \right\}.$$

Theorem 2.4.2 ([9]). *For each integer $n \geq 2$,*

$$\operatorname{spec}(P_n) = [\operatorname{val}_{\min}(P_n),\ \operatorname{val}_{\max}(P_n)] = \left[n-1, \left\lfloor \frac{n^2-2}{2} \right\rfloor \right].$$

For even integers a and b with $a < b$, let

$$E[a,b] = \{a, a+2, a+4, \dots, b\}$$

be an *even consecutive set* of integers between a and b.

Theorem 2.4.3 ([9]). *For each integer $n \geq 3$,*

$$\operatorname{spec}(C_n) = E\,[\operatorname{val}_{\min}(C_n),\ \operatorname{val}_{\max}(C_n)].$$

If f is a γ-labeling of K_n such that $f(V(K_n)) = \{a_1, a_2, \dots, a_n\}$ and

$$0 \leq a_1 < \cdots < a_n \leq \binom{n}{2} \tag{2.4.1}$$

then

$$\operatorname{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+1)(a_{n-i+1} - a_i)\,.$$

Setting

$$\alpha_i = a_{n-\lfloor \frac{n}{2} \rfloor + i} - a_{\lfloor \frac{n}{2} \rfloor} - i + 1 \ \text{ for } \ i = 1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor,$$

an increasing sequence $\{\alpha_i\}$ has following properties:

1. $\alpha_i \geq \alpha_{i-1} + 2$ for $i = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor$,
2. $\alpha_{\lfloor \frac{n}{2} \rfloor} \leq \binom{n}{2}$, and
3. $\alpha_1 \geq 2$ if n is odd, while $\alpha_1 \geq 1$ if n is even.

On the other hand, if $\{\alpha_i\}$ is an increasing sequence with properties 1-3, then there exist n integers $a_1, a_2, \ldots, a_n$ satisfying (2.4.1) such that $\alpha_i = a_{n-i+1} - a_i$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Therefore, an increasing sequence $\{\alpha_i\}$ is called a γ-*sequence* of K_n if $\{\alpha_i\}$ satisfies properties 1-3. The γ-spectrum of the complete graph K_n is then given in terms of γ-sequences of K_n.

Theorem 2.4.4 ([9]). *For each integer $n \geq 2$,*

if n is odd, then

$$\text{spec}(K_n) = \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2i\alpha_i \mid \{\alpha_i\} \text{ is a } \gamma\text{-sequence of } K_n \right\},$$

if n is even, then

$$\text{spec}(K_n) = \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i-1)\alpha_i \mid \{\alpha_i\} \text{ is a } \gamma\text{-sequence of } K_n \right\}.$$

As an illustration of Theorem 2.4.4, we see that

$$\begin{aligned} \text{spec}(K_4) &= \{3\alpha_2 + \alpha_1 \mid 6 \geq \alpha_2 \geq \alpha_1 + 2 \geq 3\} \\ &= \{10, 13, 14, 16, 17, 18, 19, 20, 21, 22\}, \\ \text{spec}(K_5) &= \{4\alpha_2 + 2\alpha_1 \mid 10 \geq \alpha_2 \geq \alpha_1 + 2 \geq 4\} \\ &= E[20, 56] - \{22\}. \end{aligned}$$

Notice that there are integers in $[\text{val}_{\min}(K_n),\ \text{val}_{\max}(K_n)]$ for which there is no γ-labeling with that value. For example, if $s \in [\text{val}_{\min}(K_n) + 1,\ \text{val}_{\min}(K_n) + n - 2]$, then there is no γ-labeling of K_n whose value is s.

5. Spans of γ-min and γ-max labelings of graphs

The *span* of a γ-labeling f of a graph G is defined as

$$\text{span}(f) = \max\{f(v) \mid v \in V(G)\} - \min\{f(v) \mid v \in V(G)\}.$$

The following results appear in [4] and [10].

Theorem 2.5.1 ([4]). *Let G be a connected graph of order n and $f : V(G) \to \mathbf{Z}$ a one-to-one function. Then there is a γ-labeling g on G with* $\text{val}(g) \leq \text{val}(f)$. *Furthermore, if* $\text{span}(f) \geq n$, *then there is a γ-labeling g with* $\text{val}(g) < \text{val}(f)$.

Theorem 2.5.2 ([4]). *If G is a connected graph of order n, then G has a γ-*min *labeling f such that $f(V(G)) = [0, n-1]$.*

Theorem 2.5.3 ([10]). *Let f be a γ-*max *labeling of a nontrivial graph G of order n and size m and let $u, w \in V(G)$ with $f(u) = \min\{f(v) \mid v \in V(G)\}$ and $f(w) = \max\{f(v) \mid v \in V(G)\}$. Then neighborhoods of u and w are not empty.*

Theorem 2.5.4 ([10]). *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

1. *If f is a γ-*min *labeling of G, then* $\text{span}(f) = n - 1$.
2. *If f is a γ-*max *labeling of G, then* $\text{span}(f) = m$.

The following results are consequences of Theorem 2.5.4.

Theorem 2.5.5 ([10]). *Let G be a nontrivial graph of order n and size m and let f be a γ-labeling of G.*

1. *If f is a γ-*min *labeling of G, then $f(V(G))$ is a consecutive subset of $[0, m]$, that is, $f(V(G)) = [k, k + (n-1)]$ for some integer k with $0 \le k \le m - (n-1)$.*
2. *If f is a γ-*max *labeling of G, then $\{0, m\} \subseteq f(V(G))$.*

Theorem 2.5.6 ([10]). *Let G be a nontrivial connected graph of order n and size m. Suppose that f and g are γ-*min *and γ-*max *labelings of G, respectively. Then* $\text{span}(f) = \text{span}(g)$ *if and only if G is a tree.*

The following open question is mentioned in [3]. For any connected graph G, does exist a γ-max labeling of G with a vertex label set that is the union of no more than two sets of consecutive numbers? The following result in [10] is related to this open question.

Theorem 2.5.7 ([10]). *Let G be a nontrivial graph of order n and size m. Then vertex label set of γ-*max *labeling of G is a set of consecutive numbers if and only if G is a tree.*

6. γ-max labelings of complete bipartite graphs and complete graphs

The characterization of γ-max labelings of complete bipartite graphs appears in [3]. Later, Fonseca, Khemmani and Zhang [10] provide alternative and improved proof of formula for $\text{val}(K_{r,s})$, that uses γ-min labelings of complete graphs.

Theorem 2.6.1 ([3], [10]). *Let f be a γ-labeling of complete bipartite graph $K_{r,s}$ with partite sets V_r and V_s of cardinality r and s, respectively. Then f is a γ*-max *labeling of $K_{r,s}$ if and only if*

1. $f(V_r) = [0, r-1]$ *and* $f(V_s) = [rs-(s-1), rs]$, *or*
2. $f(V_r) = [rs-(r-1), rs]$ *and* $f(V_s) = [0, s-1]$.

The following result provides a characterization of γ-max labelings of complete graphs which appears in [10].

Theorem 2.6.2 ([10]). *Let f be a γ-labeling of a complete graph K_n. Then f is a γ*-max *labeling of K_n if and only if*

$$f(V(K_n)) = \begin{cases} \left[0, \lfloor \frac{n}{2} \rfloor - 1\right] \cup \left[\binom{n}{2} - \lfloor \frac{n}{2} \rfloor + 1, \binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0, \lfloor \frac{n}{2} \rfloor - 1\right] \cup \left[\binom{n}{2} - \lfloor \frac{n}{2} \rfloor + 1, \binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd}, \end{cases}$$

where k is any integer in $\left[\lfloor \frac{n}{2} \rfloor, \binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right]$.

CHAPTER 3

γ-LABELINGS OF CYCLES WITH ONE CHORD

In previous chapter, we have reviewed some relevant works on the γ-labelings of graphs. This chapter containing two main sections, is to present our comprehensive work concerning the γ-labeling of a cycle with one chord. The minimum value and the maximum value of a γ-labeling of a cycle with one chord are presented in the first section. Moreover, the γ-spectrum of a cycle with one chord is presented in the second section.

1. Extremal values of γ-labelings of cycles with one chord

First, we introduce a cycle with a triangle. A *cycle with a triangle* $C_n^{\triangle}$ is a cycle with a chord joining two nonadjacent vertices but adjacent to some vertex in cycle C_n which is shown in Figure 3.

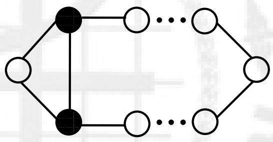

Figure 3 The cycle with a triangle $C_n^{\triangle}$

The extremal values of cycle with a triangle $C_n^{\triangle}$ is determined in [8], as we state next.

Theorem 3.1.1 ([8]). *Let $C_n^{\triangle}$ be a cycle with a triangle of order $n \geq 5$. Then*

$$\mathrm{val}_{\min}(C_n^{\triangle}) = 2n - 1$$

and

$$\mathrm{val}_{\max}(C_n^{\triangle}) = \begin{cases} 32 & \text{if } n = 6 \\ \frac{n^2+6n-10}{2} & \text{if } n \text{ is even and } n \geq 8 \\ \frac{n^2-4n+7}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Now, we consider a cycle of order $n \geq 4$ with one chord, say $C_n + e$, i.e., cycles with a chord e joining two nonadjacent vertices in cycle C_n. Therfore $C_n^{\triangle}$ is also a cycle with one chord that joins two nonadjacent vertices with distance 2 in a cycle C_n. In this section, we naturally generalize a cycle C_n with one chord e that joins two vertices with distance r in a cycle C_n where $2 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$ and determine the maximum and minimum values of a γ-labeling of $C_n + e$.

Figure 4 shows the cycle with one chord $C_8 + e$ whose chord e joins two vertices with distance 3 in the cycle C_8.

$C_8 + e$:

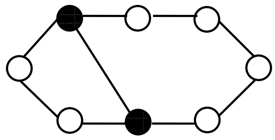

Figure 4 The cycle with one chord $C_8 + e$ where e is a chord joining two vertices with distance 3 in the cycle C_8

1.1 Minimum values of cycles with one chord

In this section we establish $\mathrm{val}_{\min}(C_n + e)$ and then $\mathrm{val}_{\max}(C_n + e)$ for cycle with one chord $C_n + e$.

Theorem 3.1.2. *For every integer $n \geq 4$,*

$$\mathrm{val}_{\min}(C_n + e) = 2n - 1.$$

Proof. By Theorem 2.2.5 and Proposition 2.1.1, we have

$$\mathrm{val}_{\min}(C_n + e) \geq 2(n-1) + 1 = 2n - 1.$$

Hence it remains to show that $\mathrm{val}_{\min}(C_n + e) \leq 2n - 1$.

Suppose that $C_n + e$ is a cycle $C_n : v_1, v_2, \ldots, v_{r-1}, v_r, v_{r+1}, \ldots, v_n, v_1$ with a chord $e = v_1 v_r$ where $3 \leq r \leq n-1$. Consider now the γ-labeling f of $C_n + e$ defined by

$$f(v_i) = \begin{cases} r-1 & \text{if } i = 1 \\ i-2 & \text{if } 2 \leq i \leq r \\ n+r-i & \text{if } r+1 \leq i \leq n. \end{cases}$$

Then

$$\begin{aligned}\text{val}(f) &= \sum_{i=3}^{r}(f(v_i)-f(v_{i-1}))+\sum_{i=r+1}^{n-1}(f(v_i)-f(v_{i+1}))+(f(v_1)-f(v_2))\\ &\quad +(f(v_n)-f(v_1))+(f(v_{r+1})-f(v_r))+(f(v_1)-f(v_r))\\ &= 2n-1.\end{aligned}$$

Therefore, $\text{val}_{\min}(C_n+e)\le 2n-1$. □

1.2 Maximum values of odd cycles with one chord

In order to discuss $\text{val}_{\max}(C_n+e)$, we first consider the maximum value of an odd cycle with one chord.

Theorem 3.1.3. *For every odd integer $n\ge 5$,*

$$\text{val}_{\max}(C_n+e)=\frac{n^2+6n-3}{2}.$$

Proof. Let C_n+e be an odd cycle with one chord of order $n=2k+1$ with $k\ge 2$, which is obtained from a cycle $C_{2k+1}: x_1,y_1,x_2,y_2,\ldots,x_r,y_r,\ldots,x_k,y_k,x_{k+1},x_1$ and a chord e. Since for each r with $2\le r\le k$, an odd cycle with one chord $C_n+x_1y_r$ is isomorphic to $C_n+x_1x_{k-r+2}$, without loss of generality, we may assume that $e=x_1y_r$. Define a γ-labeling f of C_n+e by

$$\begin{aligned} f(x_i) &= i-1 \qquad\qquad \text{if } 1\le i\le k+1\\ f(y_i) &= \begin{cases} n+1-i & \text{if } 1\le i\le r-1\\ n+1 & \text{if } i=r\\ k-r+2+i & \text{if } r+1\le i\le k.\end{cases}\end{aligned}$$

Then

$$\begin{aligned}\text{val}(f) &= 3(f(y_r)-f(x_1))+2\left(\sum_{i=1}^{r-1}f(y_i)+\sum_{i=r+1}^{k}f(y_i)-\sum_{i=2}^{k}f(x_i)\right)\\ &= \tfrac{n^2+6n-3}{2}.\end{aligned}$$

Thus $\text{val}_{\max}(C_n+e)\ge \frac{n^2+6n-3}{2}$.

It remains therefore to show that $\mathrm{val}_{\max}(C_n + e) \leq \frac{n^2+6n-3}{2}$. Let g be a γ-max labeling of $C_n + e$. Then

$$\begin{aligned}
\mathrm{val}_{\max}(C_n + e) &= \mathrm{val}(g) \\
&= \sum_{e \in E(C_n)} g'(e) + g'(x_1 y_r) \\
&\leq \mathrm{val}^1_{\max}(C_n) + (n+1) \\
&= \tfrac{(n-1)(n+2\cdot 1+3)}{2} + (n+1) \qquad \text{(by Theorem 2.3.2)} \\
&= \tfrac{n^2+6n-3}{2}.
\end{aligned}$$

Therefore, $\mathrm{val}_{\max}(C_n + e) \leq \frac{n^2+6n-3}{2}$. □

1.3 Maximum values of even cycles with one chord

In section , we considered the maximum value of odd cycle with one chord. We now compute $\mathrm{val}_{\max}(C_n + e)$ for even integer $n \geq 6$. First, we determine the maximum value of $C_n + e$ where e is a chord joining two vertices with odd distance in even cycle C_n.

Theorem 3.1.4. *For every even integer* $n \geq 6$,

$$\mathrm{val}_{\max}(C_n + e) = \frac{n^2 + 6n + 2}{2}$$

where e *is a chord joining two vertices with odd distance in even cycle* C_n.

Proof. Let $C_n + e$ be an even cycle with one chord of order $n = 2k$ with $k \geq 3$, which is obtained from a cycle $C_{2k} : x_1, y_1, x_2, y_2, \ldots, x_r, y_r, \ldots, x_k, y_k, x_1$ and a chord $e = x_1 y_r$, where $2 \leq r \leq k - 1$. Define a γ-labeling f of $C_n + e$ by

$$f(x_i) = i - 1 \qquad \text{if } 1 \leq i \leq k$$

$$f(y_i) = \begin{cases} n + 1 - i & \text{if } 1 \leq i \leq r - 1 \\ n + 1 & \text{if } i = r \\ k - r + 1 + i & \text{if } r + 1 \leq i \leq k. \end{cases}$$

Then

$$\begin{aligned}
\mathrm{val}(f) &= 3(f(y_r) - f(x_1)) + 2\left(\sum_{i=1}^{r-1} f(y_i) + \sum_{i=r+1}^{k} f(y_i) - \sum_{i=2}^{k} f(x_i)\right) \\
&= \tfrac{n^2+6n+2}{2}.
\end{aligned}$$

Thus $\text{val}_{\max}(C_n + e) \geq \frac{n^2+6n+2}{2}$.

In order to show that $\text{val}_{\max}(C_n+e) \leq \frac{n^2+6n+2}{2}$, let g be a γ-max labeling of C_n+e. Then

$$\begin{aligned}
\text{val}_{\max}(C_n + e) &= \text{val}(g) \\
&= \sum_{e \in E(C_n)} g'(e) + g'(x_1 y_r) \\
&\leq \text{val}^1_{\max}(C_n) + (n+1) \\
&= \tfrac{n(n+2\cdot 1+2)}{2} + (n+1) \qquad \text{(by Theorem 2.3.2)} \\
&= \tfrac{n^2+6n+2}{2}.
\end{aligned}$$

Therefore, $\text{val}_{\max}(C_n + e) \leq \frac{n^2+6n+2}{2}$. □

Next, we present the maximum value of even cycle with one chord, $C_n + e$ where e is a chord joining two vertices with even distance in even cycle C_n. In order to do this, for $k \geq 4$, we let $n = 2k$ and $e = x_1 x_r$ be a chord in even cycle $C_n : x_1, y_1, x_2, y_2, \ldots, x_r, y_r, \ldots, x_k, y_k, x_1$, where $2 \leq r \leq k$.

Proposition 3.1.5.

$$\begin{aligned}
\text{val}_{\max}(C_4 + e) &= 17 \\
\text{val}_{\max}(C_6 + e) &= 32
\end{aligned}$$

where e is a chord joining two vertices with even distance in cycles C_4 and C_6, respectively.

Proof. First, let $C_4 + e = C_4 + x_1 x_2$. The γ-labeling f of $C_4 + e$ is defined by

$$f(x_1) = 0, \quad f(x_2) = 1, \quad f(y_1) = 5 \text{ and } \quad f(y_2) = 4.$$

Then $\text{val}_{\max}(C_4 + e) \geq \text{val}(f) = 17$.

On the other hand, let g be a γ-max labeling of $C_4 + e$. By Theorem 2.5.5, we may assume that $g(V(C_4 + e)) = \{0, 5, a, b\}$ where $a, b \in \{1, 2, 3, 4\}$. Notice that $\deg(x_1) = \deg(x_2) = 3$ and $\deg(y_1) = \deg(y_2) = 2$. We consider four cases, according to the vertices of $C_4 + e$ labeled by 0 and 5.

Case 1. $\{0, 5\} = \{g(x_1), g(x_2)\}$. Then $\text{val}(g) = 15$.

Case 2. $\{0, 5\} = \{g(y_1), g(y_2)\}$. Then $\text{val}(g) = 10 + |a - b| \leq 13$.

Case 3. $0 \in \{g(x_1), g(x_2)\}$ *and* $5 \in \{g(y_1), g(y_2)\}$. Then

$$\mathrm{val}(g) = 10 + \max\{a, b\} + |a - b| \leq 17.$$

Case 4. $0 \in \{g(y_1), g(y_2)\}$ *and* $5 \in \{g(x_1), g(x_2)\}$. Then

$$\mathrm{val}(g) = 15 - \min\{a, b\} + |a - b| \leq 17.$$

Since $\mathrm{val}(g) \leq 17$, it follows that $\mathrm{val}_{\max}(C_4 + e) = \mathrm{val}(g) \leq 17$.

Next, we compute $\mathrm{val}_{\max}(C_6 + e)$. Since $C_6 + e = C_6 + x_1x_2$ or $C_6 + x_1x_3$, it then follows that $C_6 + e = C_6^{\triangle}$, and by Theorem 3.1.1, $\mathrm{val}_{\max}(C_6 + e) = \mathrm{val}(C_6^{\triangle}) = 32$. □

We establish a lower bound for the maximum values of $C_n + e$ of even order $n \geq 8$ where e is a chord joining two vertices with even distance in even cycle C_n.

Lemma 3.1.6. *For every even integer $n \geq 8$,*

$$\mathrm{val}_{\max}(C_n + e) \geq \frac{n^2 + 6n - 10}{2}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

Proof. Let f be a γ-labeling of $C_n + e$ defined by

$$f(x_i) = \begin{cases} i - 1 & \text{if } 1 \leq i \leq r - 1 \\ n + 1 & \text{if } i = r \\ i - 2 & \text{if } r + 1 \leq i \leq k \end{cases}$$

$$f(y_i) = \begin{cases} n + 1 - i & \text{if } 1 \leq i \leq r - 2 \\ k - r + 2 + i & \text{if } r - 1 \leq i \leq k. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f) &= 3\left(f(x_r) - f(x_1)\right) + 2\left(\sum_{i=1}^{r-2} f(y_i) + \sum_{i=r+1}^{k} f(y_i) - \sum_{i=2}^{r-1} f(x_i) - \sum_{i=r+1}^{k} f(x_i)\right) \\ &= \tfrac{n^2+6n-10}{2}. \end{aligned}$$

Therefore $\mathrm{val}_{\max}(C_n + e) \geq \mathrm{val}(f) = \frac{n^2+6n-10}{2}$. □

In order to present an upper bound for $\mathrm{val}_{\max}(C_n + e)$, where $e = x_1x_r$ is a chord in even cycle $C_n : x_1, y_1, x_2, y_2, \ldots, x_r, y_r, \ldots, x_k, y_k, x_1$ of order $n = 2k$ with $k \geq 4$ and $2 \leq r \leq k$, we need some additional notations and new definitions. Let f be a γ-max labeling of $C_n + e$. For each integer i, with $1 \leq i \leq k$, we define the 3-term sequences

$$S_i(f) = (f(x_i), f(y_i), f(x_{i+1})) \text{ and } T_i(f) = (f(y_i), f(x_{i+1}), f(y_{i+1})),$$

where the addition is taken modulo k, and let

$$ST(f) = \{S_1(f), S_2(f), \ldots, S_k(f), T_1(f), T_2(f), \ldots, T_k(f)\}$$

be a set of 3-term sequences $S_i(f)$ and $T_i(f)$, for all i with $1 \le i \le k$.

Furthermore, let $C_n(f)$ be an oriented cycle obtained from $(C_n + e) - x_1x_r$ by assigning to the edge uv the orientation (u, v) if $f(u) < f(v)$.

$C_8(f)$:

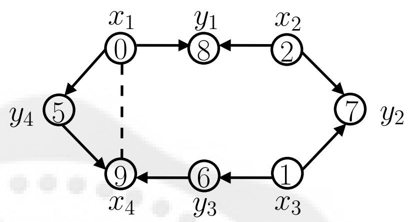

Figure 5 The oriented cycle $C_8(f)$

For example, considering the oriented cycle $C_8(f)$ defined by the γ-max labeling f of $C_8 + e$ in Figure 5, we have a set of 3-term sequences $S_i(f)$ and $T_i(f)$, for all i with $1 \le i \le 4$, as follows:

$$ST(f) = \{S_1(f), S_2(f), S_3(f), S_4(f), T_1(f), T_2(f), T_3(f), T_4(f)\}$$

where

$$S_1(f) = (f(x_1), f(y_1), f(x_2)), \quad T_1(f) = (f(y_1), f(x_2), f(y_2)),$$

$$S_2(f) = (f(x_2), f(y_2), f(x_3)), \quad T_2(f) = (f(y_2), f(x_3), f(y_3)),$$

$$S_3(f) = (f(x_3), f(y_3), f(x_4)), \quad T_3(f) = (f(y_3), f(x_4), f(y_4)),$$

$$S_4(f) = (f(x_4), f(y_4), f(x_1)), \quad T_4(f) = (f(y_4), f(x_1), f(y_1)).$$

Theorem 3.1.7.

$$\text{val}_{\max}(C_8 + e) = 51$$

where e is a chord joining two vertices with even distance in cycle C_8.

Proof. Let $e = x_1x_r$ be a chord in the cycle $C_8 : x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_1$, where $r = 2, 3, 4$. If $r = 2$ or 4, then it follows by Theorem 3.1.1, $\text{val}_{\max}(C_8 + e) = \text{val}(C_8^{\triangle}) = 51$. Assume that $r = 3$. By Lemma 3.1.6, we have $\text{val}_{\max}(C_8 + e) \ge \frac{8^2+6\cdot 8-10}{2} = 51$.

On the other hand, let f be a γ-max labeling of C_8+e. We consider the two cases according to the set $ST(f)$.

Case 1. *No element of $ST(f)$ is monotone.* Then, for each i with $1 \le i \le 4$, the vertices x_i and y_i of the oriented cycle $C_8(f)$ have

$$\textbf{either } \mathrm{id}(x_i) = 0, \mathrm{id}(y_i) = 2 \textbf{ or } \mathrm{id}(x_i) = 2, \mathrm{id}(y_i) = 0.$$

First, assume that $f(x_1) > f(x_r)$. We consider two subcases, according to whether $\mathrm{id}(x_i) = 0, \mathrm{id}(y_i) = 2$ or $\mathrm{id}(x_i) = 2, \mathrm{id}(y_i) = 0$ in the oriented cycle $C_8(f)$ for each i with $1 \le i \le 4$.

Subcase 1.1. *For each i with $1 \le i \le 4$, $\mathrm{id}(x_i) = 0$, $\mathrm{id}(y_i) = 2$ of the oriented cycle $C_8(f)$.* So, $f(x_i) < f(y_i)$ and $f(x_{i+1}) < f(y_i)$ where addition is taken modulo 4. Then

$$\mathrm{val}_{\max}(C_8+e) = \mathrm{val}(f) = 2\sum_{i=1}^{4} f(y_i) - (f(x_1) + 2f(x_2) + 3f(x_3) + 2f(x_4)).$$

Since the vertices $x_3, x_2(\text{or } x_4), x_1$ can be assigned $0, 1$ (or 2), 3, respectively and the vertices in $\{y_1, y_2, y_3, y_4\}$ can be assigned each of the labels $9, 8, 7, 6$, it follows that

$$2\sum_{i=1}^{4} f(y_i) \le 60 \quad \text{and} \quad f(x_1) + 2f(x_2) + 3f(x_3) + 2f(x_4) \ge 9.$$

Then $\mathrm{val}_{\max}(C_8+e) = \mathrm{val}(f) \le 60 - 9 = 51$.

Subcase 1.2. *For each i with $1 \le i \le 4$, $\mathrm{id}(x_i) = 2$, $\mathrm{id}(y_i) = 0$ of the oriented cycle $C_8(f)$.* So, $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$ where addition is taken modulo 4. Then

$$\mathrm{val}_{\max}(C_8+e) = \mathrm{val}(f) = (3f(x_1) + 2f(x_2) + f(x_3) + 2f(x_4)) - 2\sum_{i=1}^{4} f(y_i).$$

Since the vertices $x_1, x_2(\text{or } x_4), x_3$ can be assigned $9, 8$ (or 7), 6, respectively and the vertices in $\{y_1, y_2, y_3, y_4\}$ can be assigned each of the labels $0, 1, 2, 3$, it follows that

$$3f(x_1) + 2f(x_2) + f(x_3) + 2f(x_4) \le 63 \quad \text{and} \quad 2\sum_{i=1}^{4} f(y_i) \ge 12.$$

Then $\mathrm{val}_{\max}(C_8+e) = \mathrm{val}(f) \le 63 - 12 = 51$.

If $f(x_1) < f(x_r)$, with similar argument we can show that $\mathrm{val}_{\max}(C_8+e) \le 51$.

Case 2. *Some element of $ST(f)$ is monotone.* Then the oriented cycle $C_8(f)$ contains a directed path a, b, c of order 3. If we delete the chord $e = x_1x_r$ and vertex b from $C_8 + e$ and join the vertices a and c, the resulting graph G is isomorphic to C_7 and the restriction g of f to $V((C_8 + e) - x_1x_r) - \{b\}$ has the same value on G as f on $(C_8 + e) - x_1x_r$, that is

$$\text{val}(g) = \text{val}(f) - f'(x_1x_r) \geq \text{val}_{\max}(C_8 + e) - 9.$$

Then $\text{val}_{\max}(C_8 + e) \leq \text{val}(g) + 9$. Moreover, since g is a γ^2-labeling of a graph G that is isomorphic to C_7, it follows that $\text{val}(g) \leq \text{val}^2_{\max}(C_7)$. Therefore, by Theorem 2.3.2, $\text{val}_{\max}(C_8 + e) \leq 51$. □

As a consequence of Proposition 3.1.5 and Theorem 3.1.7, we have the following result.

Corollary 3.1.8. *For $n = 4, 6$ and 8,*

$$\text{val}_{\max}(C_n + e) = \frac{n^2 + 5n - 2}{2}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

Lemma 3.1.9. *For every even integer n, with $n \geq 8$, let f be a γ-max labeling of $C_n + e$ where e is a chord joining two vertices with even distance in even cycle C_n. If some element of $ST(f)$ is monotone, then there are exactly two monotone elements of $ST(f)$.*

Proof. Suppose that some element of $ST(f)$ is monotone. Then the oriented cycle $C_n(f)$ contains a directed path of order 3. Since the size of the oriented cycle $C_n(f)$ is even, it follows that there are ℓ directed paths of order 3 in $C_n(f)$, for some even integer l with $2 \leq l \leq n$. Next, we show that there are no more than two directed paths of order 3 in oriented cycle $C_n(f)$.

Assume, to the contrary, that there are at least 4 directed paths of order 3 in the oriented graph $C_n(f)$. For each i with $1 \leq i \leq 4$, let P_i be a directed path of order 3 having internal vertex u_i in the oriented graph $C_n(f)$. Then the cycle $(C_n + e) - x_1x_r$ is not only isomorphic to C_n but it is also a subdivision of the cycle C_{n-4}.

We can construct a graph G which is isomorphic to C_{n-4} obtained from $C_n + e$ by deleting the chord $e = x_1x_r$ and the internal vertices u_1, u_2, u_3, u_4 and then adding one or more edges to G. Then the restriction of f to $V((C_n+e)-x_1x_r)-\{u_1,u_2,u_3,u_4\}$, g, has the same value on G as f does on $(C_n+e)-x_1x_r$, that is

$$\mathrm{val}(g)=\mathrm{val}(f)-f'(x_1x_r)\geq \mathrm{val}_{\max}(C_n+e)-(n+1).$$

Then $\mathrm{val}_{\max}(C_n+e)\leq \mathrm{val}(g)+(n+1)$. Moreover, since g is a γ^5-labeling of a graph G which is isomorphic to C_{n-4}, it follows that $\mathrm{val}(g)\leq \mathrm{val}^5_{\max}(C_{n-4})$. Therefore,

$$\begin{aligned}
\mathrm{val}_{\max}(C_n+e) &\leq \mathrm{val}(g)+(n+1)\\
&\leq \mathrm{val}^5_{\max}(C_{n-4})+(n+1)\\
&= \tfrac{(n-4)((n-4)+2\cdot 5+2)}{2}+(n+1) \qquad \text{(by Theorem 2.3.2)}\\
&< \tfrac{n^2+6n-10}{2}.
\end{aligned}$$

This is a contradiction with Lemma 3.1.6. Thus, for $n\geq 8$, the oriented cycle $C_n(f)$ contains exactly two directed paths of order 3 and there are also exactly two monotone elements in $ST(f)$. □

By Theorem 3.1.7 and Lemma 3.1.9, we are able to characterize any γ-max labeling f of C_8+e, in terms of the set $ST(f)$ and the oriented cycle $C_8(f)$.

Proposition 3.1.10. *Let f be a γ-max labeling of C_8+e where e is a chord joining two vertices with even distance in cycle C_8. Then either no element of $ST(f)$ is monotone or there are exactly two monotone elements of $ST(f)$.*

Proposition 3.1.11. *For every even integer n with $n\geq 10$, let f be a γ-max labeling of C_n+e where e is a chord joining two vertices with even distance in even cycle C_n. Then $ST(f)$ contains exactly two monotone elements.*

Proof. Assume, to the contrary, that the property does not verify. Then, by Lemma 3.1.9, $ST(f)$ contains no monotone element. Hence, for the vertices x_i and y_i of the oriented cycle $C_n(f)$, **either** $\mathrm{id}(x_i)=0$, $\mathrm{id}(y_i)=2$ **or** $\mathrm{id}(x_i)=2$, $\mathrm{id}(y_i)=0$ for each i with $1\leq i\leq k$. We consider the two cases separately.

Case 1. *For each i with $1\leq i\leq k$, $id(x_i)=0$, $id(y_i)=2$ of the oriented cycle* $C_n(f)$. Thus $f(x_i)<f(y_i)$ and $f(x_{i+1})<f(y_i)$ where addition is taken modulo k. Then

$$\mathrm{val}_{\max}(C_n+e)=\mathrm{val}(f)=2\sum_{i=1}^{k}f(y_i)-2\sum_{i=1}^{k}f(x_i)+|f(x_1)-f(x_r)|.$$

If $f(x_1) > f(x_r)$, then

$$\mathrm{val}_{\max}(C_n+e) = \mathrm{val}(f) = 2\sum_{i=1}^{k} f(y_i) - \left(3f(x_r) + 2\sum_{\substack{2\le i\le k\\ i\neq r}} f(x_i) + f(x_1)\right).$$

The vertices in $\{y_1, y_2, \ldots, y_k\}$ and $\{x_1, x_2, \ldots, x_k\}$ can be assigned labels in the consecutive integers sets $\{k+2, k+3, \ldots, n+1\}$ and $\{0, 1, \ldots, k-1\}$, respectively. Moreover, x_r and x_1 can be assigned 0 and $k-1$, respectively. Hence,

$$\begin{aligned}\mathrm{val}_{\max}(C_n+e) = \mathrm{val}(f) &\le 2\sum_{i=k+2}^{n+1} i - \left(0 + 2\sum_{i=1}^{k-2} i + (k-1)\right)\\ &= n^2+3n-2k^2-k-1\\ &= \tfrac{n^2+5n-2}{2}.\end{aligned}$$

On the another hand, if $f(x_1) < f(x_r)$, with similar argument we can show that $\mathrm{val}_{\max}(C_n+e) \le \frac{n^2+5n-2}{2}$. However, by Lemma 3.1.6, we have $\mathrm{val}_{\max}(C_n+e) \ge \frac{n^2+6n-10}{2}$, which is a contradiction.

Case 2. For each i with $1 \le i \le k$, $id(x_i) = 2$, $id(y_i) = 0$ of the oriented cycle $C_n(f)$. Thus $f(x_i) > f(y_i)$ and $f(x_{i+1}) > f(y_i)$ where addition is taken modulo k. Then

$$\mathrm{val}_{\max}(C_n+e) = \mathrm{val}(f) = 2\sum_{i=1}^{k} f(x_i) - 2\sum_{i=1}^{k} f(y_i) + |f(x_1) - f(x_r)|.$$

First, if $f(x_1) > f(x_r)$. Then

$$\mathrm{val}_{\max}(C_n+e) = \mathrm{val}(f) = \left(3f(x_1) + 2\sum_{\substack{2\le i\le k\\ i\neq r}} f(x_i) + f(x_r)\right) - 2\sum_{i=1}^{k} f(y_i).$$

The vertices in $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ can be assigned labels in the consecutive integers sets $\{k+2, k+3, \ldots, n+1\}$ and $\{0, 1, \ldots, k-1\}$, respectively. Moreover, x_1 and x_r can be assigned $n+1$ and $k+2$, respectively. Therefore,

$$\begin{aligned}\mathrm{val}_{\max}(C_n+e) = \mathrm{val}(f) &\le \left(3(n+1) + 2\sum_{i=k+3}^{n} i + (k+2)\right) - 2\sum_{i=0}^{k-1} i\\ &= n^2+4n-2k^2-3k-1\\ &= \tfrac{n^2+5n-2}{2}.\end{aligned}$$

Next, if $f(x_1) < f(x_r)$, with similar argument we can show that $\mathrm{val}_{\max}(C_n+e) \le \frac{n^2+5n-2}{2}$. However, by Lemma 3.1.6, we have $\mathrm{val}_{\max}(C_n+e) \ge \frac{n^2+6n-10}{2}$, which is a contradiction. □

We are now prepared to establish a general formula for $\text{val}_{\max}(C_n + e)$, when $n \geq 10$.

Theorem 3.1.12. *For every even integer $n \geq 10$,*

$$\text{val}_{\max}(C_n + e) = \frac{n^2 + 6n - 10}{2}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

Proof. By Lemma 3.1.6, we have $\text{val}_{\max}(C_n + e) \geq \frac{n^2+6n-10}{2}$. Next, we show that $\text{val}_{\max}(C_n + e) \leq \frac{n^2+6n-10}{2}$. Let f be a γ-max labeling of $C_n + e$. By Proposition 3.1.11, the oriented cycle $C_n(f)$ contains a directed path a, b, c of order 3. If we delete the chord $e = x_1x_r$ and the vertex b from $C_n + e$ and then join the vertices a and c, the resulting graph G is isomorphic to C_{n-1} and the restriction g of f to $V((C_n + e) - x_1x_r) - \{b\}$, verifies

$$\text{val}(g) = \text{val}(f) - f'(x_1x_r) \geq \text{val}_{\max}(C_n + e) - (n + 1).$$

Therefore,

$$\text{val}_{\max}(C_n + e) \leq \text{val}(g) + (n + 1).$$

Moreover, since g is a γ^2-labeling of graph G that is isomorphic to C_{n-1}, it follows that $\text{val}(g) \leq \text{val}^2_{\max}(C_{n-1})$. Therefore,

$$\begin{aligned} \text{val}_{\max}(C_n + e) &\leq \text{val}^2_{\max}(C_{n-1}) + (n + 1) \\ &= \tfrac{((n-1)-1))((n-1)+2\cdot 2+3)}{2} + (n + 1) \\ &= \tfrac{n^2+6n-10}{2}. \end{aligned}$$

□

As a consequence of Corollary 3.1.8 and Theorem 3.1.12, we have the following result.

Corollary 3.1.13. *For every even integer $n \geq 4$,*

$$\text{val}_{\max}(C_n + e) = \begin{cases} \frac{n^2+5n-2}{2} & \text{if } n = 4, 6, 8 \\ \frac{n^2+6n-10}{2} & \text{if } n \geq 10 \end{cases}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

2. γ-spectra of cycles with one chord

In section 4 of chapter 2, Fonseca, Saenpholphat, and Zhang [4] and [9] determine the γ-spectra of some well-known classes of graphs. In this section, we study the γ-spectrum of cycle with one chord $C_n + e$. Observe that in Theorem 2.4.3, the value of any γ-labeling of cycle is always even. We show that the value of any γ-labeling of cycle with one chord is not the case.

Throughout this section, let $C_n + e$ be a cycle with one chord of order n which is obtained from

$$\text{a cycle } C_n : v_1, v_2, \ldots, v_{r-1}, v_r, v_{r+1}, \ldots, v_n, v_1 \text{ and a chord } e = v_1v_r,$$

where $3 \leq r \leq n-1$.

Since $C_n + v_1v_r$ is isomorphic to $C_n + v_1v_{n-r+2}$ for each r with $3 \leq r \leq n-1$, it is sufficient to determine that for each r with $3 \leq r \leq \lceil \frac{n+1}{2} \rceil$,

$$\text{spec}(C_n + v_1v_r) = \Big[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n + v_1v_r)\Big]. \qquad (3.2.1)$$

Observe that Theorem 3.1.2 provides that

$$\begin{aligned} \text{val}_{\min}(C_n + v_1v_r) &= (2n-2) + 1 \\ \text{val}_{\max}(C_n + v_1v_r) &= (2n-2) + \Big|\Big[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n + v_1v_r)\Big]\Big|. \end{aligned}$$

Therefore our goal is to find a γ-labeling f_l of $C_n + v_1v_r$ with $\text{val}(f_l) = (2n-2) + l$ for each integer l with

$$1 \leq l \leq \Big|\Big[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n + v_1v_r)\Big]\Big|.$$

The following propositions will be used to deduce a γ-labeling f_l of $C_n + v_1v_r$ for achieving the main result.

Proposition 3.2.1. *For every even integer $n \geq 6$,*

$$\text{spec}(C_n + e) = \Big[\text{val}_{\min}(C_n + e), \text{val}_{\max}(C_n + e)\Big] = \Big[2n-1, \frac{n^2 + 6n + 2}{2}\Big]$$

where e is a chord joining two vertices with odd distance in even cycle C_n.

Proof. By (3.2.1), we may assume that $C_n + e = C_n + v_1v_r$ where $3 \leq r \leq \frac{n}{2} + 1$ and r is even.

For each integer i with $0 \leq i \leq \frac{n}{2} - 1$, let

$$\Delta_i = \begin{cases} 2n+2 & \text{if } i = 0 \\ 2n-4i-1 & \text{if } 1 \leq i \leq \frac{n}{2} - 2 \\ \frac{n}{2}+3 & \text{if } i = \frac{n}{2} - 1. \end{cases}$$

We will show that, for each integer l with

$$1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i = \left| \left[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n + v_1v_r) \right] \right|,$$

there exists a γ-labeling f_l whose value is $(2n-2)+l$.

First, we consider $1 \leq l \leq \Delta_0$ as follows:

I. For $1 \leq l \leq 8$, we can define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} r-3 & \text{if } l = 2, 8 \\ r-2 & \text{if } l = 1, 4, 7 \\ r-1 & \text{if } l = 3, 6 \\ r & \text{if } l = 5 \end{cases}$$

$$f_l(v_i) = \begin{cases} r-i-1 & \text{if } 2 \leq i \leq r-2 \text{ and } l = 1, 3, 4, 5, 6, 7 \\ r-i-2 & \text{if } 2 \leq i \leq r-2 \text{ and } l = 2, 8 \end{cases}$$

$$f_l(v_{r-1}) = \begin{cases} 0 & \text{if } l = 1, 3, 4, 5, 6, 7 \\ r-2 & \text{if } l = 2, 8 \end{cases}$$

$$f_l(v_r) = \begin{cases} r-1 & \text{if } l = 1, 2 \\ r & \text{if } l = 3, 4 \\ r+1 & \text{if } l = 5, 6, 7, 8 \end{cases}$$

$$f_l(v_i) = f_l(v_r) + i - r \quad \text{if } r+1 \leq i \leq n \text{ and } 1 \leq l \leq 8.$$

If $l = 1, 3, 4, 5, 6$ and 7, then

$$\begin{aligned} \text{val}(f_l) &= -f_l(v_1) - 2f_l(v_{r-1}) + f_l(v_r) + 2f_l(v_n) \\ &= (2n-2)+l. \end{aligned}$$

If $l = 2$ and 8, then

$$\begin{aligned} \text{val}(f_l) &= -f_l(v_1) - 2f_l(v_{r-2}) + f_l(v_r) + 2f_l(v_n) \\ &= (2n-2) + l. \end{aligned}$$

II. For $9 \leq l \leq 2r+2$, let f_l be a γ-labeling of $C_n + v_1v_r$ defined by

$$f_l(v_1) = \begin{cases} r-2 & \text{if } l \text{ is even} \\ r-1 & \text{if } l \text{ is odd} \end{cases}$$

$$f_l(v_i) = \begin{cases} i-2 & \text{if } 2 \leq i \leq r-1 \\ r+2 & \text{if } i = r \\ i+2 & \text{if } r+1 \leq i \leq \lceil \frac{l}{2} \rceil + r - 5 \\ r+1 & \text{if } i = \lceil \frac{l}{2} \rceil + r - 4 \\ i+1 & \text{if } \lceil \frac{l}{2} \rceil + r - 3 \leq i \leq n. \end{cases}$$

If $l = 9$ and 10, then

$$\begin{aligned} \text{val}(f_l) &= -f_l(v_1) - 2f_l(v_2) + 3f_l(v_r) - 2f_l(v_{r+1}) + 2f_l(v_n) \\ &= (2n-2) + l. \end{aligned}$$

If $11 \leq l \leq 2r+2$, then

$$\begin{aligned} \text{val}(f_l) &= -f_l(v_1) - 2f_l(v_2) + f_l(v_r) + 2f_l\left(v_{\lceil \frac{l}{2} \rceil + r - 5}\right) - 2f_l\left(v_{\lceil \frac{l}{2} \rceil + r - 4}\right) + 2f_l(v_n) \\ &= (2n-2) + l. \end{aligned}$$

III. For $2r+3 \leq l \leq \Delta_0 = 2n+2$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_i) = \begin{cases} \lceil \frac{l-5}{2} \rceil & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \lceil \frac{l-6}{2} \rceil - r + i & \text{if } 3 \leq i \leq r-1 \\ \lceil \frac{l}{2} \rceil - r + i & \text{if } r \leq i \leq n + r - \lceil \frac{l+2}{2} \rceil \\ \lceil \frac{l-2}{2} \rceil - n - r + i & \text{if } n + r - \lceil \frac{l+2}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

If $2r+3 \leq l \leq 2n$, then

$$\begin{aligned} \text{val}(f_l) &= f_l(v_1) - 2f_l(v_2) + f_l(v_r) + 2f_l\left(v_{n+r-\lceil \frac{l}{2} \rceil + 1}\right) - 2f_l\left(v_{n+r-\lceil \frac{l}{2} \rceil + 2}\right) \\ &= (2n-2) + l. \end{aligned}$$

If $l = 2n+1$ and $2n+2$, then

$$\begin{aligned} \mathrm{val}(f_l) &= f_l(v_1) - 2f_l(v_2) + 3f_l(v_r) - 2f_l(v_{r+1}) \\ &= (2n-2) + l. \end{aligned}$$

Next, we consider $\Delta_0 + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, by letting (t, l_t) be a pair of integers with $1 \leq t \leq \frac{n}{2} - 1$ and $l_t = l - \sum_{i=0}^{t-1} \Delta_i$. Certainly, $1 \leq l_t \leq \Delta_t$. We construct a γ-labeling f_l as the following procedure:

IV. For $\Delta_0 + 1 \leq l \leq \sum_{i=0}^{\frac{n-r}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \neq \Delta_t - 3, \Delta_t - 1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \neq \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t - 3, \Delta_t - 1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 3 \leq i \leq r-1 \\ & \text{and } 1 \leq l_t \leq \Delta_t - (2r-1) \\ n+i-(t+r+2) & \text{if } 3 \leq i \leq f_l(v_{2t+r}) + t + r - n + 1 \\ & \text{and } \Delta_t - (2r-2) \leq l_t \leq \Delta_t - 7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r}) + t + r - n + 2 \leq i \leq r-1 \\ & \text{and } \Delta_t - (2r-2) \leq l_t \leq \Delta_t - 7 \\ n+i-(t+r+2) & \text{if } 3 \leq i \leq r-1 \text{ and } \Delta_t - 6 \leq l_t \leq \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is odd and } r \leq i \leq 2t+r+1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is even and } r \leq i \leq 2t+r-2 \end{cases}$$

$$f_l(v_{2t+r}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \leq l_t \leq \Delta_t - 5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t - 4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t - 3 \leq l_t \leq \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} i-(t+r-1) & \text{if } 2t+r+2 \le i \le n \text{ and } l_t = 1,2 \\ i-(t+r) & \text{if } 2t+r+2 \le i \le f_l(v_{2t+r})+t+r-1 \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r-1) & \text{if } f_l(v_{2t+r})+t+r \le i \le n \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r) & \text{if } 2t+r+2 \le i \le n \\ & \text{and } \Delta_t-2r \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f_l) &= f_l(v_1) - 2f_l(v_2) + 3f_l(v_r) - 2\sum_{\substack{r+1\le i\le 2t+r+1 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le 2t+r-2 \\ i \text{ is even}}} f_l(v_i) + 2f_l(v_{2t+r}) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r}) \\ &= (2n-2)+l. \end{aligned}$$

V. For $\sum_{i=0}^{\frac{n-r}{2}-1} \Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-3} \Delta_i$, now we construct a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \ne \Delta_t-3, \Delta_t-1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \ne \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t-3, \Delta_t-1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n-2+i-r}{2} & \text{if } i \text{ is even and } 3 \le i \le 2t+r-n+4 \\ \frac{n+5+r-i}{2} & \text{if } i \text{ is odd and } 3 \le i \le 2t+r-n+1 \end{cases}$$

$$f_l(v_{2t+r-n+3}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \le l_t \le \Delta_t - 5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t - 4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t - 3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 2t+r-n+5 \le i \le r-1 \\ & \text{and } 1 \le l_t \le 4 \\ n+i-(t+r+2) & \text{if } 2t+r-n+5 \le i \le f_l(v_{2t+r-n+3})+t+r-n+1 \\ & \text{and } 5 \le l_t \le \Delta_t - 7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r-n+3})+t+r-n+2 \le i \le r-1 \\ & \text{and } 5 \le l_t \le \Delta_t - 7 \\ n+i-(t+r+2) & \text{if } 2t+r-n+5 \le i \le r-1 \\ & \text{and } \Delta_t - 6 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is odd and } r \le i \le n-1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is even and } r \le i \le n-1 \end{cases}$$

$$f_l(v_n) = \begin{cases} t+3 & \text{if } l_t = 1,2 \\ t+2 & \text{if } 3 \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_l) &= f_l(v_1)+3f_l(v_r) - 2\sum_{\substack{2\le i\le 2t+r-n+4 \\ i \text{ is even}}} f_l(v_i) - 2\sum_{\substack{r+1\le i\le n-1 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{3\le i\le 2t+r-n+1 \\ i \text{ is odd}}} f_l(v_i) \\ &\quad +2f_l(v_{2t+r-n+3}) + 2\sum_{\substack{r+2\le i\le n-2 \\ i \text{ is even}}} f_l(v_i) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r-n+3}) \\ &= (2n-2)+l. \end{aligned}$$

VI. For $\sum_{i=0}^{\frac{n}{2}-3}\Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-2}\Delta_i$, observe that $\Delta_{\frac{n}{2}-2} = 7$. We define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} \frac{n}{2} - l_t & \text{if } l_t = 1,2,3 \\ \frac{n}{2} + 4 - l_t & \text{if } l_t = 4,5,6,7 \end{cases}$$

$$f_l(v_2) = \begin{cases} 0 & \text{if } l_t = 2,5 \\ \frac{n}{2} - 2 & \text{if } l_t = 1,3 \\ \frac{n}{2} - 1 & \text{if } l_t = 4,6,7 \end{cases}$$

$$f_l(v_3) = \begin{cases} 0 & \text{if } l_t = 1,4 \\ \frac{n}{2} - 1 & \text{if } l_t = 2,3 \\ \frac{n}{2} & \text{if } l_t = 5,6,7 \end{cases}$$

$$f_l(v_i) = \frac{n+4+r-i}{2} \quad \text{if } i \text{ is even and } 4 \le i \le r-2$$

$$f_l(v_i) = n + 1 + \left(\frac{i-r}{2}\right) \quad \text{if } i \text{ is even and } r \le i \le n$$

$$f_l(v_i) = \begin{cases} \frac{n}{2} & \text{if } i = 5 \text{ and } l_t = 1,2,3 \\ \frac{i-5}{2} & \text{if } i \text{ is odd, } 7 \le i \le n-3 \text{ and } l_t = 1,2,3 \\ \frac{i-3}{2} & \text{if } i \text{ is odd, } 5 \le i \le n-3 \text{ and } l_t = 4,5,6 \\ \frac{i-3}{2} & \text{if } i \text{ is odd, } 5 \le i \le n-5 \text{ and } l_t = 7 \\ 0 & \text{if } i = n-3 \text{ and } l_t = 7 \end{cases}$$

$$f_l(v_{n-1}) = \begin{cases} 0 & \text{if } l_t = 3,6 \\ \frac{n}{2} - 3 & \text{if } l_t = 1,2 \\ \frac{n}{2} - 2 & \text{if } l_t = 4,5,7. \end{cases}$$

Since $l_t = l - \sum_{i=0}^{\frac{n}{2}-3}\Delta_i = l - \frac{n^2}{2} - \frac{n}{2} + 7$, it follows that for each l with $\sum_{i=0}^{\frac{n}{2}-3}\Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-2}\Delta_i$,

$$\begin{aligned} \text{val}(f_l) &= \frac{n^2}{2} + \frac{5n}{2} - 9 + l_t \\ &= (2n-2) + l. \end{aligned}$$

VII. For $\sum_{i=0}^{\frac{n}{2}-2} \Delta_i + 1 \leq l \leq \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} \frac{n}{2} - 1 - l_t & \text{if } 1 \leq l_t \leq 3 \\ \frac{n}{2} + 3 - l_t & \text{if } 4 \leq l_t \leq \Delta_{\frac{n}{2}-1} \end{cases}$$

$$f_l(v_2) = \begin{cases} \frac{n}{2} & \text{if } 1 \leq l_t \leq 3 \\ \frac{n}{2} + 2 & \text{if } 4 \leq l_t \leq \Delta_{\frac{n}{2}-1} \end{cases}$$

$$f_l(v_i) = \tfrac{i-1}{2} \qquad \text{if } i \text{ is odd, } 3 \leq i \leq n-1 \text{ and } i \neq 2f_l(v_1)+1$$

$$f_l(v_{2f_l(v_1)+1}) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n+4+r-i}{2} & \text{if } i \text{ is even, } 4 \leq i \leq r-2 \text{ and } 1 \leq l_t \leq 3 \\ \frac{n+2+r-i}{2} & \text{if } i \text{ is even, } 4 \leq i \leq r-2 \text{ and } 4 \leq l_t \leq \Delta_{\frac{n}{2}} - 1 \end{cases}$$

$$f_l(v_i) = n + 1 - \left(\tfrac{r-i}{2}\right) \qquad \text{if } i \text{ is even and } r \leq i \leq n.$$

Then

$$\begin{aligned} \text{val}(f_l) &= -3f_l(v_1) + 2f_l(v_2) + 3f_l(v_r) - 2\sum_{\substack{3\leq i\leq n-1 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{4\leq i\leq r-2 \\ i \text{ is even}}} f_l(v_i) + 2\sum_{\substack{r+2\leq i\leq n \\ i \text{ is even}}} f_l(v_i) \\ &= -\tfrac{n^2}{4} + \tfrac{3n}{2} + 1 - f_l(v_1) + 2f_l(v_2) + 2\sum_{\substack{4\leq i\leq n \\ i \text{ is even}}} f(v_i) \\ &= (2n-2) + l. \end{aligned}$$

□

We now illustrate the proof of Proposition 3.2.1. Table 1 shows all variables in the proof of Proposition 3.2.1 that we use to find spec$(C_8 + v_1v_4)$.

Table 1 $\text{spec}(C_8 + v_1v_4)$

$\text{val}(f_l) = (2n-2) + l$ **of** $C_8 + v_1v_4 \in \left[2n-1, \frac{n^2+6n+2}{2}\right] = [15, 57]$

$\Delta_0 = 2n+2 = 18, 1 \leq l \leq 18$

l	γ-labeling f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\text{val}(f_l)$
1	2	1	0	3	4	5	6	7	15
2	1	0	2	3	4	5	6	7	16
3	3	1	0	4	5	6	7	8	17
4	2	1	0	4	5	6	7	8	18
5	4	1	0	5	6	7	8	9	19
6	3	1	0	5	6	7	8	9	20
7	2	1	0	5	6	7	8	9	21
8	1	0	2	5	6	7	8	9	22
9	3	0	1	6	5	7	8	9	23
10	2	0	1	6	5	7	8	9	24
11	3	0	2	6	7	8	9	1	25
12	4	0	2	6	7	8	9	1	26
13	4	0	3	7	8	9	1	2	27
14	5	0	3	7	8	9	1	2	28
15	5	0	4	8	9	1	2	3	29
16	6	0	4	8	9	1	2	3	30
17	6	0	5	9	1	2	3	4	31
18	7	0	5	9	1	2	3	4	32

$\Delta_1 = 2n - 4(1) - 1 = 11, 19 \leq l \leq 29$

l	l_1	γ-labeling f_l : v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\text{val}(f_l)$
19	1	6	0	5	9	1	3	2	4	33
20	2	7	0	5	9	1	3	2	4	34
21	3	6	0	5	9	1	4	2	3	35
22	4	7	0	5	9	1	4	2	3	36
23	5	6	0	4	9	1	5	2	3	37
24	6	7	0	4	9	1	5	2	3	38
25	7	8	0	4	9	1	5	2	3	39
26	8	5	0	4	9	1	7	2	3	40
27	9	6	0	4	9	1	7	2	3	41
28	10	5	0	4	9	1	8	2	3	42
29	11	6	0	4	9	1	8	2	3	43

$\text{val}(f_l) = (2n-2)+l$ **of** $C_8 + v_1v_4 \in \left[2n-1, \frac{n^2+6n+2}{2}\right] = [15, 57]$										
$\Delta_2 = 2n - 4(2) - 1 = 7, 30 \le l \le 36$										
l	l_2	*γ-labeling* f_l :	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\text{val}(f_l)$
30	1		3	2	0	9	4	8	1	7	44
31	2		2	0	3	9	4	8	1	7	45
32	3		1	2	3	9	4	8	0	7	46
33	4		4	3	0	9	1	8	2	7	47
34	5		3	0	4	9	1	8	2	7	48
35	6		2	3	4	9	1	8	0	7	49
36	7		1	3	4	9	0	8	2	7	50
$\Delta_3 = \frac{n}{2} + 3 = 7, 37 \le l \le 43$											
l	l_3	*γ-labeling* f_l :	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$\text{val}(f_l)$
37	1		2	4	1	9	0	8	3	7	51
38	2		1	4	0	9	2	8	3	7	52
39	3		0	4	1	9	2	8	3	7	53
40	4		3	6	1	9	2	8	0	7	54
41	5		2	6	1	9	0	8	3	7	55
42	6		1	6	0	9	2	8	3	7	56
43	7		0	6	1	9	2	8	3	7	57

Proposition 3.2.2. *For every even integer $n \ge 4$,*

$$\begin{aligned}\text{spec}(C_n + e) &= \left[\text{val}_{\min}(C_n+e), \text{val}_{\max}(C_n+e)\right] \\ &= \begin{cases} \left[2n-1, \frac{n^2+5n-2}{2}\right] & \text{if } n = 4,6,8 \\ \left[2n-1, \frac{n^2+6n-10}{2}\right] & \text{if } n \ge 10 \end{cases}\end{aligned}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

Proof. Assume by (3.2.1) that $C_n + e = C_n + v_1v_r$ where $3 \le r \le \frac{n}{2} + 1$ and r is odd. First we consider $C_n + v_1v_r$ when $n \ge 8$. For each integer i with $0 \le i \le \frac{n}{2} - 1$, let

$$\Delta_i = \begin{cases} 2n+2 & \text{if } i = 0 \\ 2n - 4i - 1 & \text{if } 1 \le i \le \frac{n}{2} - 2 \\ \frac{n}{2} - 3 & \text{if } i = \frac{n}{2} - 1. \end{cases}$$

For each integer l with

$$1 \le l \le \sum_{i=0}^{\frac{n}{2}-1} \Delta_i = \left|\left[\text{val}_{\min}(C_n + v_1v_r), \text{val}_{\max}(C_n+v_1v_r)\right]\right|,$$

we show that there is a γ-labeling f_l whose value is $(2n-2)+l$.

1. For $1 \le l \le \Delta_0 = 2n+2$, a γ-labeling f_l is defined in similar way as described in **I.**, **II.**, and **III.** of Proposition 3.2.1.

Now we consider $\Delta_0 + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, by letting (t, l_t) be a pair of integers with $1 \le t \le \frac{n}{2} - 1$ and $l_t = l - \sum_{i=0}^{t-1} \Delta_i$, that is, $1 \le l_t \le \Delta_t$. A γ-labeling f_l can be constructed as the following procedure:

2. For $\Delta_0 + 1 \le l \le \sum_{i=0}^{\frac{n-r-1}{2}} \Delta_i$ of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \neq \Delta_t - 3, \Delta_t - 1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \neq \Delta_t - 4 \\ n-(t+2) & \text{if } l_t = \Delta_t - 3, \Delta_t - 1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 3 \le i \le r-1 \text{ and } 1 \le l_t \le \Delta_t - (2r-1) \\ n+i-(t+r+2) & \text{if } 3 \le i \le f_l(v_{2t+r}) + t + r - n + 1 \\ & \text{and } \Delta_t - (2r-2) \le l_t \le \Delta_t - 7 \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r}) + t + r - n + 2 \le i \le r-1 \\ & \text{and } \Delta_t - (2r-2) \le l_t \le \Delta_t - 7 \\ n+i-(t+r+2) & \text{if } 3 \le i \le r-1 \text{ and } \Delta_t - 6 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is even and } r \le i \le 2t+r+1 \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is odd and } r \le i \le 2t+r-2 \end{cases}$$

$$f_l(v_{2t+r}) = \begin{cases} \left\lceil \frac{l_t+2}{2} \right\rceil + t & \text{if } 1 \le l_t \le \Delta_t - 5 \\ \left\lceil \frac{l_t}{2} \right\rceil + t & \text{if } l_t = \Delta_t - 4 \\ \left\lceil \frac{l_t+3}{2} \right\rceil + t & \text{if } \Delta_t - 3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} i-(t+r-1) & \text{if } 2t+r+2 \le i \le n \text{ and } l_t = 1,2 \\ i-(t+r) & \text{if } 2t+r+2 \le i \le f_l(v_{2t+r})+t+r-1 \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r-1) & \text{if } f_l(v_{2t+r})+t+r \le i \le n \\ & \text{and } 3 \le l_t \le \Delta_t-(2r+1) \\ i-(t+r) & \text{if } 2t+r+2 \le i \le n \text{ and } \Delta_t-2r \le l_t \le \Delta_t. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f_l) &= f_l(v_1)-2f_l(v_2)+3f_l(v_r) - 2\sum_{\substack{r+1\le i\le 2t+r+1 \\ i \text{ is even}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le 2t+r-2 \\ i \text{ is odd}}} f_l(v_i) + 2f_l(v_{2t+r}) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r}) \\ &= (2n-2)+l. \end{aligned}$$

3. For $\sum_{i=0}^{\frac{n-r-1}{2}} \Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-2} \Delta_i$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$f_l(v_1) = \begin{cases} n-(t-1) & \text{if } l_t = \Delta_t - 4 \\ n-t & \text{if } l_t \text{ is even and } l_t \ne \Delta_t-3, \Delta_t-1 \\ n-(t+1) & \text{if } l_t \text{ is odd and } l_t \ne \Delta_t-4 \\ n-(t+2) & \text{if } l_t = \Delta_t-3, \Delta_t-1 \end{cases}$$

$$f_l(v_2) = 0$$

$$f_l(v_i) = \begin{cases} \frac{n-1+i-r}{2} & \text{if } i \text{ is even and } 3 \le i \le 2t+r-n+3 \\ \frac{n+4+r-i}{2} & \text{if } i \text{ is odd and } 3 \le i \le 2t+r-n \end{cases}$$

$$f_l(v_{2t+r-n+2}) = \begin{cases} \lceil \frac{l_t+2}{2} \rceil + t & \text{if } 1 \le l_t \le \Delta_t-5 \\ \lceil \frac{l_t}{2} \rceil + t & \text{if } l_t = \Delta_t-4 \\ \lceil \frac{l_t+3}{2} \rceil + t & \text{if } \Delta_t-3 \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} n+i-(t+r+1) & \text{if } 2t+r-n+4 \le i \le r-1 \text{ and } l_t = 1,2 \\ n+i-(t+r+2) & \text{if } 2t+r-n+4 \le i \le f_l(v_{2t+r-n+2})+t+r-n+1 \\ & \text{and } 3 \le l_t \le \Delta_t - r \\ n+i-(t+r+1) & \text{if } f_l(v_{2t+r-n+2})+t+r-n+2 \le i \le r-1 \\ & \text{and } 3 \le l_t \le \Delta_t - r \\ n+i-(t+r+2) & \text{if } 2t+r-n+4 \le i \le r-1 \\ & \text{and } \Delta_t-(r-1) \le l_t \le \Delta_t \end{cases}$$

$$f_l(v_i) = \begin{cases} \frac{i-r+1}{2} & \text{if } i \text{ is even and } r \le i \le n \\ n+1-\left(\frac{i-r}{2}\right) & \text{if } i \text{ is odd and } r \le i \le n. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f_l) &= f_l(v_1)+3f_l(v_r) - 2\sum_{\substack{2\le i\le 2t+r-n+3 \\ i \text{ is even}}} f_l(v_i) - 2\sum_{\substack{r+1\le i\le n \\ i \text{ is even}}} f_l(v_i) + 2\sum_{\substack{3\le i\le 2t+r-n \\ i \text{ is odd}}} f_l(v_i) \\ &= 2nt+n-2t^2-1+f_l(v_1)+2f_l(v_{2t+r-n+2}) \\ &= (2n-2)+l. \end{aligned}$$

4. For $\sum_{i=0}^{\frac{n}{2}-2} \Delta_i + 1 \le l \le \sum_{i=0}^{\frac{n}{2}-1} \Delta_i$, define a γ-labeling f_l of $C_n + v_1v_r$ by

$$\begin{aligned} f_l(v_1) &= \tfrac{n}{2}-3-l_t \\ f_l(v_2) &= \tfrac{n}{2} \\ f_l(v_i) &= \tfrac{n+4+r-i}{2} && \text{if } i \text{ is odd, } 3 \le i \le r-2 \\ f_l(v_i) &= \tfrac{i}{2}-1 && \text{if } i \text{ is even, } 4 \le i \le n \text{ and } i \ne 2f_l(v_1)+2 \\ f_l(v_{2f_l(v_1)+2}) &= 0 \\ f_l(v_i) &= n+1+\tfrac{r-i}{2} && \text{if } i \text{ is odd, } r \le i \le n-1. \end{aligned}$$

Then

$$\begin{aligned} \text{val}(f_l) &= -3f_l(v_1)+3f_l(v_r) + 2\sum_{\substack{3\le i\le r-2 \\ i \text{ is odd}}} f_l(v_i) + 2\sum_{\substack{r+2\le i\le n-1 \\ i \text{ is odd}}} f_l(v_i) - 2\sum_{\substack{4\le i\le n-2 \\ i \text{ is even}}} f_l(v_i) \\ &= \tfrac{n^2}{2}+\tfrac{5n}{2}-2+l-\tfrac{n^2}{2}-\tfrac{n}{2} \\ &= (2n-2)+l. \end{aligned}$$

Next, we consider $C_4+v_1v_3$, by letting $\Delta_0 = 10$ and $\Delta_1 = 1$. Therefore $\Delta_0+\Delta_1 = |[\mathrm{val}_{\min}(C_4 + v_1v_3), \mathrm{val}_{\max}(C_4 + v_1v_3)]|$.

If $1 \le l \le \Delta_0 = 10$, then we define a γ-labeling f_l similar to γ-labeling f_l in **1.** of $C_n + v_1v_r$ when $n \ge 8$.

Otherwise, if $l = \Delta_0 + \Delta_1 = 11$, then a γ-labeling f_l of $C_4 + v_1v_3$ defined by $f_l(v_1) = 4, f_l(v_2) = 0, f_l(v_3) = 5,$ and $f_l(v_4) = 1$ has $\mathrm{val}(f_l) = (2n - 2) + l$.

Last, we consider $C_6 + v_1v_3$, by letting $\Delta_0 = 14, \Delta_1 = 7$ and $\Delta_2 = 1$. Then $\Delta_0 + \Delta_1 + \Delta_2 = |[\mathrm{val}_{\min}(C_6 + v_1v_3), \mathrm{val}_{\max}(C_6 + v_1v_3)]|$.

If $1 \le l \le \Delta_0 + \Delta_1 = 21$, then we define a γ-labeling f_l similar to γ-labeling f_l in **1.** of $C_n + v_1v_r$ when $n \ge 8$.

Otherwise, if $l = \Delta_0 + \Delta_1 + \Delta_2 = 22$, then define a γ-labeling f_l of $C_6 + v_1v_3$ by $f_l(v_1) = 5, f_l(v_2) = 0, f_l(v_3) = 7, f_l(v_4) = 1, f_l(v_5) = 6,$ and $f_l(v_6) = 2$ having $\mathrm{val}(f_l) = (2n - 2) + l$. □

Proposition 3.2.3. *For every odd integer $n \ge 5$,*

$$\mathrm{spec}(C_n + e) = \Big[\mathrm{val}_{\min}(C_n + e), \mathrm{val}_{\max}(C_n + e)\Big] = \left[2n - 1, \frac{n^2 + 6n - 3}{2}\right].$$

The proof of Proposition 3.2.3 is similar to Propositions 3.2.1 and 3.2.2 and is therefore omitted.

As a consequence of Propositions 3.2.1 - 3.2.3, we have the main result.

Theorem 3.2.4. *For every integer $n \ge 4$,*

$$\mathrm{spec}(C_n + e) = \Big[\mathrm{val}_{\min}(C_n + e), \mathrm{val}_{\max}(C_n + e)\Big].$$

CHAPTER 4

CHARACTERIZATION OF γ-LABELINGS OF GRAPHS

This chapter containing three main sections, is to present our comprehensive work concerning the γ-labelings of graphs. The first section, we characterize the γ-max labeling of a graph with exterior major vertices, and also determine the maximum value of a γ-labeling of a tree with a unique exterior major vertex and a tree T of $\text{ma}(T) = 2$ with adjacent exterior major vertices. For the second section, we give an alternative proof by mathematical induction to achieve the formulae for $\text{val}_{\max}(K_{r,s})$ and $\text{val}_{\max}(K_n)$. In the last section, we study a connected graph having the unique γ-min labeling. The minimum value of a γ-labeling is determined for some classes of trees. Spontaneously, we are able to find that they have no unique γ-min labeling.

First, let us introduce some additional definitions and notation.

A vertex of degree at least 3 in a graph G is called a *major vertex* of G. The *major degree* $\text{ma}(G)$ of a graph G is the number of major vertices of G. Any end-vertex z of G is said to be a *terminal vertex* of a major vertex v of G if $d(z, v) < d(z, w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of a graph G is an *exterior major vertex* of G if it has positive terminal degree.

For an exterior major vertex v in G, an *exneighbor* of v is a vertex that is in neighborhood $N(v)$ and also in a $v - z$ path for some terminal vertex z of v in G. The *exneighborhood* $N^e(v)$ of v is a set of exneighbors of v. For an exterior major vertex v in G, an *exbranch* G_v of v is a minimal subgraph of G containing all paths from v to every its terminal vertex.

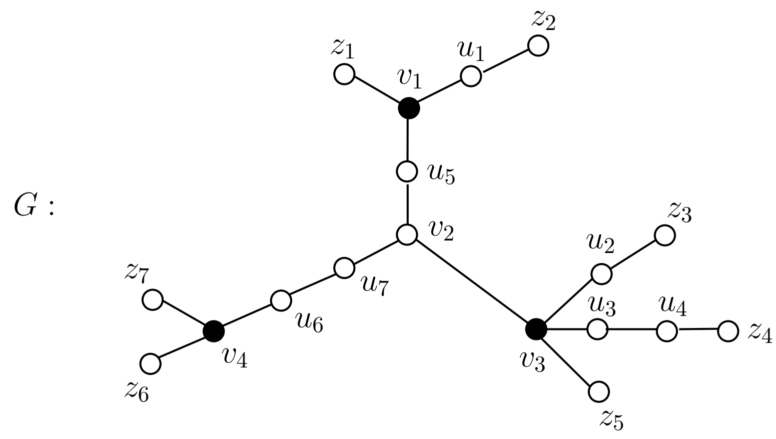

Figure 6 The graph G

For example, the graph G of Figure 6 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are z_1 and z_2, the terminal vertices of v_3 are z_3, z_4 and z_5, and the terminal vertices of v_4 are z_6 and z_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of G. Thus G has three exterior major vertices v_1, v_3 and v_4, where $ter(v_1) = 2, ter(v_3) = 3$ and $ter(v_4) = 2$. The exneighborhoods $N^e(v_1) = \{z_1, u_1\}, N^e(v_3) = \{u_2, u_3, z_5\}$ and $N^e(v_4) = \{z_6, z_7\}$. The exbranches G_{v_1}, G_{v_3} and G_{v_4} are shown in Figure 7

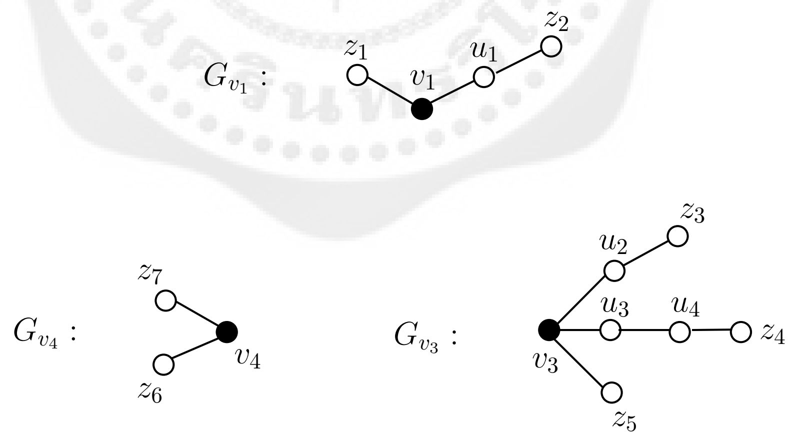

Figure 7 The exbranches G_{v_1}, G_{v_3} and G_{v_4}

Let D be an oriented graph of a graph G. A vertex of indegree 0 in D is called a *transmitter* of D. A vertex of outdegree 0 in D is called a *receiver* of D. The *outneighbor* of a vertex v in D is a vertex that is adjacent from v in D, and the *outneighborhood* $N^-(v)$ of v is a set of outneighbors of v. Then $N^-(v) = \{x \mid (v,x) \in E(D)\}$. The *inneighbor* of a vertex v in D is a vertex that is adjacent to v in D, and the *inneighborhood* $N^+(v)$ of v is a set of inneighbors of v. Then $N^+(v) = \{x \mid (x,v) \in E(D)\}$. So, if v is a transmitter of D, then $N^+(v) = \emptyset$, while if v is a receiver of D, then $N^-(v) = \emptyset$.

For a γ-labeling f of a graph G, a γ*-orientation* $D(f)$ of f is an oriented graph derived from a γ-labeling f of G, by assigning to each edge xy the orientation (x,y) if $f(x) < f(y)$.

$D(f)$:

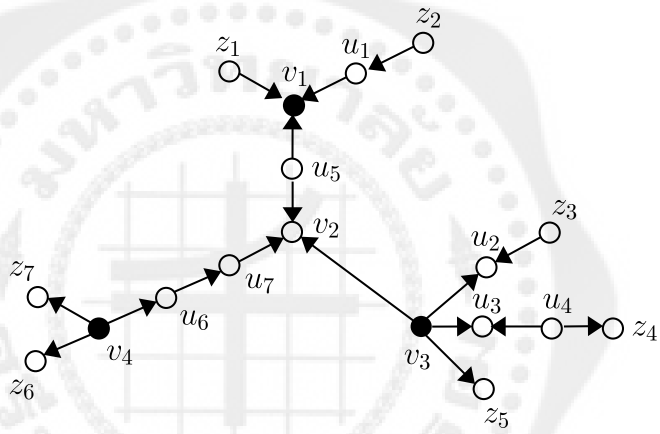

Figure 8 The γ-orientation $D(f)$

For example, the γ-orientation $D(f)$ of Figure 8 has seven transmitters, namely, $u_4, u_5, v_3, v_4, z_1, z_2, z_3$ and $N^-(u_4) = \{u_3, z_4\}, N^-(u_5) = \{v_1, v_2\}, N^-(v_3) = \{u_2, u_3, v_2, z_5\}$, $N^-(v_4) = \{u_6, z_6, z_7\}, N^-(z_1) = \{v_1\}, N^-(z_2) = \{u_1\}, N^-(z_3) = \{u_2\}$ and $N^+(u) = \emptyset$ where $u = u_4, u_5, v_3, v_4, z_1, z_2, z_3$. The γ-orientation $D(f)$ has eight receivers, namely, $u_2, u_3, v_1, v_2, z_4, z_5, z_6, z_7$ and $N^-(u) = \emptyset$ where $u = u_2, u_3, v_1, v_2, z_4, z_5, z_6, z_7$ and $N^+(u_2) = \{v_3, z_3\}, N^+(u_3) = \{u_4, v_3\}, N^+(v_1) = \{u_1, u_5, z_1\}, N^+(v_2) = \{u_5, u_7, v_3\}, N^+(z_4) = \{u_4\}$, $N^+(z_5) = \{v_3\}, N^+(z_6) = N^+(z_7) = \{v_4\}$. However, u_1, u_6, u_7 are neither transmitters nor receivers in $D(f)$, and $N^-(u_1) = \{v_1\}, N^+(u_1) = \{z_2\}, N^-(u_6) = \{u_7\}, N^+(u_6) = \{v_4\}$ and $N^-(u_7) = \{v_2\}, N^+(u_7) = \{u_6\}$.

We have a following result.

Theorem A *Let $D(f)$ be a γ-orientation of a γ-labeling f of a graph G. Then*

$$\text{val}(f) = \sum_{v \in V(D(f))} (\text{id}\, v - \text{od}\, v) f(v)\,.$$

1. γ-max labelings of graphs with exterior major vertices

In this section, we characterize a γ-orientation $D(f)$ of a γ-max labeling f of a graph G with exterior major vertices. Furthermore, we determine the maximum value of a γ-labeling of a tree with a unique exterior major vertex and also a tree T of $\text{ma}(T) = 2$ with adjacent exterior major vertices.

1.1 γ-orientations of γ-max labelings of graphs with exterior major vertices

First, we consider a γ-orientation $D(f)$ of a γ-max labeling f of a graph G with exterior major vertices according to the kind of vertices of G as follows.

Proposition 4.1.1. *Let $D(f)$ be a γ-orientation of a γ-*max *labeling f of a graph G. For every pair of an exterior major vertex v and its terminal vertex z, an internal vertex of $v - z$ path in G is either a transmitter or a receiver in $D(f)$.*

Proof. Let $P : v = u_0, u_1, \ldots, u_d = z$ be a $v - z$ path in a graph G where v is an exterior major vertex and z is a terminal vertex of v. Assume, to the contrary, that there is an internal vertex u_t for some integer $1 \le t \le d-1$ such that $\text{id}\, u_t = 1$ and $\text{od}\, u_t = 1$ in a γ-orientation $D(f)$ of a γ-max labeling f of G. Let g be a γ-labeling of G defined by

$$g(a) = \begin{cases} f(a) & \text{if } a \in V(G) - \{u_i \mid t \le i \le d\} \\ f(u_{i+1}) & \text{if } a = u_i \text{ and } t \le i \le d-1 \\ f(u_t) & \text{if } a = u_d\,. \end{cases}$$

Since $f(u_{t-1}) < f(u_t) < f(u_{t+1})$ or $f(u_{t+1}) < f(u_t) < f(u_{t-1})$, it follows that $\text{val}(g) = \text{val}(f) + |f(u_t) - f(u_d)| > \text{val}(f)$, which is a contradiction. $\square$

Observation 4.1.2. *For any γ-orientation $D(f)$ of a γ-labeling f of a graph G, each terminal vertex of G is either a transmitter or a receiver in $D(f)$.*

Proposition 4.1.3. *Let v be an exterior major vertex of a graph G. For any γ-orientation $D(f)$ of a γ*-max *labeling f of G, all vertices of exneighborhood $N^e(v)$ are either transmitters or receivers in $D(f)$, that is,*

$$N^e(v) \subseteq N^+(v) \text{ or } N^e(v) \subseteq N^-(v) \text{ in } D(f).$$

Proof. Let $D(f)$ be a γ-orientation of a γ-max labeling f of a graph G. Assume, to the contrary, that for an exterior major vertex v of G, $N^-(v) \cap N^e(v) \neq \emptyset$ and $N^+(v) \cap N^e(v) \neq \emptyset$ in $D(f)$. By Observation 1.3.1, we may let $1 \leq k = \operatorname{id} v \leq \operatorname{od} v = l$ in $D(f)$. Let $N^+(v) = \{x_1, x_2, \ldots, x_{k_1}, y_1, y_2, \ldots, y_{k_2}, z_1, z_2, \ldots, z_{k_3}\}$ in $D(f)$ whose all of vertices $x_1, x_2, \ldots, x_{k_1}$ are in $N^+(v) \cap N^e(v)$ with

$$f(x_1) < f(x_2) < \cdots < f(x_{k_1})$$

and all of vertices $y_1, y_2, \ldots, y_{k_2}, z_1, z_2, \ldots, z_{k_3}$ are in $N^+(v) - N^e(v)$ with

$$f(x_1) < f(y_1) < f(y_2) < \cdots < f(y_{k_2})$$

and

$$f(z_1) < f(z_2) < \cdots < f(z_{k_3}) < f(x_1)$$

where $k = k_1 + k_2 + k_3$, $k_1 \geq 1$ and $k_2, k_3 \geq 0$. Let $P : v = u_0, x_1 = u_1, u_2, \ldots, u_d$ be a $v - u_d$ geodesic containing a vertex x_1 and a terminal vertex u_d of v in G. Let g be a γ-labeling of G defined by

$$g(a) = \begin{cases} f(a) & \text{if } a \in V(G) - V(P) \\ f(u_{i+1}) & \text{if } a = u_i \text{ and } 0 \leq i \leq d-1 \\ f(v) & \text{if } a = u_d\,. \end{cases}$$

Then

$$
\begin{aligned}
\mathrm{val}(g) &= \mathrm{val}(f) + l(f(v) - f(x_1)) - (f(v) - f(x_1)) + |f(v) - f(u_d)| \\
&\quad - \sum_{i=2}^{k_1}(f(v) - f(x_i)) + \sum_{i=2}^{k_1}(f(x_i) - f(x_1)) \\
&\quad - \sum_{i=1}^{k_2}(f(v) - f(y_i)) + \sum_{i=1}^{k_2}(f(y_i) - f(x_1)) - k_3(f(v) - f(x_1)) \\
&> \mathrm{val}(f) + (l-1)(f(v) - f(x_1)) + |f(v) - f(u_d)| \\
&\quad -(k_1 - 1)f(v) + (k_1 - 1)f(x_1) + (k_1 - 1)f(x_1) - (k_1 - 1)f(x_1) \\
&\quad -k_2 f(v) + k_2 f(x_1) + k_2 f(x_1) - k_2 f(x_1) - k_3 f(v) + k_3 f(x_1) \\
&= \mathrm{val}(f) + (l - 1 - k_1 + 1 - k_2 - k_3)f(v) \\
&\quad -(l - 1 - k_1 + 1 - k_2 - k_3)f(x_1) + |f(v) - f(u_d)| \\
&= \mathrm{val}(f) + (l - k)(f(v) - f(x_1)) + |f(v) - f(u_d)| \\
&> \mathrm{val}(f)\,,
\end{aligned}
$$

which contradicts the fact that $\mathrm{val}(f) = \mathrm{val}_{\max}(G)$. □

On the other hand, Proposition 4.1.3 shows that for any γ-orientation $D(f)$ of a γ-max labeling f of a graph G, an exterior major vertex v in G is either a transmitter or a receiver in $D(f|_{V(G_v)})$.

Combining Propositions 4.1.1 and 4.1.3 and Observation 4.1.2, we have the following theorem providing a characterization of a γ-orientation of a γ-max labeling of a graph with exterior major vertices.

Theorem 4.1.4. *Let $D(f)$ be a γ-orientation of a γ-*max *labeling f of a graph G. Then each vertex in an exbranch G_v of an exterior major vertex v is either a transmitter or a receiver in $D(f)$.*

An immediate consequence of Theorem 4.1.4 is stated next.

Corollary 4.1.5. *Let $D(f)$ be a γ-orientation of a γ-*max *labeling f of a graph G. For an exbranch G_v of an exterior major vertex v in G, let*

$$X = \{x \in V(G_v) \mid \mathrm{id}\, x \le \mathrm{od}\, x \ \textit{in } D(f)\}$$

and

$$Y = \{y \in V(G_v) \mid \mathrm{od}\, y < \mathrm{id}\, y \ \textit{in } D(f)\}.$$

Then $f(x) < f(y)$ for each pair of vertices $x \in X$ and $y \in Y$.

Proof. Let $D(f)$ be a γ-orientation of a γ-max labeling f of a graph G. Let $x \in X$ and $y \in Y$. If x and y are adjacent vertices, by Theorem 4.1.4, it is clear that $f(x) < f(y)$. We may assume that x and y are non-adjacent vertices. Assume, to the contrary, that $f(y) < f(x)$. By Theorem 4.1.4, every vertex in $G_v - v$ is either a transmitter or a receiver in $D(f)$. Note that a transmitter in $G_v - v$ is in X and a receiver in $G_v - v$ is in Y. Moreover, the vertex v is in X if $\operatorname{id} v \le \operatorname{od} v$ or the vertex v is in Y if $\operatorname{od} v < \operatorname{id} v$. By Observation 1.3.1, we may let $k = \operatorname{id} v \le \operatorname{od} v = l$. Then $v \in X$. Let g be a γ-labeling of G defined by

$$g(a) = \begin{cases} f(a) & \text{if } a \neq x, y \\ f(y) & \text{if } a = x \\ f(x) & \text{if } a = y\,. \end{cases}$$

If $x \neq v$, then $\operatorname{val}(g) = \operatorname{val}(f) + (\deg x + \deg y)(f(x) - f(y)) > \operatorname{val}(f)$, which is a contradiction. Assume that $x = v$. Let $N^+(v) = \{y_1, y_2, \ldots, y_{k_1}, z_1, z_2, \ldots, z_{k_2}\}$ in $D(f)$ such that

$$f(y_i) < f(y) < f(z_j) \text{ for all integers } 1 \le i \le k_1, 1 \le j \le k_2$$

where $k = k_1 + k_2$ and $k_1, k_2 \ge 0$.

Then

$$\begin{aligned}
\operatorname{val}(g) &= \operatorname{val}(f) + l(f(v) - f(y)) + \deg y(f(v) - f(y)) - \sum_{i=1}^{k_1}(f(v) - f(y_i)) \\
&\quad + \sum_{i=1}^{k_1}(f(y) - f(y_i)) - \sum_{i=1}^{k_2}(f(v) - f(z_i)) + \sum_{i=1}^{k_2}(f(z_i) - f(y)) \\
&= \operatorname{val}(f) + (l + \deg y)(f(v) - f(y)) - k_1 f(v) + \sum_{i=1}^{k_1} f(y_i) + k_1 f(y) \\
&\quad - \sum_{i=1}^{k_1} f(y_i) - k_2 f(v) + \sum_{i=1}^{k_2} f(z_i) + \sum_{i=1}^{k_2} f(z_i) - k_2 f(y) \\
&> \operatorname{val}(f) + (l + \deg y)(f(v) - f(y)) - k_1(f(v) - f(y)) - k_2 f(v) + k_2 f(y) \\
&\quad + k_2 f(y) - k_2 f(y) \\
&= \operatorname{val}(f) + (l + \deg y)(f(v) - f(y)) - k_1(f(v) - f(y)) - k_2(f(v) - f(y)) \\
&= \operatorname{val}(f) + (l + \deg y)(f(v) - f(y)) - (k_1 + k_2)(f(v) - f(y)) \\
&= \operatorname{val}(f) + (l + \deg y)(f(v) - f(y)) - k(f(v) - f(y)) \\
&= \operatorname{val}(f) + (l - k + \deg y)(f(v) - f(y)) \\
&> \operatorname{val}(f)\,, \text{ which produces a contradiction.}
\end{aligned}$$

□

1.2 Maximum values of trees with a unique exterior major vertex

First, we present the minimum value of a tree T with a unique exterior major vertex of terminal degree 3 in [5].

Theorem 4.1.6 ([5])**.** *Let T be a tree of order n with a unique exterior major vertex of terminal degree 3. If d is the distance between v and a nearest end-vertex, then*

$$\mathrm{val}_{\min}(T) = n + d - 1\,.$$

We now present the lower bound of maximum value of a tree with a unique exterior major vertex.

Proposition 4.1.7. *Let T be a tree of order n with a unique exterior major vertex of terminal degree 3. Then*

$$\mathrm{val}_{\max}(T) \geq \begin{cases} \frac{n^2+n-8}{2} & \text{if the distances between } v \text{ and} \\ & \text{all its terminal vertices are even or odd} \\ \frac{n^2+n-6}{2} & \text{otherwise}\,. \end{cases}$$

Proof. Since a tree T is a connected bipartite graph, it is observed that there are unique partite sets of a connected bipartite graph T, say U and W. Without loss of generality, let $v \in W$. Since T has a unique exterior major vertex v of terminal degree 3, let x, y and z be terminal vertices of v.

We consider four cases according to the distances between v and all its terminal vertices. Note that for the pair $u, w \in V(T)$, u and w are in the same partite set if and only if the distance between u and w is even. In other words, u and w are in different partite sets if and only if the distance between u and w is odd.

Case 1. *The distances between v and all its terminal vertices are even.*
That is, $x, y, z \in W$. Then $|W| = \lceil \frac{n}{2} \rceil$ and $|U| = \lfloor \frac{n}{2} \rfloor$.
Let $W = \{v = w_1, w_2, \ldots, w_{\frac{n-5}{2}}, x, y, z\}$ and $U = \{u_1, u_2, \ldots, u_{\frac{n-1}{2}}\}$.

Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} n-i & \text{if } a = w_i \text{ and } 1 \le i \le \frac{n-5}{2} \\ i-1 & \text{if } a = u_i \text{ and } 1 \le i \le \frac{n-1}{2} \\ \frac{n+3}{2} & \text{if } a = x \\ \frac{n+1}{2} & \text{if } a = y \\ \frac{n-1}{2} & \text{if } a = z\,. \end{cases}$$

Then $f(u) < f(w)$ for all $u \in U$ and $w \in W$. Thus,

$$\begin{aligned} \text{val}(f) &= \sum_{w \in W} \deg w \cdot f(w) - \sum_{u \in U} \deg u \cdot f(u) \\ &= \left(2 \sum_{i=1}^{\frac{n-5}{2}} (n-i) + (n-1) + \left(\tfrac{n+3}{2}\right) + \left(\tfrac{n+1}{2}\right) + \left(\tfrac{n-1}{2}\right) \right) - 2 \sum_{i=1}^{\frac{n-1}{2}} (i-1) \\ &= \tfrac{n^2+n-8}{2}\,. \end{aligned}$$

Therefore $\text{val}_{\max}(T) \ge \text{val}(f) = \frac{n^2+n-8}{2}$.

Case 2. *The distances between v and all its terminal vertices are odd.*

That is, $x, y, z \in U$. Then $|W| = \frac{n}{2} - 1$ and $|U| = \frac{n}{2} + 1$.

Let $W = \{v = w_1, w_2, \ldots, w_{\frac{n}{2}-1}\}$ and $U = \{u_1, u_2, \ldots, u_{\frac{n}{2}-2}, x, y, z\}$.

Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} n-i & \text{if } a = w_i \text{ and } 1 \le i \le \frac{n}{2} - 1 \\ i-1 & \text{if } a = u_i \text{ and } 1 \le i \le \frac{n}{2} - 2 \\ \frac{n}{2} - 2 & \text{if } a = x \\ \frac{n}{2} - 1 & \text{if } a = y \\ \frac{n}{2} & \text{if } a = z\,. \end{cases}$$

Then $f(u) < f(w)$ for all $u \in U$ and $w \in W$. Thus,

$$\begin{aligned} \text{val}(f) &= \sum_{w \in W} \deg w \cdot f(w) - \sum_{u \in U} \deg u \cdot f(u) \\ &= \left(2 \sum_{i=1}^{\frac{n}{2}-1} (n-i) + (n-1) \right) - \left(2 \sum_{i=1}^{\frac{n}{2}-2} (i-1) + \left(\tfrac{n}{2} - 2\right) + \left(\tfrac{n}{2} - 1\right) + \tfrac{n}{2} \right) \\ &= \tfrac{n^2+n-8}{2}\,. \end{aligned}$$

Therefore $\text{val}_{\max}(T) \ge \text{val}(f) = \frac{n^2+n-8}{2}$.

Case 3. *The distances between v and exactly two its terminal vertices are even.* That is, exactly two terminal vertices of v are in W and exactly one terminal vertex of v is in U. Without loss of generality, let $x, y \in W$ and $z \in U$. Then $|W| = \frac{n}{2}$ and $|U| = \frac{n}{2}$.
Let $W = \{v = w_1, w_2, \ldots, w_{\frac{n}{2}-2}, x, y\}$ and $U = \{u_1, u_2, \ldots, u_{\frac{n}{2}-1}, z\}$.
Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} n-i & \text{if } a = w_i \text{ and } 1 \leq i \leq \frac{n}{2}-2 \\ i-1 & \text{if } a = u_i \text{ and } 1 \leq i \leq \frac{n}{2}-1 \\ \frac{n}{2}+1 & \text{if } a = x \\ \frac{n}{2} & \text{if } a = y \\ \frac{n}{2}-1 & \text{if } a = z\,. \end{cases}$$

Then $f(u) < f(w)$ for all $u \in U$ and $w \in W$. Thus,

$$\begin{aligned} \text{val}(f) &= \sum_{w \in W} \deg w \cdot f(w) - \sum_{u \in U} \deg u \cdot f(u) \\ &= \left(2\sum_{i=1}^{\frac{n}{2}-2}(n-i) + (n-1) + \left(\tfrac{n}{2}+1\right) + \tfrac{n}{2}\right) - \left(2\sum_{i=1}^{\frac{n}{2}-1}(i-1) + \left(\tfrac{n}{2}-1\right)\right) \\ &= \tfrac{n^2+n-6}{2}\,. \end{aligned}$$

Therefore $\text{val}_{\max}(T) \geq \text{val}(f) = \frac{n^2+n-6}{2}$.

Case 4. *The distances between v and exactly two its terminal vertices are odd.* Then there are two terminal vertices of v in U and one terminal vertex of v in W. Without loss of generality, let $x, y \in U$ and $z \in W$. Then $|W| = \lfloor \frac{n}{2} \rfloor$ and $|U| = \lceil \frac{n}{2} \rceil$.
Let $W = \{v = w_1, w_2, \ldots, w_{\frac{n-3}{2}}, z\}$ and $U = \{u_1, u_2, \ldots, u_{\frac{n-3}{2}}, x, y\}$.
Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} n-i & \text{if } a = w_i \text{ and } 1 \leq i \leq \frac{n-3}{2} \\ i-1 & \text{if } a = u_i \text{ and } 1 \leq i \leq \frac{n-3}{2} \\ \frac{n-3}{2} & \text{if } a = x \\ \frac{n-1}{2} & \text{if } a = y \\ \frac{n+1}{2} & \text{if } a = z\,. \end{cases}$$

Then $f(u) < f(w)$ for all $u \in U$ and $w \in W$. Thus,

$$\begin{aligned}
\mathrm{val}(f) &= \sum_{w\in W} \deg w \cdot f(w) - \sum_{u\in U} \deg u \cdot f(u) \\
&= \left(2\sum_{i=1}^{\frac{n-3}{2}} (n-i) + (n-1) + \left(\tfrac{n+1}{2}\right)\right) - \left(2\sum_{i=1}^{\frac{n-3}{2}} (i-1) + \left(\tfrac{n-3}{2}\right) + \left(\tfrac{n-1}{2}\right)\right) \\
&= \tfrac{n^2+n-6}{2}\,.
\end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) \geq \mathrm{val}(f) = \frac{n^2+n-6}{2}$. □

The consequence of Theorem 4.1.4 gives us a characterization of a γ-orientation of a γ-max labeling of a tree with a unique exterior major vertex as state next.

Corollary 4.1.8. *Let T be a tree with a unique exterior major vertex. For any γ-orientation $D(f)$ of a γ-max labeling f of T, each vertex in T is either a transmitter or a receiver in $D(f)$.*

With aid of Corollary 4.1.8 we are able to show the upper bound of maximum value of a tree with a unique exterior major vertex of terminal degree 3.

Proposition 4.1.9. *Let T be a tree of order n with a unique exterior major vertex of terminal degree 3. Then*

$$\mathrm{val}_{\max}(T) \leq \begin{cases} \frac{n^2+n-8}{2} & \text{if the distances between } v \text{ and all its terminal vertices are even or odd} \\ \frac{n^2+n-6}{2} & \text{otherwise}. \end{cases}$$

Proof. Let f be a γ-max labeling of T. By Corollary 4.1.8, the labeling f induces a partition of $V(T)$ into two independent sets, $R(f)$ and $L(f)$, such that for every edge uw joining a vertex w in $R(f)$ to a vertex u in $L(f)$, we have $f(u) < f(w)$. We may assume that $v \in R(f)$. Since if $v \in L(f)$, then $v \in R(\bar{f})$, it follows that we can consider a γ-max labeling $\bar{f}$ of T. By applying Theorem A, it is immediate that

$$\mathrm{val}(f) = \sum_{w\in R(f)} \deg w \cdot f(w) - \sum_{u\in L(f)} \deg u \cdot f(u)\,.$$

Since $\sum_{w\in R(f)} \deg w \cdot f(w)$ can be no greater than the quantity obtained by assigning the greatest possible values to $R(f)$ and $\sum_{u\in L(f)} \deg u \cdot f(u)$ can be no less than the quantity obtained by assigning the least possible values to $L(f)$, we have upper bound for $\mathrm{val}(f)$

by considering in four cases, according to the distances between v and all its terminal vertices, as follows.

Case 1. *The distances between v and all its terminal vertices are even.* That is, all terminal vertices of v are in $R(f)$. Then $|R(f)| = \lceil \frac{n}{2} \rceil$ and $|L(f)| = \lfloor \frac{n}{2} \rfloor$. Since each of vertices in $R(f)$ can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n-1}{2}$, and each of vertices in $L(f)$ can be assigned by each of the labels $0, 1, \ldots, \frac{n-3}{2}$, it follows that

$$\sum_{w \in R(f)} \deg w \cdot f(w) \leq 2 \sum_{i=1}^{\frac{n-5}{2}} (n-i) + (n-1) + \left(\frac{n+3}{2}\right) + \left(\frac{n+1}{2}\right) + \left(\frac{n-1}{2}\right)$$

and

$$\sum_{u \in L(f)} \deg u \cdot f(u) \geq 2 \sum_{i=1}^{\frac{n-1}{2}} (i-1)\,.$$

Thus

$$\begin{aligned} \mathrm{val}(f) &\leq \left(2 \sum_{i=1}^{\frac{n-5}{2}} (n-i) + (n-1) + \left(\tfrac{n+3}{2}\right) + \left(\tfrac{n+1}{2}\right) + \left(\tfrac{n-1}{2}\right)\right) - 2 \sum_{i=1}^{\frac{n-1}{2}} (i-1) \\ &= \tfrac{n^2+n-8}{2}\,. \end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) = \mathrm{val}(f) \leq \frac{n^2+n-8}{2}$.

Case 2. *The distances between v and all its terminal vertices are odd.* That is, all terminal vertices of v are in $L(f)$. Then $|R(f)| = \frac{n}{2} - 1$ and $|L(f)| = \frac{n}{2} + 1$. Since each of vertices in $R(f)$ can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n}{2}+1$, and each of vertices in $L(f)$ can be assigned by each of the labels $0, 1, \ldots, \frac{n}{2}$, it follows that

$$\sum_{w \in R(f)} \deg w \cdot f(w) \leq 2 \sum_{i=1}^{\frac{n}{2}-1} (n-i) + (n-1)$$

and

$$\sum_{u \in L(f)} \deg u \cdot f(u) \geq 2 \sum_{i=1}^{\frac{n}{2}-2} (i-1) + \left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2}\,.$$

Thus

$$\begin{aligned} \mathrm{val}(f) &\leq \left(2 \sum_{i=1}^{\frac{n}{2}-1} (n-i) + (n-1)\right) - \left(2 \sum_{i=1}^{\frac{n}{2}-2} (i-1) + \left(\tfrac{n}{2} - 2\right) + \left(\tfrac{n}{2} - 1\right) + \tfrac{n}{2}\right) \\ &= \tfrac{n^2+n-8}{2}\,. \end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) = \mathrm{val}(f) \leq \frac{n^2+n-8}{2}$.

Case 3. *The distances between v and exactly two its terminal vertices are even.* That is, exactly two terminal vertices of v are in $R(f)$ and exactly one terminal vertex of v is in $L(f)$. Then $|R(f)| = \frac{n}{2}$ and $|L(f)| = \frac{n}{2}$. Since each of vertices in $R(f)$ can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n}{2}$, and each of vertices in $L(f)$ can be assigned by each of the labels $0, 1, \ldots, \frac{n}{2}-1$, it follows that

$$\sum_{w \in R(f)} \deg w \cdot f(w) \leq 2\sum_{i=1}^{\frac{n}{2}-2}(n-i) + (n-1) + \left(\frac{n}{2}+1\right) + \frac{n}{2}$$

and

$$\sum_{u \in L(f)} \deg u \cdot f(u) \geq 2\sum_{i=1}^{\frac{n}{2}-1}(i-1) + \left(\frac{n}{2}-1\right) .$$

Thus

$$\begin{aligned} \mathrm{val}(f) &\leq \left(2\sum_{i=1}^{\frac{n}{2}-2}(n-i) + (n-1) + \left(\tfrac{n}{2}+1\right) + \tfrac{n}{2}\right) - \left(2\sum_{i=1}^{\frac{n}{2}-1}(i-1) + \left(\tfrac{n}{2}-1\right)\right) \\ &= \tfrac{n^2+n-6}{2} . \end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) = \mathrm{val}(f) \leq \frac{n^2+n-6}{2}$.

Case 4. *The distances between v and exactly two its terminal vertices are odd.* That is, there are two terminal vertices of v in $L(f)$ and one terminal vertex of v in $R(f)$. Then $|R(f)| = \lfloor \frac{n}{2} \rfloor$ and $|L(f)| = \lceil \frac{n}{2} \rceil$. Since each of vertices in $R(f)$ can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n+1}{2}$, and each of vertices in $L(f)$ can be assigned by each of the labels $0, 1, \ldots, \frac{n-1}{2}$, it follows that

$$\sum_{w \in R(f)} \deg w \cdot f(w) \leq 2\sum_{i=1}^{\frac{n-3}{2}}(n-i) + (n-1) + \left(\frac{n+1}{2}\right)$$

and

$$\sum_{u \in L(f)} \deg u \cdot f(u) \geq 2\sum_{i=1}^{\frac{n-3}{2}}(i-1) + \left(\frac{n-3}{2}\right) + \left(\frac{n-1}{2}\right) .$$

Thus

$$\begin{aligned} \mathrm{val}(f) &\leq \left(2\sum_{i=1}^{\frac{n-3}{2}}(n-i) + (n-1) + \left(\tfrac{n+1}{2}\right)\right) - \left(2\sum_{i=1}^{\frac{n-3}{2}}(i-1) + \left(\tfrac{n-3}{2}\right) + \left(\tfrac{n-1}{2}\right)\right) \\ &= \tfrac{n^2+n-6}{2} . \end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) = \mathrm{val}(f) \leq \frac{n^2+n-6}{2}$. □

Combining Propositions 4.1.7 and 4.1.9, we have the following.

Theorem 4.1.10. *Let T be a tree of order n with a unique exterior major vertex of terminal degree 3. Then*

$$\mathrm{val}_{\max}(T) = \begin{cases} \frac{n^2+n-8}{2} & \text{if the distances between } v \text{ and} \\ & \text{all its terminal vertices are even or odd} \\ \frac{n^2+n-6}{2} & \text{otherwise.} \end{cases}$$

1.3 Maximum values of trees with exterior major vertices

In this section, we investigate a maximum value of a tree T of $\mathrm{ma}(T) = 2$ with adjacent exterior major vertices. In order to do so, we need some additional notation.

For an exterior major vertex v of a graph G, we define

$$Ter(v) = \{z \in V(G) \mid z \text{ is a terminal vertex of } v\}.$$

For a γ-max labeling f of a graph G, let

$$Ter^-(v) = \{z \in Ter(v) \mid z \text{ is a transmitter in } D(f)\}$$

and

$$Ter^+(v) = \{z \in Ter(v) \mid z \text{ is a receiver in } D(f)\}.$$

Their cardinalities are denoted by $ter^-(v)$ and $ter^+(v)$, respectively.

Furthermore, for nonnegative integers p, q, r, s, n, let
$\alpha(p,q,r,s,n) = \frac{1}{2}(n^2 - p^2 - q^2 - r^2 - s^2 - pq - rs - pr - qs + (n-1)(p+q+r+s-2))$.

Proposition 4.1.11. *Let T be a tree of order n and $\mathrm{ma}(T) = 2$ with adjacent exterior major vertices v_1 and v_2. Then*

$$\mathrm{val}_{\max}(T) \geq \alpha(ter^-(v_1), ter^+(v_1), ter^-(v_2), ter^+(v_2), n)\,.$$

Proof. Let U and W be unique partite sets of a connected bipartite graph T. Without loss of generality, let $v_1 \in U$ and $v_2 \in W$.

Let $Ter(v_1) \cap U = \{y_1, y_2, \ldots, y_p\}$, $Ter(v_1) \cap W = \{y'_1, y'_2, \ldots, y'_q\}$,

$Ter(v_2) \cap U = \{z_1, z_2, \ldots, z_r\}$ and $Ter(v_2) \cap W = \{z'_1, z'_2, \ldots, z'_s\}$

where p, q, r, s are nonnegative integers.

Then $|U| = \frac{n-q+r}{2}$ and $|W| = \frac{n+q-r}{2}$.

Let $U = \{v_1 = u_1, u_2, \ldots, u_{\frac{n-q-r}{2}-p}, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_r\}$ and
$W = \{v_2 = w_1, w_2, \ldots, w_{\frac{n-q-r}{2}-s}, y'_1, y'_2, \ldots, y'_q, z'_1, z'_2, \ldots, z'_s\}$.
Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} n-i & \text{if } a = w_i \text{ and } 1 \le i \le \frac{n-q-r}{2} - s \\ i-1 & \text{if } a = u_i \text{ and } 1 \le i \le \frac{n-q-r}{2} - p \\ \frac{n+q+r}{2} + s - i & \text{if } a = y'_i \text{ and } 1 \le i \le q \\ \frac{n-q+r}{2} + s - i & \text{if } a = z'_i \text{ and } 1 \le i \le s \\ \frac{n-q-r}{2} - p + (i-1) & \text{if } a = y_i \text{ and } 1 \le i \le p \\ \frac{n-q-r}{2} + (i-1) & \text{if } a = z_i \text{ and } 1 \le i \le r\,. \end{cases}$$

Then $f(u) < f(w)$ for all $u \in U$ and $w \in W$. Thus,

$$\begin{aligned} \operatorname{val}(f) &= \sum_{w\in W} \deg w \cdot f(w) - \sum_{u\in U} \deg u \cdot f(u) \\ &= \left(2 \sum_{i=1}^{\frac{n-q-r}{2}-s} (n-i) + ((r+s)-1)(n-1) + \sum_{i=1}^{q+s} (\tfrac{n+q+r}{2} + s - i)\right) \\ &\quad - \left(2 \sum_{i=1}^{\frac{n-q-r}{2}-p} (i-1) + ((p+q)-1)(0) + \sum_{i=1}^{p+r} (\tfrac{n-q-r}{2} - p - 1 + i)\right) \\ &= \tfrac{1}{2}(n^2 - p^2 - q^2 - r^2 - s^2 - pq - rs - pr - qs + (n-1)(p+q+r+s-2)) \\ &= \alpha(p, q, r, s, n)\,. \end{aligned}$$

Therefore $\operatorname{val}_{\max}(T) \ge \operatorname{val}(f) = \alpha(p, q, r, s, n)$,
where $p = ter^-(v_1)$, $q = ter^+(v_1)$, $r = ter^-(v_2)$ and $s = ter^+(v_2)$. □

Proposition 4.1.12. *Let T be a tree of $\operatorname{ma}(T) = 2$ with adjacent exterior major vertices and f a γ-max labeling of T. Then an exterior major vertex of T is either a transmitter or a receiver in $D(f)$.*

Proof. Let v_1 and v_2 be adjacent exterior major vertices of T. By Theorem 4.1.4, each vertex in exbranches G_{v_1} and G_{v_2} is either a transmitter or a receiver in $D(f)$. Furthermore, by Proposition 4.1.3, all vertices of exneighborhoods $N^e(v_1)$ and $N^e(v_2)$ are either transmitters or receivers in $D(f)$. By Observation 1.3.1, we may let $f(v_1) < f(v_2)$. In order to show that an exterior major vertex of T is either a transmitter or a receiver in $D(f)$, we claim that all vertices of $N^e(v_1)$ are receivers and all vertices of $N^e(v_2)$ are transmitters. To do so, assume, to the contrary, that the statement is false. We consider two cases according to $N^e(v_1)$ and $N^e(v_2)$ in $D(f)$.

Case 1. *all vertices of $N^e(v_1)$ are transmitters and all vertices of $N^e(v_2)$ are receivers in $D(f)$.*

Let g be a γ-labeling of T defined by

$$g(a) = \begin{cases} f(a) & \text{if } a \neq v_1, v_2 \\ f(v_2) & \text{if } a = v_1 \\ f(v_1) & \text{if } a = v_2\,. \end{cases}$$

Then $\text{val}(g) = \text{val}(f) + (\text{ter}(v_1) + \text{ter}(v_2))(f(v_2) - f(v_1)) > \text{val}(f)$, which is a contradiction.

Case 2. *either all vertices of $N^e(v_1)$ and $N^e(v_2)$ are receivers or all vertices of $N^e(v_1)$ and $N^e(v_2)$ are transmitters in $D(f)$.*

By Observation 1.3.1, we may assume that all vertices of $N^e(v_1)$ and $N^e(v_2)$ are receivers in $D(f)$. Then a γ-max labeling f induces a partition of $V(T)$ into two sets, X and Y, where

$$X = \{x \in V(T) \mid x = v_2 \text{ or } x \text{ is a transmitter in } D(f)\}$$

and

$$Y = \{y \in V(T) \mid y \text{ is a receiver in } D(f)\}.$$

By Theorem A, it is immediate that

$$\begin{aligned} \text{val}(f) &= \sum_{y \in Y} \text{id}\, y \cdot f(y) - \sum_{x \in X - \{v_2\}} \text{od}\, x \cdot f(x) + (\text{id}\, v_2 - \text{od}\, v_2) f(v_2) \\ &= \sum_{y \in Y} \text{id}\, y \cdot f(y) - \left(\sum_{x \in X - \{v_2\}} \text{od}\, x \cdot f(x) + (\text{ter}(v_2) - 1) f(v_2) \right). \end{aligned}$$

We have $|X| = \frac{n+q+s}{2} - 1$ and $|Y| = \frac{n-q-s}{2} + 1$, where $p = ter^-(v_1)$, $q = ter^+(v_1)$, $r = ter^-(v_2)$ and $s = ter^+(v_2)$. Since $\sum_{y \in Y} \text{id}\, y \cdot f(y)$ can be no greater than the quantity obtained by assigning the greatest possible value to Y and $\sum_{x \in X - \{v_2\}} \text{od}\, x \cdot f(x)$ can be no less than the quantity obtained by assigning the least possible value to X, it follows that each of vertices in Y can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n-q-s}{2} + 1$, and each of vertices in X can be assigned by each of the labels $0, 1, \ldots, \frac{n-q-s}{2}$, it follows that an upper bound for $\text{val}_{\max}(T)$ can be considered into two cases according to $ter(v_2)$.

Subcase 2.1. $ter(v_2) = 2$.

Since each of vertices in Y can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n-q-s}{2}+1$, and each of vertices in X can be assigned by each of the labels $0, 1, \ldots, \frac{n-q-s}{2}$, it follows that

$$\sum_{y\in Y} \operatorname{id} y \cdot f(y) \leq 2 \sum_{i=1}^{\frac{n-q-s}{2}-1} (n-i) + \sum_{i=1}^{q+s} (\tfrac{n+q+r}{2} + 1 - i)$$

and

$$\sum_{x\in X-\{v_2\}} \operatorname{od} x \cdot f(x) + f(v_2)$$

$$\geq 2 \sum_{i=1}^{\frac{n-q-s}{2}-p-r} (i-1) + ((p+q)-1)(0) + \sum_{i=1}^{p+r+1} (\tfrac{n-q-s}{2} - p - r - 1 + i)\,.$$

Thus

$$\begin{aligned}
\text{val}(f) &\leq \left(2 \sum_{i=1}^{\frac{n-q-s}{2}-1} (n-i) + \sum_{i=1}^{q+s} (\tfrac{n+q+r}{2} + 1 - i)\right) \\
&\quad - \left(2 \sum_{i=1}^{\frac{n-q-s}{2}-p-r} (i-1) + ((p+q)-1)(0) + \sum_{i=1}^{p+r+1} (\tfrac{n-q-s}{2} - p - r - 1 + i)\right) \\
&= \tfrac{1}{2}(n^2 - p^2 - q^2 - r^2 - s^2 - pq - rs - pr - qs + (n-1)(p+q+r+s-2)) \\
&\quad -\tfrac{1}{2}(n + pr + qs + ps + qr + q + s + 2) \\
&= \alpha(p, q, r, s, n) - \tfrac{1}{2}(n + pr + qs + ps + qr + q + s + 2)\,.
\end{aligned}$$

Therefore $\text{val}_{\max}(T) = \text{val}(f) < \alpha(p, q, r, s, n)$, which contradicts Proposition 4.1.11.

Subcase 2.2. $ter(v_2) \geq 3$.

Since each of vertices in Y can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n-q-s}{2}+1$, each of vertices in X can be assigned by each of the labels $0, 1, \ldots, \frac{n-q-s}{2}$, it follows that

$$\sum_{y\in Y} \operatorname{id} y \cdot f(y) \leq 2 \sum_{i=1}^{\frac{n-q-s}{2}-1} (n-i) + \sum_{i=1}^{q+s} (\tfrac{n+q+s}{2} + 1 - i)$$

and

$$\sum_{x\in X-\{v_2\}} \operatorname{od} x \cdot f(x) + (ter(v_2) - 1) f(v_2)$$

$$\geq 2 \sum_{i=1}^{\frac{n-q-s}{2}-p-r+1} (i-1) + ((p+q)-1)(0) + ((r+s)-3)(1) + \sum_{i=1}^{p+r} (\tfrac{n-q-s}{2} - p - r + i)\,.$$

Thus

$$\mathrm{val}(f) \leq \left(2\sum_{i=1}^{\frac{n-q-s}{2}-1}(n-i)+\sum_{i=1}^{q+s}(\tfrac{n+q+s}{2}+1-i)\right)$$

$$-\left(2\sum_{i=1}^{\frac{n-q-s}{2}-p-r+1}(i-1)+((p+q)-1)(0)+((r+s)-3)(1)+\sum_{i=1}^{p+r}(\tfrac{n-q-s}{2}-p-r+i)\right)$$

$$= \tfrac{1}{2}(n^2-p^2-q^2-r^2-s^2-pq-rs-pr-qs+(n-1)(p+q+r+s-2))$$

$$-\tfrac{1}{2}(2n+pr+qs+ps+qr-2p-2q-4)$$

$$= \alpha(p,q,r,s,n)-\tfrac{1}{2}(2n+pr+qs+ps+qr-2p-2q-4)\,.$$

Therefore, $\mathrm{val}_{\max}(T)=\mathrm{val}(f)<\alpha(p,q,r,s,n)$, which contradicts Proposition 4.1.11. □

Combining Propositions 4.1.1 and 4.1.12 and Observation 4.1.2, we have the following.

Proposition 4.1.13. *Let T be a tree of $\mathrm{ma}(T)=2$ with adjacent exterior major vertices and f a γ-max labeling of T. Then every vertex in T is either a transmitter or a receiver in $D(f)$.*

Proposition 4.1.14. *Let T be a tree of order n and $\mathrm{ma}(T)=2$ with adjacent exterior major vertices v_1 and v_2. Then*

$$\mathrm{val}_{\max}(T)\leq\alpha(ter^-(v_1),ter^+(v_1),ter^-(v_2),ter^+(v_2),n)\,.$$

Proof. Let f be a γ-max labeling of T. By Proposition 4.1.13, a γ-max labeling f induces a partition of $V(T)$ into two independent sets, $R(f)$ and $L(f)$ such that for every edge uw joining a vertex w in $R(f)$ to a vertex u in $L(f)$, we have $f(u)<f(w)$. By Observation 1.3.1, we may assume that $v_1\in L(f)$ and hence $v_2\in R(f)$. By Theorem A, it is immediate that

$$\mathrm{val}(f)=\sum_{w\in R(f)}\deg w\cdot f(w)-\sum_{u\in L(f)}\deg u\cdot f(u)\,.$$

Since $\sum_{w\in R(f)}\deg w\cdot f(w)$ can be no greater than the quantity obtained by assigning the greatest possible values to $R(f)$ and $\sum_{u\in L(f)}\deg u\cdot f(u)$ can be no less than the quantity obtained by assigning the least possible values to $L(f)$, we have upper bound for $\mathrm{val}(f)$ as follows.

Let $p = ter^{-}(v_1)$, $q = ter^{+}(v_1)$, $r = ter^{-}(v_2)$ and $s = ter^{+}(v_2)$. Then $|R(f)| = \frac{n+q-r}{2}$ and $|L(f)| = \frac{n-q+r}{2}$. Since each of vertices in $R(f)$ can be assigned by each of the labels $n-1, n-2, \ldots, \frac{n-q+r}{2}$, and each of vertices in $L(f)$ can be assigned by each of the labels $0, 1, \ldots, \frac{n-q+r}{2} - 1$, it follows that

$$\sum_{w \in R(f)} \deg w \cdot f(w) \leq 2 \sum_{i=1}^{\frac{n-q-r}{2}-s} (n-i) + ((r+s)-1)(n-1) + \sum_{i=1}^{q+s} (\frac{n+q+r}{2} + s - i)$$

and

$$\sum_{u \in L(f)} \deg u \cdot f(u) \geq 2 \sum_{i=1}^{\frac{n-q-r}{2}-p} (i-1) + ((p+q)-1)(0) + \sum_{i=1}^{p+r} (\frac{n-q-r}{2} - p - 1 + i)\,.$$

Thus

$$\begin{aligned}
\mathrm{val}(f) \quad &\leq \quad \left(2 \sum_{i=1}^{\frac{n-q-r}{2}-s} (n-i) + ((r+s)-1)(n-1) + \sum_{i=1}^{q+s} (\tfrac{n+q+r}{2} + s - i)\right) \\
&\quad - \left(2 \sum_{i=1}^{\frac{n-q-r}{2}-p} (i-1) + ((p+q)-1)(0) + \sum_{i=1}^{p+r} (\tfrac{n-q-r}{2} - p - 1 + i)\right) \\
&= \quad \tfrac{1}{2}(n^2 - p^2 - q^2 - r^2 - s^2 - pq - rs - pr - qs + (n-1)(p+q+r+s-2)) \\
&= \quad \alpha(p,q,r,s,n)\,.
\end{aligned}$$

Therefore $\mathrm{val}_{\max}(T) = \mathrm{val}(f) \leq \alpha(p,q,r,s,n)\,.$ □

Combining Propositions 4.1.11 and 4.1.14, we have the following.

Theorem 4.1.15. *Let T be a tree of order n and $\mathrm{ma}(T) = 2$ with adjacent exterior major vertices. Let v_1 and v_2 be exterior major vertices of T. Then*

$$\mathrm{val}_{\max}(T) = \alpha(ter^{-}(v_1), ter^{+}(v_1), ter^{-}(v_2), ter^{+}(v_2), n)\,.$$

As we have seen earlier, each vertex of a tree T with a unique exterior major vertex or exactly two adjacent exterior major vertices, is either a transmitter or a receiver in a γ-orientation of a γ-max labeling of T. However, it is not necessarily true for any tree. For example, the γ-orientation $D(f)$ of the γ-max labeling f of the tree with two non-adjacent exterior major vertices containing a vertex that is neither a transmitter nor a receiver are shown in Figure 9.

$D(f)$:

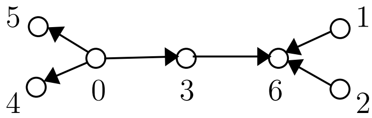

Figure 9 The γ-orientation $D(f)$ of the γ-max labeling f of the tree containing a vertex that is neither a transmitter nor a receiver

Last, we show the result by considering an internal vertex of a $v_1 - v_2$ path containing no major vertex as its internal vertex in a graph G where v_1 and v_2 are exterior major vertices of G.

Proposition 4.1.16. *Let v_1 and v_2 be exterior major vertices of a graph G. Let P be a v_1-v_2 path containing no major vertex as its internal vertex in G. For any γ-orientation $D(f)$ of a γ-max labeling f of G, each internal vertex of P, except at most one, is either a transmitter or a receiver in $D(f)$.*

Proof. Assume that the lemma is false. Let $P : v_1 = u_0, u_1, \ldots, u_d = v_2$ be $v_1 - v_2$ path containing no major vertex as its internal vertex in G. Among all γ-max labelings of G, let g be one having shortest distance between two internal vertices of P that are neither transmitters nor receivers in $D(g)$, namely u_i and u_j with $1 \le i < j \le d-1$. By Observation 1.3.1, we may assume that $g(u_{i-1}) < g(u_i) < g(u_{i+1})$. Let g_1 be a γ-labeling of G defined by

$$g_1(a) = \begin{cases} g(a) & \text{if } a \neq u_i, u_{i+1} \\ g(u_{i+1}) & \text{if } a = u_i \\ g(u_i) & \text{if } a = u_{i+1}\,. \end{cases}$$

Then

$$\begin{aligned} \text{val}(g_1) &= \text{val}(g) - (g(u_{i+1}) - g(u_{i+2})) + (g(u_{i+1}) - g(u_i)) + |g(u_{i+2}) - g(u_i)| \\ &= \text{val}(g) + (g(u_{i+2}) - g(u_i)) + |g(u_{i+2}) - g(u_i)|\,. \end{aligned}$$

If $g(u_i) < g(u_{i+2})$, then $\text{val}(g_1) = \text{val}(g) + 2(g(u_{i+2}) - g(u_i)) > \text{val}(g)$, which is a contradiction. Assume that $g(u_{i+2}) < g(u_i)$. Then $\text{val}(g_1) = \text{val}(g) = \text{val}_{\max}(G)$. We have g_1 is a γ-max labeling of G and also internal vertices u_{i+1} and u_j of P are neither transmitters nor receivers in $D(g_1)$. This contradicts the defining property of g. □

2. γ-max labelings of graphs

In section 6 of chapter 2, we present a characterization of γ-max labelings of complete bipartite graphs and complete graphs which appeared in [3] and [10]. In this section, we provide an alternative approach to formulae for $\text{val}_{\max}(K_{r,s})$ and $\text{val}_{\max}(K_n)$ proved by mathematical induction.

For any positive integers n, m, Δ with $\Delta \geq m$, let G be a nontrivial graph of order n and size m, a γ^{Δ}-*labeling* of G is defined as a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, m, m+1, \ldots, \Delta\}$ that induces an *edge-labeling* $f'\colon E(G) \to \{1, 2, \ldots, \Delta\}$ on G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The *value* of f is defined by $\text{val}(f) = \sum_{e \in E(G)} f'(e)$.

The *maximum value* of a γ^{Δ}-labeling of G is

$$\text{val}^{\Delta}_{\max}(G) = \max\{\text{val}(f) \mid f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}$$

and the *minimum value* of a γ^{Δ}-labeling of G is

$$\text{val}^{\Delta}_{\min}(G) = \min\{\text{val}(f) \mid f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}.$$

A γ^{Δ}-labeling g of G is a γ^{Δ}-max *labeling* if

$$\text{val}(g) = \text{val}^{\Delta}_{\max}(G)$$

and a γ^{Δ}-labeling h of G is a γ^{Δ}-min *labeling* if

$$\text{val}(h) = \text{val}^{\Delta}_{\min}(G).$$

Note that $\text{val}_{\max}(G) = \text{val}^{\Delta}_{\max}(G)$ and $\text{val}_{\min}(G) = \text{val}^{\Delta}_{\min}(G)$ when $\Delta = m$.

We first make the following observation for γ^{Δ}-max and γ^{Δ}-min labelings of graphs.

Observation 4.2.1. *Let f be a γ^{Δ}-labeling of a graph G. Then f is a γ^{Δ}-*max *labeling* (γ^{Δ}-min *labeling*) *of G if and only if $\bar{f}$ is a γ^{Δ}-*max *labeling* (γ^{Δ}-min *labeling*) *of G.*

2.1 γ^{Δ}-max labelings of graphs

In this section, we begin our investigation for γ^{Δ}-max labeling of any nontrivial graph by presenting a useful lemma.

Lemma 4.2.2. *Let f be a γ^{Δ}-max labeling of a nontrivial graph G of order n and size m. Let $u, w \in V(G)$ with $f(u) = \min\{f(v) \mid v \in V(G)\}$ and $f(w) = \max\{f(v) \mid v \in V(G)\}$. Then neighborhoods of u and w are not empty.*

Proof. For any nontrivial connected graph G, it is obvious that $|N(u)|$ and $|N(w)|$ are not empty. Let G be a disconnected graph. We will show that $|N(u)| \neq 0$ and $|N(w)| \neq 0$ Assume, to the contrary, that $|N(u)| = 0$ or $|N(w)| = 0$.

Case 1. $|N(u)| = 0$.

Then u is an isolated vertex of G. Since G has a γ^{Δ}-max labeling, $\Delta \geq m \geq n-1$. Therefore, there is a component G_1 of G with $|V(G_1)| \geq 2$. Let $x \in V(G_1)$ with $f(x) = \min\{f(v) \mid v \in V(G_1)\}$. Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} f(x) & \text{if } v = u \\ f(u) & \text{if } v = x \\ f(v) & \text{if } v \neq u, x. \end{cases}$$

Then

$$\begin{aligned} \text{val}(g) &= \text{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (g(v) - g(x)) \\ &= \text{val}(f) - \sum_{v \in N(x)} (f(v) - f(x)) + \sum_{v \in N(x)} (f(v) - f(u)) \\ &= \text{val}(f) + |N(x)|(f(x) - f(u)) \\ &> \text{val}(f), \end{aligned}$$

which is a contradiction.

Case 2. $|N(w)| = 0$.

Then w is an isolated vertex of G. Since G has a γ^{Δ}-max labeling, $\Delta \geq m \geq n-1$. Therefore, there is a component G_1 of G with $|V(G_1)| \geq 2$. Let $y \in V(G_1)$ with $f(y) = \max\{f(v) \mid v \in V(G)\}$. Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} f(y) & \text{if } v = w \\ f(w) & \text{if } v = y \\ f(v) & \text{if } v \neq w, y. \end{cases}$$

Then

$$\begin{aligned}\text{val}(g) &= \text{val}(f) - \sum_{v\in N(y)} (f(y) - f(v)) + \sum_{v\in N(y)} (g(y) - g(v))\\ &= \text{val}(f) - \sum_{v\in N(y)} (f(y) - f(v)) + \sum_{v\in N(y)} (f(w) - f(v))\\ &= \text{val}(f) + |N(y)|(f(w) - f(y))\\ &> \text{val}(f)\,,\end{aligned}$$

which is a contradiction. □

We now show formula for span of γ^{Δ}-max labelings of graphs.

Proposition 4.2.3. *Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-*max *labeling of G, then* $\text{span}(f) = \Delta$.

Proof. Let f be a γ^{Δ}-max labeling of G. Let $u, w \in V(G)$ with $f(u) = \min\{f(v) \mid v \in V(G)\}$ and $f(w) = \max\{f(v) \mid v \in V(G)\}$. Then $f(u) \geq 0$ and $f(w) \leq \Delta$. Assume, to the contrary, that $span(f) < \Delta$. Then $f(w) - f(u) < \Delta$. Therefore, $f(u) > 0$ or $\Delta - f(w) > 0$.

Case 1. $f(u) > 0$.

Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} 0 & \text{if } v = u \\ f(v) & \text{if } v \neq u. \end{cases}$$

Then

$$\begin{aligned}\text{val}(g) &= \text{val}(f) - \sum_{v\in N(u)} (f(v) - f(u)) + \sum_{v\in N(u)} (g(v) - g(u))\\ &= \text{val}(f) - \sum_{v\in N(u)} (f(v) - f(u)) + \sum_{v\in N(u)} (f(v) - 0)\\ &= \text{val}(f) + |N(u)|(f(u) - 0)\\ &> \text{val}(f) \quad \text{(by Lemma 4.2.2)},\end{aligned}$$

which is a contradiction.

Case 2. $\Delta - f(w) > 0$.

Let g be a γ^{Δ}-labeling of G defined by

$$g(v) = \begin{cases} \Delta & \text{if } v = w \\ f(v) & \text{if } v \neq w. \end{cases}$$

Then

$$\begin{aligned}\mathrm{val}(g) &= \mathrm{val}(f) - \sum_{v\in N(w)}(f(w)-f(v)) + \sum_{v\in N(w)}(g(w)-g(v))\\ &= \mathrm{val}(f) - \sum_{v\in N(w)}(f(w)-f(v)) + \sum_{v\in N(w)}(\Delta - f(v))\\ &= \mathrm{val}(f) + |N(w)|(\Delta - f(w))\\ &> \mathrm{val}(f) \quad \text{(by Lemma 4.2.2)},\end{aligned}$$

which is a contradiction. □

This also provides the following corollary.

Corollary 4.2.4. *Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-max labeling of G, then $\{0,\Delta\} \subseteq f(V(G))$.*

2.2 γ-max labelings of complete bipartite graphs

We define the γ^{Δ}-*spectrum* of a graph G by

$$\mathrm{spec}^{\Delta}(G) = \{\mathrm{val}(f) \mid f \text{ is a } \gamma^{\Delta}\text{-labeling of } G\}.$$

Consequently, $\{\mathrm{val}^{\Delta}_{\min}(G), \mathrm{val}^{\Delta}_{\max}(G)\} \subseteq \mathrm{spec}^{\Delta}(G)$ for every graph G. As an illustration, we now establish the γ^{Δ}-spectrum of a star $K_{1,s}$.

Proposition 4.2.5. *For positive integers s, Δ with $\Delta \geq s$,*

$$\mathrm{spec}^{\Delta}(K_{1,s}) = \left\{\binom{\Delta-k+1}{2} - \binom{\Delta-s+1}{2} + \binom{k+1}{2} \mid 0 \leq k \leq \Delta\right\}.$$

Proof. Let $K_{1,s}$ be a star with $V(K_{1,s}) = \{v\} \cup V_s$ where v is a central vertex and $V_s = \{v_1, v_2, \ldots, v_s\}$ and f a γ^{Δ}-labeling of a graph $K_{1,s}$ with $f(v) = k$ where $0 \leq k \leq \Delta$. If $k = 0$, then we may assume that $f(v_i) = \Delta - (s-i)$ for all $1 \leq i \leq s$. Then

$$\mathrm{val}(f) = \sum_{i=1}^{s} |f(v_i) - f(v)| = \sum_{i=1}^{s}(\Delta - (s-i)) = \binom{\Delta+1}{2} - \binom{\Delta-s+1}{2}.$$

If $k = \Delta$, then by Observation 1.3.1,

$$\mathrm{val}(f) = \binom{\Delta+1}{2} - \binom{\Delta-s+1}{2}.$$

If $0 < k < \Delta$, then we may assume that

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \leq i \leq k \\ \Delta - (s-i) & \text{if } k+1 \leq i \leq s\,. \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{val}(f) &= (k+(k-1)+\cdots+1)+((\Delta-(s-1))+(\Delta-(s-2))+\cdots+(\Delta-k)) \\ &= \tbinom{k+1}{2}+\tbinom{\Delta-k+1}{2}-\tbinom{\Delta-s+1}{2}, \end{aligned}$$

as desired. □

In Proposition 4.2.5, we considered γ^{Δ}-spectrum of a star $K_{1,s}$. We are now ready to compute the maximum value of a γ^{Δ}-labeling of $K_{1,s}$.

Corollary 4.2.6. *For positive integers s, Δ with $\Delta \geq s$,*

$$\mathrm{val}^{\Delta}_{\max}(K_{1,s}) = \binom{\Delta+1}{2} - \binom{\Delta-s+1}{2}.$$

Moreover, let f be a γ^{Δ}-labeling of $K_{1,s}$ with

$$f(v)=0 \quad \textit{and} \quad f(V_s)=[\Delta-(s-1),\Delta].$$

*Then f and $\bar{f}$ are only γ^{Δ}-*max *labelings of $K_{1,s}$.*

Next, we show an alternative and yet simple proof employing mathematical induction of Theorem 2.6.1 which is proposed by Bullington, Eroh and Winters [3] in 2010 and by Fonseca, Khemmani and Zhang [9] in 2015. In order to do this, first, let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, where $1 \leq r \leq s$, $V_r = \{u_1, u_2, \ldots, u_r\}$ and $V_s = \{v_1, v_2, \ldots, v_s\}$ and then we discuss γ^{Δ}-max labelings of $K_{r,s}$ as follows.

Theorem 4.2.7. *Let f be a γ^{Δ}-labeling of a complete bipartite graph $K_{r,s}$ with*

$$f(V_r)=[0,r-1] \quad \textit{and} \quad f(V_s)=[\Delta-(s-1),\Delta],$$

*where $\Delta \geq rs$. Then f and $\bar{f}$ are only two γ^{Δ}-*max *labelings of $K_{r,s}$.*

Proof. We proceed by induction on $r+s$. The result is certainly true for $r+s=2$. Assume that $r+s \geq 3$ and the result holds for $K_{r',s'}$ when $2 \leq r'+s' < r+s$. By Corollary 4.2.6, hence the theorem holds when $r=1$. Suppose that $r \geq 2$. Let f be a γ^{Δ}-max labeling of $K_{r,s}$ with $f(u_1) < f(u_2) < \cdots < f(u_r)$ and $f(v_1) < f(v_2) < \cdots < f(v_s)$.

Case 1. $f(u_1) < f(v_1)$.

By Corollary 4.2.4, $f(u_1) = 0$. Furthermore, for each $j \in \{1, 2, \ldots, s\}$ it follows that $f(v_j) \leq \Delta - (s-j)$. Let $K_{r-1,s}$ be a complete bipartite graph with vertex set $V(K_{r-1,s}) = V(K_{r,s}) - \{u_1\}$ and partite sets $V_{r-1} = V_r - \{u_1\}$ and V_s. Consequently, let f_1 be a $\gamma^{\Delta-1}$-labeling of $K_{r-1,s}$ defined by

$$f_1(u) = f(u) - 1 \text{ for each } u \in V(K_{r-1,s}).$$

Then

$$\text{val}(f_1) = \sum_{\substack{2 \leq i \leq r \\ 1 \leq j \leq s}} f_1'(u_i v_j) \leq \text{val}_{\max}^{\Delta-1}(K_{r-1,s}).$$

Let g_1 be a $\gamma^{\Delta-1}$-max labeling of $K_{r-1,s}$. Since $\Delta - 1 \geq (r-1)s$, by induction hypothesis, we have

$$g_1(V_{r-1}) = [0, r-2] \text{ and } g_1(V_s) = [(\Delta-1) - (s-1), (\Delta-1)].$$

We can extend g_1 to a γ^{Δ}-labeling g of $K_{r,s}$ defined by

$$g(u) = \begin{cases} 0 & \text{if } u = u_1 \\ g_1(u) + 1 & \text{otherwise}. \end{cases}$$

Since

$$\begin{aligned} \text{val}_{\max}^{\Delta}(K_{r,s}) &= \text{val}(f) \\ &= \sum_{\substack{2 \leq i \leq r \\ 1 \leq j \leq s}} f'(u_i v_j) + \sum_{j=1}^{s} |f(v_j) - f(u_1)| \\ &\leq \text{val}(f_1) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &\leq \text{val}_{\max}^{\Delta-1}(K_{r-1,s}) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &= \text{val}(g_1) + \sum_{j=1}^{s} |\Delta - (s-j) - 0| \\ &= \text{val}(g) \\ &\leq \text{val}_{\max}^{\Delta}(K_{r,s}), \end{aligned}$$

it follows that

$$\sum_{j=1}^{s} |f(v_j) - f(u_1)| = \sum_{j=1}^{s} |\Delta - (s-j) - 0| \tag{4.2.1}$$

and

$$\text{val}(f_1) = \text{val}_{\max}^{\Delta-1}(K_{r-1,s}). \tag{4.2.2}$$

From (4.2.1), we have

$$f(V_s) = [\Delta - (s-1), \Delta].$$

From (4.2.2), we have $f_1(V_{r-1}) = [0, r-2]$, hence

$$f(V_{r-1}) = [1, r-1]$$

and we have $f(u_1) = 0$. Therefore,

$$f(V_r) = [0, r-1] \;\text{ and }\; f(V_s) = [\Delta - (s-1), \Delta]\,.$$

Case 2. $f(v_1) < f(u_1)$.

By Corollary 4.2.4, $f(v_1) = 0$. Furthermore, for each $i \in \{1, 2, \ldots, r\}$ it follows that $f(u_i) \leq \Delta - (r-i)$. Let $K_{r,s-1}$ be a complete bipartite graph with vertex set $V(K_{r,s-1}) = V(K_{r,s}) - \{v_1\}$ and partite sets V_r and $V_{s-1} = V_s - \{v_1\}$. Consequently, let f_1 be a $\gamma^{\Delta-1}$-labeling of $K_{r,s-1}$ defined by

$$f_1(u) = f(u) - 1 \;\text{ for each }\; u \in V(K_{r,s-1})\,.$$

Then

$$\mathrm{val}(f_1) = \sum_{\substack{1\leq i\leq r \\ 2\leq j\leq s}} f_1'(u_i v_j) \leq \mathrm{val}_{\max}^{\Delta-1}(K_{r,s-1})\,.$$

Let g_1 be a $\gamma^{\Delta-1}$-max labeling of $K_{r,s-1}$. Since $\Delta - 1 \geq r(s-1)$, by the induction hypothesis, we have

$$g_1(V_{s-1}) = [0, s-2] \;\text{ and }\; g_1(V_r) = [(\Delta-1) - (r-1), (\Delta-1)]\,.$$

We can extend g_1 to a γ^{Δ}-labeling g of $K_{r,s}$ defined by

$$g(u) = \begin{cases} 0 & \text{if } u = v_1 \\ g_1(u) + 1 & \text{otherwise}\,. \end{cases}$$

Since

$$\begin{aligned}
\mathrm{val}^{\Delta}_{\max}(K_{r,s}) &= \mathrm{val}(f)\\
&= \sum_{\substack{1\le i\le r\\ 2\le j\le s}} f'(u_iv_j) + \sum_{i=1}^{r} |f(u_i) - f(v_1)|\\
&\le \mathrm{val}(f_1) + \sum_{i=1}^{r} |\Delta - (r-i) - 0|\\
&\le \mathrm{val}^{\Delta-1}_{\max}(K_{r,s-1}) + \sum_{i=1}^{r} |\Delta - (r-i) - 0|\\
&= \mathrm{val}(g_1) + \sum_{i=1}^{r} |\Delta - (r-i) - 0|\\
&= \mathrm{val}(g)\\
&\le \mathrm{val}^{\Delta}_{\max}(K_{r,s})\,,
\end{aligned}$$

it follows that

$$\sum_{i=1}^{r} |f(u_i) - f(v_1)| = \sum_{i=1}^{r} |\Delta - (r-i) - 0| \tag{4.2.3}$$

and

$$\mathrm{val}(f_1) = \mathrm{val}^{\Delta-1}_{\max}(K_{r,s-1})\,. \tag{4.2.4}$$

From (4.2.3), we have

$$f(V_r) = [\Delta - (r-1), \Delta]\,.$$

From (4.2.4), we have $f_1(V_{s-1}) = [0, s-2]$, hence

$$f(V_{s-1}) = [1, s-1]$$

and we have $f(v_1) = 0$. Therefore,

$$f(V_s) = [0, s-1] \quad \text{and} \quad f(V_r) = [\Delta - (r-1), \Delta]\,. \qquad \square$$

The following result is the consequence of Theorem 4.2.7 when $\Delta = rs$.

Theorem 4.2.8. *Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, let f be a γ-labeling of $K_{r,s}$ with*

$$f(V_r) = [0, r-1] \quad \textit{and} \quad f(V_s) = [rs - (s-1), rs]\,.$$

*Then f and $\bar{f}$ are only two γ-*max *labelings of $K_{r,s}$.*

2.3 γ-max labelings of complete graphs

The γ-max labelings of complete graphs K_n are characterized in [9]. In this section, we present characterization of γ^{Δ}-max labelings and γ-max labeling of complete graphs K_n, by applying a similar fashion to the one used in the proof of Theorem 4.2.7.

Theorem 4.2.9. *Let f be a γ^{Δ}-labeling of a complete graph K_n with*

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

where $\Delta \geq \binom{n}{2}$ and $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \Delta - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ^{Δ}-max labelings of K_n.

Proof. Let K_n be a complete graph with $V(K_n) = \{u_1, u_2, \ldots, u_n\}$.

Case 1. *n is even.*
We use mathematical induction on n. When $n = 2$, the result is obvious. Assume that $n \geq 4$ and the result holds for $K_{n'}$ when n' is even and $2 \leq n' < n$. Let f be a γ^{Δ}-max labeling of K_n with $f(u_1) < f(u_2) < \cdots < f(u_n)$. By Corollary 4.2.4, $f(u_1) = 0$ and $f(u_n) = \Delta$. Let f_1 be a $\gamma^{\Delta-2}$-labeling of a complete graph K_{n-2} with vertex set $V(K_{n-2}) = \{u_2, u_3, \ldots, u_{n-1}\}$ defined by

$$f_1(u_i) = f(u_i) - 1 \text{ for each } 2 \leq i \leq n-1\,.$$

Let g_1 be a $\gamma^{\Delta-2}$-max labeling of K_{n-2}. Since $\Delta - 2 \geq \binom{n-2}{2}$, by induction hypothesis, we have

$$g_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta - 2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta - 2)\right].$$

We can extend g_1 to a γ^{Δ}-labeling g of K_n defined by

$$g(u) = \begin{cases} 0 & \text{if } u = u_1 \\ \Delta & \text{if } u = u_n \\ g_1(u) + 1 & \text{if } u \neq u_1, u_n\,. \end{cases}$$

Since

$$\begin{aligned}
\mathrm{val}_{\max}^{\Delta}(K_n) &= \mathrm{val}(f) \\
&= \mathrm{val}(f_1) + \sum_{i=2}^{n-1}(f(u_i) - f(u_1)) + \sum_{i=2}^{n-1}(f(u_n) - f(u_i)) + (f(u_n) - f(u_1)) \\
&\leq \mathrm{val}_{\max}^{\Delta-2}(K_{n-2}) + \sum_{i=2}^{n-1}(\Delta - 0) + (\Delta - 0) \\
&= \mathrm{val}(g_1) + \sum_{i=2}^{n-1}(g(u_i) - g(u_1)) + \sum_{i=2}^{n-1}(g(u_n) - g(u_i)) + (g(u_n) - g(u_1)) \\
&= \mathrm{val}(g) \\
&\leq \mathrm{val}_{\max}^{\Delta}(K_n)\,,
\end{aligned}$$

it follows that

$$\mathrm{val}(f_1) = \mathrm{val}_{\max}^{\Delta-2}(K_{n-2})\,.$$

Thus

$$f_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2)\right].$$

Hence

$$f(V(K_{n-2})) = \left[1, \left\lfloor \frac{n-2}{2} \right\rfloor\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 2, (\Delta-1)\right]$$

and we have $f(u_1) = 0$, $f(u_n) = \Delta$. Therefore,

$$f(V(K_n)) = \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right].$$

Case 2. *n is odd.*

Again, we use mathematical induction on n. When $n = 3$, the result can be easily verified. Assume that $n \geq 5$ and the result holds for $K_{n'}$ when n' is odd and $3 \leq n' < n$. Let f be a γ^{Δ}-max labeling of K_n with $f(u_1) < f(u_2) < \cdots < f(u_n)$. By Corollary 4.2.4, $f(u_1) = 0$ and $f(u_n) = \Delta$. Let f_1 be a $\gamma^{\Delta-2}$-labeling of a complete graph K_{n-2} with vertex set $V(K_{n-2}) = \{u_2, u_3, \ldots, u_{n-1}\}$ defined by

$$f_1(u_i) = f(u_i) - 1 \text{ for each } 2 \leq i \leq n-1\,.$$

Let g_1 be a $\gamma^{\Delta-2}$-max labeling of K_{n-2}. Since $\Delta - 2 \geq \binom{n-2}{2}$, by induction hypothesis, we have

$$g_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2)\right] \cup \{k\}$$

where $k \in \left[\left\lfloor \frac{n-2}{2} \right\rfloor, (\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor\right]$.

We can extend g_1 to a γ^{Δ}-labeling g of K_n defined by

$$g(u) = \begin{cases} 0 & \text{if } u = u_1 \\ \Delta & \text{if } u = u_n \\ g_1(u) + 1 & \text{if } u \neq u_1, u_n \,. \end{cases}$$

Since

$$\begin{aligned} \mathrm{val}^{\Delta}_{\max}(K_n) &= \mathrm{val}(f) \\ &= \mathrm{val}(f_1) + \sum_{i=2}^{n-1}(f(u_i) - f(u_1)) + \sum_{i=2}^{n-1}(f(u_n) - f(u_i)) + (f(u_n) - f(u_1)) \\ &\leq \mathrm{val}^{\Delta-2}_{\max}(K_{n-2}) + \sum_{i=2}^{n-1}(\Delta - 0) + (\Delta - 0) \\ &= \mathrm{val}(g_1) + \sum_{i=2}^{n-1}(g(u_i) - g(u_1)) + \sum_{i=2}^{n-1}(g(u_n) - g(u_i)) + (g(u_n) - g(u_1)) \\ &= \mathrm{val}(g) \\ &\leq \mathrm{val}^{\Delta}_{\max}(K_n)\,, \end{aligned}$$

it follows that

$$\mathrm{val}(f_1) = \mathrm{val}^{\Delta-2}_{\max}(K_{n-2})\,.$$

Thus

$$f_1(V(K_{n-2})) = \left[0, \left\lfloor \frac{n-2}{2} \right\rfloor - 1\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2)\right] \cup \{k\}$$

where $k \in \left[\left\lfloor \frac{n-2}{2} \right\rfloor, (\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor\right]$.

Hence

$$f(V(K_{n-2})) = \left[1, \left\lfloor \frac{n-2}{2} \right\rfloor\right] \cup \left[(\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 2, (\Delta-1)\right] \cup \{k\}$$

where $k \in \left[\left\lfloor \frac{n-2}{2} \right\rfloor + 1, (\Delta-2) - \left\lfloor \frac{n-2}{2} \right\rfloor + 1\right]$,

and we have $f(u_1) = 0$, $f(u_n) = \Delta$. Therefore,

$$f(V(K_n)) = \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] \cup \{k\}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \Delta - \left\lfloor \frac{n}{2} \right\rfloor\right]$. □

The following result is the consequence of Theorem 4.2.9 when $\Delta = \binom{n}{2}$.

Theorem 4.2.10. *Let f be a γ-labeling of a complete graph K_n with*

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ-max labelings of K_n.

3. Unique γ-min labelings of graphs

Let G be a connected graph and f a γ-min labeling of G. Then G has a *unique* γ-min *labeling* if f and $\bar{f}$ are only two γ-min labelings of G. Consider Figure 1, since the γ-labelings f_1 and $\bar{f}_1$ are only two γ-min labelings of the path P_5 in Figure 1, it follows that the path P_5 has a unique γ-min labeling.

The goal of this section is to study a connected graph having the unique γ-min labeling. We also determine the minimum values of γ-labelings of some generalized trees with exterior major vertices. It is shown that they have no unique γ-min labeling, but not so for a path.

3.1 Unique γ-min labelings of graphs

Let G be a connected graph of order n and size m and f a γ-labeling of G. For each integer k with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$, let $f^k : V(G) \to \{0, 1, 2, \ldots, m\}$ be a γ-labeling of G defined by

$$f^k(v) = f(v) + k\,, \quad \text{for each } v \in V(G)\,.$$

Note that $f^k = f$ when $k = 0$.

Theorem 4.3.1. *Let G be a connected graph of order n and size m and f a γ-labeling of G. Then for each integer k with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$, $\mathrm{val}(f^k) = \mathrm{val}(f)$.*

Proof. Let k be an integer with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$.
Since $|f^k(u) - f^k(v)| = |(f(u) + k) - (f(v) + k)| = |f(u) - f(v)|$ for each $e = uv \in E(G)$, $\mathrm{val}(f^k) = \mathrm{val}(f)$. □

This also provides the following corollary.

Corollary 4.3.2. *Let G be a connected graph of order n and size m and f a γ-labeling of G. Then f is a γ-*max *labeling (γ-*min *labeling) of G if and only if f^k is a γ-*max *labeling (γ-*min *labeling) of G for each integer k with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$.*

By Theorem 2.5.2 and Corollary 4.3.2, we can verify that none of graphs with cycle has a unique γ-min labeling.

Theorem 4.3.3. *If a connected graph G has the unique γ-*min *labeling, then G is a tree.*

Proof. Let G be a connected graph of order n and size m. Assume that G contains a cycle. Then $m \geq n$. By Theorem 2.5.2, G has a γ-min labeling f such that $f(V(G)) = [0, n-1]$. Since $m \geq n$, $m - (n-1) \geq 1$. Thus G has a γ- labeling f^1. By Corollary 4.3.2, f^1 is a γ-min labeling of G. Since $f^1(V(G)) = [1, n]$, $f^1 \neq f$ and $f^1 \neq \bar{f}$. Therefore G has no unique γ-min labeling. □

Next, we determine that every path P_n of order n has a unique γ-min labeling. This starts by characterizing γ-min labelings of a path P_n.

Theorem 4.3.4. *Let f be a γ-labeling of a path $P_n : v_1, v_2, \ldots, v_n$ defined by*

$$f(v_i) = i - 1\,, \quad \text{for each integer } i \text{ with } 1 \leq i \leq n\,.$$

*Then f and $\bar{f}$ are only two γ-*min *labelings of P_n.*

Proof. By Theorem 2.2.3, we have $\text{val}_{\min}(P_n) = n - 1$. Since $\text{val}(f) = \text{val}(\bar{f}) = n - 1$, f and $\bar{f}$ are γ-min labelings of P_n. Let f_1 be a γ-min labelings of P_n. Then $\text{val}(f_1) = \text{val}_{\min}(P_n) = n - 1 =$ the size of P_n. Since $f_1'(e) = 1$ for each edge e in P_n, it follows that $|f_1(v_{i+1}) - f_1(v_i)|$ for each i, $1 \leq i \leq n-1$. Thus either $f_1 = f$ or $f_1 = \bar{f}$. Therefore f and $\bar{f}$ are only two γ-min labelings of P_n. □

Corollary 4.3.5. *A path has a unique γ-*min *labeling.*

The following result shows that there are many trees that fail to have unique γ-min labeling.

Theorem 4.3.6. *Let T be a tree with exterior major vertices. If there are at least two terminal vertices z_1 and z_2 of some exterior major vertex v of T such that $d(v, z_1) = d(v, z_2)$, then T has no unique γ-*min *labeling.*

Proof. Assume that there are at least two terminal vertices z_1 and z_2 of some exterior major vertex v of T such that $d(v, z_1) = d(v, z_2)$. By Theorem 2.5.2, T has a γ-min labeling f such that $f(V(T)) = [0, n-1]$. Let $P : v = u_0, u_1, \ldots, u_d = z_1$ be a $v - z_1$ path in T and $Q : v = w_0, w_1, \ldots, w_d = z_2$ be a $v - z_2$ path in T. Let f_1 be a γ-labeling of T defined by

$$f_1(a) = \begin{cases} f(a) & \text{if } a \in V(T) - \{u_i, w_j | 1 \leq i, j \leq d\} \\ f(w_i) & \text{if } a = u_i \quad \text{with } 1 \leq i \leq d \\ f(u_j) & \text{if } a = w_j \quad \text{with } 1 \leq j \leq d\,. \end{cases}$$

Then $\text{val}(f_1) = \text{val}(f) = \text{val}_{\min}(T)$. Thus f_1 is a γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$. Therefore T has no unique γ-min labeling. □

3.2 γ-min labelings of trees with exterior major vertices of degree 3

The *maximum degree* of a graph G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$. A *caterpillar* is a tree of order at least 3, the removal of whose end-vertices produces a path. We present the minimum value of a γ-labeling of a caterpillar T with $\Delta(T) = 3$ having an arbitrary number of exterior major vertices in [5].

Theorem 4.3.7 ([5]). *If T is a caterpillar of order $n \geq 4$ such that $\Delta(T) = 3$ and T has exactly k exterior major vertices, then*

$$\text{val}_{\min}(T) = n + k - 1\,.$$

Note that if a tree T is a caterpillar, then $d(v, z) = 1$ for each terminal vertex z of an exterior major vertex v of T. Next, we generalize a caterpillar of Theorem 4.3.7 to a tree T having $\Delta(T) = 3$ and $d(v, z) \geq 1$ for each terminal vertex z of an exterior major vertex v of T, and then formulate a generalized $\text{val}_{\min}(T)$.

Proposition 4.3.8. *Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \text{diam}(T)$. Then*

$$\text{val}_{\min}(T) \leq 2n - d - 2\,.$$

Proof. Let $P : v_0, v_1, \ldots, v_d$ be a path containing all exterior major vertices in T. Let $v_{l_1}, v_{l_2}, \ldots, v_{l_k}$ be all exterior major vertices in T such that $1 \leq l_1 < l_2 < \cdots < l_k \leq d-1$.

For each $1 \leq j \leq k$, let z_j be the terminal vertices of v_{l_j} not on P and $Q_j : v_{l_j} = u_{j0}, u_{j1}, \ldots, u_{jd_j} = z_j$ the $v_{l_j} - z_j$ path in T. Let f be a γ-labeling of T defined by

$$f(a) = \begin{cases} i & \text{if } a = v_i \text{ with } 0 \leq i \leq l_1 \\ \sum_{r=1}^{s} d_r + i & \text{if } a = v_i \text{ with } l_s + 1 \leq i \leq l_{s+1},\ 1 \leq s \leq k-1 \\ n - d - 1 + i & \text{if } a = v_i \text{ with } l_k + 1 \leq i \leq d \\ l_1 + i & \text{if } a = u_{1i} \text{ with } 1 \leq i \leq d_1 \\ \sum_{r=1}^{j-1} d_r + l_j + i & \text{if } a = u_{ji} \text{ with } 1 \leq i \leq d_j,\ 2 \leq j \leq k. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f) &= \sum_{e \in E(P)} f'(e) + \left(\sum_{e \in E(Q_1)} f'(e) + \sum_{e \in E(Q_2)} f'(e) + \cdots + \sum_{e \in E(Q_k)} f'(e) \right) \\ &= n - 1 + \sum_{i=1}^{k} d_i \\ &= 2n - d - 2\,. \end{aligned}$$

Therefore $\text{val}_{\min}(T) \leq \text{val}(f) = 2n - d - 2\,.$ □

We now establish the lower bound of minimum value of a tree with $\Delta(T) = 3$ having an arbitrary number of exterior major vertices of degree 3, as follows.

Proposition 4.3.9. *Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \text{diam}(T)$. Then*

$$\text{val}_{\min}(T) \geq 2n - d - 2\,.$$

Proof. Let g be an arbitrary γ-labeling of T. Since T has exactly $n-1$ edges, there are vertices $u, w \in V(T)$ with $g(u) = 0$ and $g(w) = n-1$. Let Q be a $u - w$ path in T. By Theorem 2.1.5,

$$\sum_{e \in E(Q)} g'(e) \geq |g(u) - g(w)| = n - 1.$$

Since the length of Q is at most $\text{diam}(T) = d$, there are at least $n - d - 1$ edges of T not on Q, and hence

$$\sum_{e \in E(T) - E(Q)} g'(e) \geq n - d - 1.$$

Thus

$$\begin{aligned} \mathrm{val}(g) &= \sum_{e\in E(Q)} g'(e) + \sum_{e\in E(T)-E(Q)} g'(e) \\ &\geq (n-1)+(n-d-1) \\ &= 2n-d-2\,. \end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \geq 2n-d-2$. □

Combining Propositions 4.3.8 and 4.3.9, we have the following.

Theorem 4.3.10. *Let T be a tree of order n with $\Delta(T)=3$ whose all major vertices are exterior major vertices and lie on the same path of length $d=\mathrm{diam}(T)$. Then*

$$\mathrm{val}_{\min}(T) = 2n-d-2\,.$$

With aid of Theorem 4.3.10 we are able to show that a tree in Theorem 4.3.10 has no unique γ-min labeling.

Theorem 4.3.11. *If T is a tree with $\Delta(T)=3$ whose all major vertices are exterior major vertices and lie on the same path of length* $\mathrm{diam}(T)$*, then T has no unique γ-*min *labeling.*

Proof. Let T be a tree with $\Delta(T)=3$ whose all major vertices are exterior major vertices and lie on the same path of length $\mathrm{diam}(T)$. Let $P: v_0, v_1, \ldots, v_d$ be a path containing all exterior major vertices in T. Let $v_{l_1}, v_{l_2}, \ldots, v_{l_k}$ be exterior major vertices in T such that $1 \leq l_1 < l_2 < \cdots < l_k \leq d-1$. For each $1 \leq j \leq k$, let z_j be the terminal vertices of v_{l_j} not on P and $Q_j : v_{l_j} = u_{j0}, u_{j1}, \ldots, u_{jd_j} = z_j$ the $v_{l_j} - z_j$ path in T. Let f_1 be a γ-labeling of T defined by

$$f_1(a) = \begin{cases} i & \text{if } a = v_i \ \text{ with } 0 \leq i \leq l_1 - 1 \\ \sum\limits_{r=1}^{s} d_r + i & \text{if } a = v_i \ \text{ with } l_s + 1 \leq i \leq l_{s+1},\ 1 \leq s \leq k-1 \\ n-d-1+i & \text{if } a = v_i \ \text{ with } l_k + 1 \leq i \leq d \\ l_1 + d_1 - i & \text{if } a = u_{1i} \text{ with } 0 \leq i \leq d_1 \\ \sum\limits_{r=1}^{j-1} d_r + l_j + i & \text{if } a = u_{ji} \text{ with } 1 \leq i \leq d_j,\ 2 \leq j \leq k. \end{cases}$$

Then

$$\begin{aligned}
\text{val}(f_1) &= \sum_{e\in E(P)} f_1'(e) + \left(\sum_{e\in E(Q_1)} f_1'(e) + \sum_{e\in E(Q_2)} f_1'(e) + \cdots + \sum_{e\in E(Q_k)} f_1'(e) \right) \\
&= n - 1 + \sum_{i=1}^{k} d_i \\
&= \text{val}_{\min}(T) \qquad\qquad \text{(by Theorem 4.3.10).}
\end{aligned}$$

Thus not only f_1 is a γ-min labeling of T, but the γ-labeling f in Proposition 4.3.8 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$. Therefore T has no unique γ-min labeling. □

3.3 γ-min labelings of trees with a unique exterior major vertex

In this section, we establish a minimum value of a γ-labeling of a tree with a unique exterior major vertex of an arbitrary degree. In order to do this, we recall the minimum value of a γ-labeling of a tree with a unique exterior major vertex of degree 3 shown in [5].

Theorem 4.3.12 ([5]). *Let T be a tree of order n with a unique exterior major vertex v of degree 3. If $d = \min\{d(v,z) \mid z$ is a terminal vertex of $v\}$, then*

$$\text{val}_{\min}(T) = n + d - 1\,.$$

Next, we generalize Theorem 4.3.12 to a tree T with a unique exterior major vertex of an arbitrary degree. We are now prepared to present the upper bound of minimum value of a γ-labeling of such a tree.

Proposition 4.3.13. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\text{val}_{\min}(T) \leq \begin{cases} n - 1 + \displaystyle\sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i & \text{if } \Delta(T) \text{ is even} \\ n - 1 + \displaystyle\sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i & \text{if } \Delta(T) \text{ is odd}. \end{cases}$$

Proof. Let $z_1, z_2, \ldots, z_{\Delta(T)}$ be the terminal vertices of an exterior major vertex v. For each $1 \leq i \leq \Delta(T)$, let $Q_i : v = v_{i0}, v_{i1}, \ldots, v_{id_i} = z_i$ be the $v - z_i$ path in T.

Case 1. *$\Delta(T)$ is even.*

Let f be a γ-labeling of T defined by

$$f(v_{ij}) = \begin{cases} \sum\limits_{\substack{i \le k \le \Delta(T) \\ k \text{ is even}}} d_k - j & \text{if } i \text{ is even, } 2 \le i \le \Delta(T) \text{ and } 1 \le j \le d_i \\ n - 1 + j - \sum\limits_{\substack{i \le k \le \Delta(T)-1 \\ k \text{ is odd}}} d_k & \text{if } i \text{ is odd, } 1 \le i \le \Delta(T) - 1 \text{ and } 1 \le j \le d_i \\ \sum\limits_{\substack{1 \le k \le \Delta(T) \\ k \text{ is even}}} d_k & \text{if } v_{ij} = v\,. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f) &= \left(\sum_{e \in E(Q_1)} f'(e) + \sum_{e \in E(Q_3)} f'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)-1})} f'(e) \right) \\ &\quad + \left(\sum_{e \in E(Q_2)} f'(e) + \sum_{e \in E(Q_4)} f'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)})} f'(e) \right) \\ &= (d_1 + (d_3 + d_1) + (d_5 + d_3 + d_1) + \cdots + (d_{\Delta(T)-1} + d_{\Delta(T)-3} + \cdots + d_1)) \\ &\quad + (d_2 + (d_4 + d_2) + (d_6 + d_4 + d_2) + \cdots + (d_{\Delta(T)} + d_{\Delta(T)-2} + \cdots + d_2)) \\ &= (d_1 + d_2 + \cdots + d_{\Delta(T)}) + (d_1 + d_2) + (d_1 + d_2 + d_3 + d_4) \\ &\quad + \cdots + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) \\ &= n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i\,. \end{aligned}$$

Therefore $\mathrm{val}_{\min}(T) \le \mathrm{val}(f) = n - 1 + \sum\limits_{j=1}^{\frac{\Delta(T)}{2}-1} \sum\limits_{i=1}^{2j} d_i\,.$

Case 2. *$\Delta(T)$ is odd.*

Let f be a γ-labeling of T defined by

$$f(v_{ij}) = \begin{cases} \sum\limits_{\substack{i \le k \le \Delta(T) \\ k \text{ is odd}}} d_k - j & \text{if } i \text{ is odd, } 1 \le i \le \Delta(T) \text{ and } 1 \le j \le d_i \\ n - 1 + j - \sum\limits_{\substack{i \le k \le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } i \text{ is even, } 2 \le i \le \Delta(T) - 1 \text{ and } 1 \le j \le d_i \\ \sum\limits_{\substack{1 \le k \le \Delta(T) \\ k \text{ is odd}}} d_k & \text{if } v_{ij} = v\,. \end{cases}$$

Then

$$\begin{aligned}
\text{val}(f) &= \left(\sum_{e \in E(Q_1)} f'(e) + \sum_{e \in E(Q_3)} f'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)})} f'(e) \right) \\
&\quad + \left(\sum_{e \in E(Q_2)} f'(e) + \sum_{e \in E(Q_4)} f'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)-1})} f'(e) \right) \\
&= (d_1 + (d_3 + d_1) + (d_5 + d_3 + d_1) + \cdots + (d_{\Delta(T)} + d_{\Delta(T)-2} + \cdots + d_1)) \\
&\quad + (d_2 + (d_4 + d_2) + (d_6 + d_4 + d_2) + \cdots + (d_{\Delta(T)-1} + d_{\Delta(T)-3} + \cdots + d_2)) \\
&= (d_1 + d_2 + \cdots + d_{\Delta(T)}) + d_1 + (d_1 + d_2 + d_3) \\
&\quad + \cdots + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) \\
&= n - 1 + \sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i \,.
\end{aligned}$$

Therefore $\text{val}_{\min}(T) \leq \text{val}(f) = n - 1 + \sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i$. □

We are able to show the lower bound of minimum value of a tree with a unique exterior major vertex of an arbitrary degree.

Proposition 4.3.14. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\text{val}_{\min}(T) \geq \begin{cases} n - 1 + \displaystyle\sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i & \text{if } \Delta(T) \text{ is even} \\ n - 1 + \displaystyle\sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i & \text{if } \Delta(T) \text{ is odd}. \end{cases}$$

Proof. Let g be an arbitrary γ-labeling of T. Since T has exactly $n-1$ edges, there are vertices $u_1, w_1 \in V(T)$ with $g(u_1) = 0$ and $g(w_1) = n - 1$. Let Q_1 be a $u_1 - w_1$ path in T. By Theorem 2.1.5,

$$\sum_{e \in E(Q_1)} g'(e) \geq |g(u_1) - g(w_1)| = n - 1.$$

Let $u_2, w_2 \in V(T)$ with

$$g(u_2) = \min\{g(x) \mid x \notin V(Q_1)\} \text{ and } g(w_2) = \max\{g(x) \mid x \notin V(Q_1)\}.$$

Let Q_2 be a $u_2 - w_2$ path in T. By Theorem 2.1.5,

$$\sum_{e \in E(Q_2)} g'(e) \geq |g(u_2) - g(w_2)| = g(w_2) - g(u_2).$$

Since the length of Q_1 is at most $\text{diam}(T) = d_{\Delta(T)} + d_{\Delta(T)-1}$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1}$ edges of T not on Q_1, and hence

$$g(w_2) - g(u_2) \geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1}.$$

Thus

$$\sum_{e \in E(Q_2)} g'(e) \geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} = d_1 + d_2 + \cdots + d_{\Delta(T)-2}.$$

Let $u_3, w_3 \in V(T)$ with

$$g(u_3) = \min\{g(x) \mid x \notin V(Q_1) \cup V(Q_2)\} \text{ and } g(w_3) = \max\{g(x) \mid x \notin V(Q_1) \cup V(Q_2)\}.$$

Let Q_3 be a $u_3 - w_3$ path in T. By Theorem 2.1.5,

$$\sum_{e \in E(Q_3)} g'(e) \geq |g(u_3) - g(w_3)| = g(w_3) - g(u_3).$$

Since the sum of the length of Q_1 and Q_2 is at most $d_{\Delta(T)} + d_{\Delta(T)-1} + d_{\Delta(T)-2} + d_{\Delta(T)-3}$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3}$ edges of T not on Q_1 and Q_2, and hence

$$g(w_3) - g(u_3) \geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3}.$$

Thus

$$\begin{aligned}\sum_{e \in E(Q_3)} g'(e) &\geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - d_{\Delta(T)-3} \\ &= d_1 + d_2 + \cdots + d_{\Delta(T)-4}.\end{aligned}$$

Continue until we have for each $1 \leq j \leq \left\lfloor \frac{\Delta(T)}{2} \right\rfloor$, let $u_j, w_j \in V(T)$ with

$$g(u_j) = \min\{g(x) \mid x \notin \bigcup_{i=1}^{j-1} V(Q_i)\} \text{ and } g(w_j) = \max\{g(x) \mid x \notin \bigcup_{i=1}^{j-1} V(Q_i)\}$$

and let Q_j be a $u_j - w_j$ path in T. Then

$$\begin{aligned}\sum_{e \in E(Q_j)} g'(e) &\geq (n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - \cdots - d_{\Delta(T)-2j+3} \\ &= d_1 + d_2 + \cdots + d_{\Delta(T)-2j+2}.\end{aligned}$$

Case 1. *$\Delta(T)$ is even.*

Then $\left\lfloor \frac{\Delta(T)}{2} \right\rfloor = \frac{\Delta(T)}{2}$. We have $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) = \emptyset$ or $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) \neq \emptyset$.

If $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) = \emptyset$, then

$$\begin{aligned}
\text{val}(g) &= \sum_{e \in E(Q_1)} g'(e) + \sum_{e \in E(Q_2)} g'(e) + \cdots + \sum_{e \in E\left(Q_{\frac{\Delta(T)}{2}}\right)} g'(e) \\
&\geq (n-1) + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) + (d_1 + d_2 + \cdots + d_{\Delta(T)-4}) \\
&\quad + \cdots + (d_1 + d_2 + d_3 + d_4) + (d_1 + d_2) \\
&= n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i \, .
\end{aligned}$$

If $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j) \neq \emptyset$, then

$$\begin{aligned}
\text{val}(g) &= \left(\sum_{e \in E(Q_1)} g'(e) + \sum_{e \in E(Q_2)} g'(e) + \cdots + \sum_{e \in E\left(Q_{\frac{\Delta(T)}{2}}\right)} g'(e) \right) + \sum_{e \in E(T) - \bigcup_{j=1}^{\frac{\Delta(T)}{2}} E(Q_j)} g'(e) \\
&\geq ((n-1) + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) + (d_1 + d_2 + \cdots + d_{\Delta(T)-4}) \\
&\quad + \cdots + (d_1 + d_2)) + 1 \\
&= n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i + 1 \\
&> n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i \, .
\end{aligned}$$

In general, $\text{val}(g) \geq n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i$. Therefore $\text{val}_{\min}(T) \geq n - 1 + \sum_{j=1}^{\frac{\Delta(T)}{2}-1} \sum_{i=1}^{2j} d_i$.

Case 2. *$\Delta(T)$ is odd.*

Then $\left\lfloor \frac{\Delta(T)}{2} \right\rfloor = \frac{\Delta(T)-1}{2}$, and so $E(T) - \bigcup_{j=1}^{\frac{\Delta(T)-1}{2}} E(Q_j) \neq \emptyset$.

Since the sum of the length of Q_j for all $1 \leq j \leq \frac{\Delta(T)-1}{2}$ is at most $d_{\Delta(T)} + d_{\Delta(T)-1} + d_{\Delta(T)-2} + \cdots + d_2$, there are at least $(n-1) - d_{\Delta(T)} - d_{\Delta(T)-1} - d_{\Delta(T)-2} - \cdots - d_2 = d_1$ edges of T not on Q_j for all $1 \leq j \leq \frac{\Delta(T)-1}{2}$.

Thus

$$\begin{aligned}
\operatorname{val}(g) &= \left(\sum_{e\in E(Q_1)} g'(e) + \sum_{e\in E(Q_2)} g'(e) + \cdots + \sum_{e\in E\left(Q_{\frac{\Delta(T)-1}{2}}\right)} g'(e) \right) + \sum_{e\in E(T)-\bigcup_{j=1}^{\frac{\Delta(T)-1}{2}} E(Q_j)} g'(e) \\
&\geq ((n-1) + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) + (d_1 + d_2 + \cdots + d_{\Delta(T)-4}) \\
&\quad + \cdots + (d_1 + d_2 + d_3)) + d_1 \\
&= n - 1 + \sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i \,.
\end{aligned}$$

Therefore $\operatorname{val}_{\min}(T) \geq n - 1 + \sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i \,.$ □

We compute the minimum value of a γ-labeling of a tree with a unique exterior major vertex of an arbitrary degree by combining Propositions 4.3.13 and 4.3.14 as follows.

Theorem 4.3.15. *Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then*

$$\operatorname{val}_{\min}(T) = n - 1 + \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} \left(\left\lfloor \frac{\Delta(T)}{2} \right\rfloor - i \right) (d_{2i-1} + d_{2i}) + \delta_\Delta \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} d_{2i-1}$$

where

$$\delta_\Delta = \begin{cases} 0 & \text{if } \Delta(T) \text{ is even} \\ 1 & \text{if } \Delta(T) \text{ is odd.} \end{cases}$$

We are now able to apply Theorem 4.3.15 to show that a tree with a unique exterior major vertex of an arbitrary degree has no unique γ-min labeling.

Theorem 4.3.16. *If T is a tree with a unique exterior major vertex, then T has no unique γ-min labeling.*

Proof. Let T be a tree with a unique exterior major vertex v. Let $z_1, z_2, \ldots, z_{\Delta(T)}$ be the terminal vertices of v. Let $Q_i : v = v_{i0}, v_{i1}, \ldots, v_{id_i} = z_i$ be the $v - z_i$ path of T for each $1 \leq i \leq \Delta(T)$.

Case 1. *$\Delta(T)$ is even.*

Let f_1 be a γ-labeling of T defined by

$$f_1(v_{ij}) = \begin{cases} \displaystyle\sum_{\substack{i\le k\le \Delta(T)\\ k \text{ is even}}} d_k - j & \text{if } i \text{ is even, } 4 \le i \le \Delta(T) \text{ and } 1 \le j \le d_i \\ n-1+j-\displaystyle\sum_{\substack{i\le k\le \Delta(T)-1\\ k \text{ is odd}}} d_k & \text{if } i \text{ is odd, } 3 \le i \le \Delta(T)-1 \text{ and } 1 \le j \le d_i \\ n-1-d_2+j-\displaystyle\sum_{\substack{3\le k\le \Delta(T)-1\\ k \text{ is odd}}} d_k & \text{if } i = 2 \text{ and } 1 \le j \le d_2 \\ d_1+\displaystyle\sum_{\substack{4\le k\le \Delta(T)\\ k \text{ is even}}} d_k - j & \text{if } i = 1 \text{ and } 1 \le j \le d_1 \\ d_1+\displaystyle\sum_{\substack{4\le k\le \Delta(T)\\ k \text{ is even}}} d_k & \text{if } v_{ij} = v\,. \end{cases}$$

Then

$$\begin{aligned} \mathrm{val}(f_1) &= \left(\sum_{e\in E(Q_1)} f_1'(e) + \sum_{e\in E(Q_3)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)-1})} f_1'(e)\right) \\ &\quad + \left(\sum_{e\in E(Q_2)} f_1'(e) + \sum_{e\in E(Q_4)} f_1'(e) + \cdots + \sum_{e\in E(Q_{\Delta(T)})} f_1'(e)\right) \\ &= (d_1 + (d_3+d_2) + (d_5+d_3+d_2) + \cdots + (d_{\Delta(T)-1} + d_{\Delta(T)-3} + \cdots + d_3 + d_2)) \\ &\quad + (d_2 + (d_4+d_1) + (d_6+d_4+d_1) + \cdots + (d_{\Delta(T)} + d_{\Delta(T)-2} + \cdots + d_4 + d_1)) \\ &= (d_1+d_2+\cdots+d_{\Delta(T)}) + (d_1+d_2) + (d_1+d_2+d_3+d_4) \\ &\quad + \cdots + (d_1+d_2+\cdots+d_{\Delta(T)-2}) \\ &= n-1+\sum_{j=1}^{\frac{\Delta(T)}{2}-1}\sum_{i=1}^{2j} d_i \\ &= \mathrm{val}_{\min}(T) \qquad \text{(by Theorem 4.3.15).} \end{aligned}$$

Thus f_1 is a γ-min labeling of T. Since the γ-labeling f in Case 1 of Proposition 4.3.13 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$, it follows that T has no unique γ-min labeling.

Case 2. *$\Delta(T)$ is odd.*

Let f_1 be a γ-labeling of T defined by

$$f_1(v_{ij}) = \begin{cases} \displaystyle\sum_{\substack{i \le k \le \Delta(T) \\ k \text{ is odd}}} d_k - j & \text{if } i \text{ is odd, } 3 \le i \le \Delta(T) \text{ and } 1 \le j \le d_i \\ n - 1 + j - \displaystyle\sum_{\substack{i \le k \le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } i \text{ is even, } 2 \le i \le \Delta(T) - 1 \text{ and } 1 \le j \le d_i \\ n - 1 - d_1 + j - \displaystyle\sum_{\substack{2 \le k \le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } i = 1 \text{ and } 1 \le j \le d_1 \\ n - 1 - d_1 - \displaystyle\sum_{\substack{2 \le k \le \Delta(T)-1 \\ k \text{ is even}}} d_k & \text{if } v_{ij} = v\,. \end{cases}$$

Then

$$\begin{aligned} \text{val}(f_1) &= \left(\sum_{e \in E(Q_1)} f_1'(e) + \sum_{e \in E(Q_3)} f_1'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)})} f_1'(e) \right) \\ &\quad + \left(\sum_{e \in E(Q_2)} f_1'(e) + \sum_{e \in E(Q_4)} f_1'(e) + \cdots + \sum_{e \in E(Q_{\Delta(T)-1})} f_1'(e) \right) \\ &= (d_1 + d_3 + (d_5 + d_3) + \cdots + (d_{\Delta(T)} + d_{\Delta(T)-2} + \cdots + d_5 + d_3)) \\ &\quad + ((d_2 + d_1) + (d_4 + d_2 + d_1) + \cdots + (d_{\Delta(T)-1} + d_{\Delta(T)-3} + \cdots + d_2 + d_1)) \\ &= (d_1 + d_2 + \cdots + d_{\Delta(T)}) + d_1 + (d_1 + d_2 + d_3) \\ &\quad + \cdots + (d_1 + d_2 + \cdots + d_{\Delta(T)-2}) \\ &= n - 1 + \sum_{j=1}^{\frac{\Delta(T)-1}{2}} \sum_{i=1}^{2j-1} d_i \\ &= \text{val}_{\min}(T) \qquad \text{(by Theorem 4.3.15).} \end{aligned}$$

Thus f_1 is a γ-min labeling of T, however the γ-labeling f in Case 2 of Proposition 4.3.13 is also γ-min labeling of T such that $f_1 \neq f$ and $f_1 \neq \bar{f}$. Therefore T has no unique γ-min labeling. □

CHAPTER 5

CONCLUSION AND OPEN PROBLEMS

We conclude main results of this work and give some open problems for future work in this chapter.

1. Conclusion

This section is to present our comprehensive work concerning the γ-labelings of graphs. The main results are as follows.

1.1 γ-labelings of cycles with one chord

Extremal values of γ-labelings of cycles with one chord

1. For every integer $n \geq 4$,

$$\mathrm{val}_{\min}(C_n + e) = 2n - 1\,.$$

2. For every odd integer $n \geq 5$,

$$\mathrm{val}_{\max}(C_n + e) = \frac{n^2 + 6n - 3}{2}.$$

3. For every even integer $n \geq 6$,

$$\mathrm{val}_{\max}(C_n + e) = \frac{n^2 + 6n + 2}{2}$$

where e is a chord joining two vertices with odd distance in even cycle C_n.

4. Let f be a γ-max labeling of $C_8 + e$ where e is a chord joining two vertices with even distance in cycle C_8. Then either no element of $ST(f)$ is monotone or there are exactly two monotone elements of $ST(f)$.

5. For every even integer n with $n \geq 10$, let f be a γ-max labeling of $C_n + e$ where e is a chord joining two vertices with even distance in even cycle C_n. Then $ST(f)$ contains exactly two monotone elements.

6. For every even integer $n \geq 4$,

$$\text{val}_{\max}(C_n + e) = \begin{cases} \frac{n^2+5n-2}{2} & \text{if } n = 4, 6, 8 \\ \frac{n^2+6n-10}{2} & \text{if } n \geq 10 \end{cases}$$

where e is a chord joining two vertices with even distance in even cycle C_n.

γ-spectra of cycles with one chord

For every integer $n \geq 4$,

$$\text{spec}(C_n + e) = \Big[\text{val}_{\min}(C_n + e), \text{val}_{\max}(C_n + e)\Big].$$

1.2 γ-max labelings of graphs with exterior major vertices

γ-orientations of γ-max labelings of graphs with exterior major vertices

1. Let $D(f)$ be a γ-orientation of a γ-max labeling f of a graph G. For every pair of exterior major vertex v and its terminal vertex z, an internal vertex of $v - z$ path in G is either a transmitter or a receiver in $D(f)$.

2. For any γ-orientation $D(f)$ of a γ-labeling f of a graph G, each terminal vertex of G is either a transmitter or a receiver in $D(f)$.

3. Let v be an exterior major vertex of a graph G. For any γ-orientation $D(f)$ of a γ-max labeling f of G, all vertices of exneighborhood $N^e(v)$ are either transmitters or receivers in $D(f)$, that is,

$$N^e(v) \subseteq N^+(v) \text{ or } N^e(v) \subseteq N^-(v) \text{ in } D(f).$$

4. Let $D(f)$ be a γ-orientation of a γ-max labeling f of a graph G. Then each vertex in an exbranch G_v of an exterior major vertex v is either a transmitter or a receiver in $D(f)$.

5. Let $D(f)$ be a γ-orientation of a γ-max labeling f of a graph G. For an exbranch G_v of an exterior major vertex v in G, let

$$X = \{x \in V(G_v) \mid \operatorname{id} x \leq \operatorname{od} x \text{ in } D(f)\}$$

and

$$Y = \{y \in V(G_v) \mid \operatorname{od} y < \operatorname{id} y \text{ in } D(f)\}.$$

Then $f(x) < f(y)$ for each pair of vertices $x \in X$ and $y \in Y$.

Maximum values of trees with a unique exterior major vertex

1. Let T be a tree with a unique exterior major vertex. For any γ-orientation $D(f)$ of a γ-max labeling f of T, each vertex in T is either a transmitter or a receiver in $D(f)$.

2. Let T be a tree of order n with a unique exterior major vertex of terminal degree 3. Then

$$\operatorname{val}_{\max}(T) = \begin{cases} \frac{n^2+n-8}{2} & \text{if the distances between } v \text{ and} \\ & \text{all its terminal vertices are even or odd} \\ \frac{n^2+n-6}{2} & \text{otherwise}. \end{cases}$$

Maximum values of trees with exterior major vertices

1. Let T be a tree of $\operatorname{ma}(T) = 2$ with adjacent exterior major vertices and f a γ-max labeling of T. Then an exterior major vertex of T is either a transmitter or a receiver in $D(f)$.

2. Let T be a tree of $\operatorname{ma}(T) = 2$ with adjacent exterior major vertices and f a γ-max labeling of T. Then every vertex in T is either a transmitter or a receiver in $D(f)$.

3. Let T be a tree of order n and $\operatorname{ma}(T) = 2$ with adjacent exterior major vertices. Let v_1 and v_2 be exterior major vertices of T. Then

$$\operatorname{val}_{\max}(T) = \alpha(ter^-(v_1), ter^+(v_1), ter^-(v_2), ter^+(v_2), n).$$

4. Let v_1 and v_2 be exterior major vertices of a graph G. Let P be a $v_1 - v_2$ path containing no major vertex as its internal vertex in G. For any γ-orientation $D(f)$ of a γ-max labeling f of G, each internal vertex of P, except at most one, is either a transmitter or a receiver in $D(f)$.

1.3 γ-max labelings of graphs

γ^{Δ}-max labelings of graphs

1. Let f be a γ^{Δ}-max labeling of a nontrivial graph G of order n and size m. Let $u, w \in V(G)$ with $f(u) = \min\{f(v) \mid v \in V(G)\}$ and $f(w) = \max\{f(v) \mid v \in V(G)\}$. Then neighborhoods of u and w are not empty.

2. Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-max labeling of G, then $\text{span}(f) = \Delta$.

3. Let G be a nontrivial graph of order n and size m and f a γ^{Δ}-labeling of G. If f is a γ^{Δ}-max labeling of G, then $\{0, \Delta\} \subseteq f(V(G))$.

γ-max labelings of complete bipartite graphs

1. Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, let f be a γ^{Δ}-labeling of $K_{r,s}$ with

$$f(V_r) = [0, r-1] \text{ and } f(V_s) = [\Delta - (s-1), \Delta],$$

where $\Delta \geq rs$. Then f and $\bar{f}$ are only two γ^{Δ}-max labelings of $K_{r,s}$.

2. Let $K_{r,s}$ be a complete bipartite graph with partite sets V_r and V_s of cardinalities r and s, respectively, let f be a γ-labeling of $K_{r,s}$ with

$$f(V_r) = [0, r-1] \text{ and } f(V_s) = [rs - (s-1), rs].$$

Then f and $\bar{f}$ are only two γ-max labelings of $K_{r,s}$.

γ-max labelings of complete graphs

1. Let f be a γ^{Δ}-labeling of a complete graph K_n with

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\Delta - \left\lfloor \frac{n}{2} \right\rfloor + 1, \Delta\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

where $\Delta \geq \binom{n}{2}$ and $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \Delta - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ^{Δ}-max labelings of K_n.

2. Let f be a γ-labeling of a complete graph K_n with

$$f(V(K_n)) = \begin{cases} \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] & \text{if } n \text{ is even} \\ \left[0, \left\lfloor \frac{n}{2} \right\rfloor - 1\right] \cup \left[\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2}\right] \cup \{k\} & \text{if } n \text{ is odd} \end{cases}$$

where $k \in \left[\left\lfloor \frac{n}{2} \right\rfloor, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right]$. Then f and $\bar{f}$ are only two γ-max labelings of K_n.

1.4 Unique γ-min labelings of graphs

Unique γ-min labelings of graphs

1. Let G be a connected graph of order n and size m and f a γ-labeling of G. Then for each integer k with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$, $\text{val}(f^k) = \text{val}(f)$.

2. Let G be a connected graph of order n and size m and f a γ-labeling of G. Then f is a γ-max labeling (γ-min labeling) of G if and only if f^k is a γ-max labeling (γ-min labeling) of G for each integer k with $0 \leq k \leq m - \max\{f(v) \mid v \in V(G)\}$.

3. If a connected graph G has the unique γ-min labeling, then G is a tree.

4. Let f be a γ-labeling of a path $P_n : v_1, v_2, \ldots, v_n$ defined by

$$f(v_i) = i - 1\,, \quad \text{for each integer i with } 1 \leq i \leq n\,.$$

Then f and $\bar{f}$ are only two γ-min labelings of P_n.

5. A path has a unique γ-min labeling.

6. Let T be a tree with exterior major vertices. If there are at least two terminal vertices z_1 and z_2 of some exterior major vertex v of T such that $d(v, z_1) = d(v, z_2)$, then T has no unique γ-min labeling.

γ-min labelings of trees with exterior major vertices of degree 3

1. Let T be a tree of order n with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $d = \text{diam}(T)$. Then

$$\text{val}_{\min}(T) = 2n - d - 2\,.$$

2. If T is a tree with $\Delta(T) = 3$ whose all major vertices are exterior major vertices and lie on the same path of length $\text{diam}(T)$, then T has no unique γ-min labeling.

γ-min labelings of trees with a unique exterior major vertex

1. Let T be a tree of order n with a unique exterior major vertex v. If $d_1, d_2, \ldots, d_{\Delta(T)}$ are the distances between v and all its terminal vertices with $d_1 \leq d_2 \leq \cdots \leq d_{\Delta(T)}$, then

$$\mathrm{val}_{\min}(T) = n - 1 + \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} \left(\left\lfloor \frac{\Delta(T)}{2} \right\rfloor - i \right) (d_{2i-1} + d_{2i}) + \delta_\Delta \sum_{i=1}^{\left\lfloor \frac{\Delta(T)}{2} \right\rfloor} d_{2i-1}$$

where

$$\delta_\Delta = \begin{cases} 0 & \text{if } \Delta(T) \text{ is even} \\ 1 & \text{if } \Delta(T) \text{ is odd}. \end{cases}$$

2. If T is a tree with a unique exterior major vertex, then T has no unique γ-min labeling.

2. Open Problems

We have some open problems for future work as follows.

1. In chapter 3, we have established the extremal values of a γ-labeling of a cycle with one arbitrary chord and the γ-spectrum of a cycle with one arbitrary chord. A natural question arises how to determine the extremal values of a γ-labeling of a cycle with two or more chords and then the γ-spectrum of a cycle with two or more chords.

2. In sections 1 of chapter 4, we determine the maximum value of a γ-labeling of a tree with at most 2 exterior major vertices. It would be interesting to study the maximum value of γ-labeling of any tree and unicyclic graph.

3. The characterization of γ-max labelings of $K_{r,s}$ and K_n are determined. The main open question is to characterize γ-min labelings of those graphs.

4. Theorems 4.3.6, 4.3.11 and 4.3.16 show that some trees with exterior major vertices have no unique γ-min labeling. However, Corollary 4.3.5 shows that a path has a unique γ-min labeling. All such results lead us to the conjecture:

"A connected graph G has the unique γ-min labeling if and only if G is a path."

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