A FRAMEWORK FOR CHARACTERIZING LOWER SECONDARY SCHOOL STUDENTS' ALGEBRAIC THINKING

A DISSERTATION

BY

NATCHA KAMOL

Presented in Partial Fulfillment of the Requirements for the

Doctor of Education Degree in Mathematics Education

at Srinakharinwirot University

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A FRAMEWORK FOR CHARACTERIZING LOWER SECONDARY SCHOOL STUDENTS' ALGEBRAIC THINKING

AN ABSTRACT

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Natcha Kamol. (2548), *A framework for characterizing lower secondary school students' algebraic thinking*. Dissertation, Ed.D. (Mathematics Education). Bangkok: Graduate School, Srinakharinwirot University. Advisor Committee: Assoc. Prof. Dr. Piyavadee Wongyai, Assoc. Prof. Dr. Chaweewan Sawetamalya, Assoc. Prof. Dr. Siriporn Thipkong.

The purpose of this study was to formulate and validate a framework for characterizing lower secondary school students' algebraic thinking. Such framework was based on previous studies and the SOLO model of Biggs and Collis (1982 / 1991). The three key indicators representing algebraic thinking in this framework were pattern, representation, and variable. For each of these indicators, four levels of algebraic thinking were identified: Level 1 (prestructural), Level 2 (unistructural), Level 3 (multistructural), and Level 4 (relational). Descriptors of characteristic of algebraic thinking are developed across four levels of thinking as an initial framework and used to design algebraic problems. The framework was refined and validated through data obtained from paper works and interviews of 24 students across Mathayomsuksa one through three, eight case studies for each grade.

Results of the study indicated that students illustrated through four levels of thinking within each algebraic indicator and also coincided with the prestructural, unistructural, multistructural, and relational levels of the model of Biggs and Collis (1991). The profile of students' levels of algebraic thinking showed strong stability across the three indicators. It confirms that the framework provided a cohesive picture of lower secondary school students' algebraic thinking. The distribution of the algebraic thinking levels over the three lower secondary grades produces little difference among the three grade levels.

กรอบแสดงลักษณะการคิดเชิงพีชคณิตสำหรับนักเรียน
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บทคัดย่อ

ของ

ณัชชา กมล

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของ
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วัตถุประสงค์ของการวิจัยเพื่อพัฒนาและปรับกรอบแสดงลักษณะการคิดเชิงพีชคณิตสำหรับนักเรียนชั้นมัธยมศึกษาตอนต้น ซึ่งกรอบแสดงลักษณะการคิดดังกล่าวพัฒนามาจากการสังเคราะห์งานวิจัยและโซโลโมเดล (Structure of the Observing Learning Outcome) ของบิกส์และคอลลิส (1982 / 1991) โดยกรอบการคิดแบ่งเป็น 4 ระดับคือระดับที่ 1 (prestructural) ระดับที่ 2 (unistructural) ระดับที่ 3 (multistructural) และระดับที่ 4 (relational) ส่วนแนวคิดเชิงพีชคณิตใช้แนวคิดใน 3 เรื่อง ได้แก่ แบบรูป การนำเสนอ และตัวแปร เป็นตัวแทนที่บ่งชี้ถึงลักษณะการคิดเชิงพีชคณิตของนักเรียนชั้นมัธยมศึกษาตอนต้น แต่ละเรื่องจะมีคำอธิบายลักษณะการคิดเชิงพีชคณิตที่คาดว่านักเรียนจะแสดงออกในแต่ระดับการคิด ซึ่งคำอธิบายลักษณะการคิดเชิงพีชคณิตที่พัฒนาขึ้นจะนำมาเป็นข้อมูลพื้นฐานในการสร้างโจทย์ปัญหาในแบบทดสอบพีชคณิต ในแบบทดสอบดังกล่าวประกอบด้วย โจทย์ปัญหาด้านแบบรูป 4 ข้อ ด้านการนำเสนอ 3 ข้อ และด้านตัวแปร ที่มีคำถามย่อย 7 คำถาม การวิเคราะห์ข้อมูลเพื่อปรับคำอธิบายกรอบแสดงลักษณะการคิดเชิงพีชคณิตในแต่ละระดับได้จากผลการทดสอบและสัมภาษณ์นักเรียนชั้นมัธยมศึกษาปีที่ 1 ถึง 3 จำนวน 24 คน ชั้นละ 8 คน

ผลจากการวิจัยพบว่า นักเรียนแสดงลักษณะการคิดตรงตามที่ผู้วิจัยคาดหวังในกรอบแสดงลักษณะการคิดเชิงพีชคณิตสำหรับนักเรียนชั้นมัธยมศึกษาตอนต้นครบทั้ง 4 ระดับ ของแต่ละตัวบ่งชี้ ซึ่งตรงกับการจัดระดับการคิดของนักวิจัยหลายท่าน เช่น บิกส์และคอลลิส ที่ระบุไว้ นักเรียนส่วนใหญ่ยังมีระดับการคิดในระดับเดียวกันในทั้งสามตัวบ่งชี้ ซึ่งทำให้เห็นว่ากรอบแสดงลักษณะการคิดที่ได้สามารถนำไปใช้อธิบายการคิดเชิงพีชคณิตของนักเรียนชั้นมัธยมศึกษาตอนต้นได้และจากการศึกษายังพบว่า นักเรียนทั้งสามชั้นมีระดับการคิดเชิงพีชคณิตต่างกันเพียงเล็กน้อย

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CHAPTER 1

INTRODUCTION

Background

In improving students' learning in mathematics, it is necessary to understand the developmental mode of their thinking and reasoning. With the nature of mathematics that deals with abstract entities, students have difficulty in understanding the mathematics content, especially algebra. Therefore, thinking or in particular algebraic thinking is a tool for understanding abstraction (Russell. 1999: 1). Students are capable of powerful mathematical thinking but traditional instruction usually focuses on the memorization of facts and computational procedures rather than building on children's natural ways of thinking about mathematics. Students cannot access and understand the content exactly. This has an effect on the ability of students in learning mathematics. Mullis and others (2000: 98-112) reported on the finding of the Third International Mathematics and Science Study-Repeated (TIMSS-R). These findings revealed the average achievement of eighth grade students in mathematics content areas of 38 countries participating in this study. It is noted that the average achievement of Thai students in mathematics was lower than the average achievement of eighth-graders from other countries. In particular, Thai students achieved the lowest percent of correct responses in algebra among all mathematical content areas. Similarly, the findings of other countries indicated that algebra was a major stumbling block for many students in secondary schools (Herscovics. 1989: 60).

There are several studies related to algebraic thinking. It was found that students could develop their algebraic thinking by using pictorial representations or questions which teachers regularly asked their students. Problem-solving techniques, technology including spreadsheets, Logo programming language, the student-teacher discourse as well as Cognitively Guided Instruction also helped students to illustrate their algebraic thinking (Clements; & Sarama. 1999: 22-30; Lubinski; & Otto. 1999: 99-105; Ferrini-Mundy; Lappan; & Phillips. 1999: 112-119; Carpenter; & Levi. 2000: Online; Steele. 2000: 92-96). Patterns are tools in investigating algebraic thinking as children search for collecting, organizing, and graphing data (Enright. 1998: 174-178; Herbert; & Brown. 1999: 123-128). Students use equations to describe and represent problem situations (Swafford; & Langrall.

2000: 89-112), this is considered the theme of most research studies related to algebraic thinking.

Although the previous research contributed substantial evidence on students' algebraic thinking and some elements of students' algebraic thinking have been identified (Bishop. 2000: 107-126; Orton; & Orton. 1999: 104-120; Pegg; & Redden. 1990: 386-391; Zack. 1995: 106-113), research on students' algebraic thinking is not well established. Existing research on students' algebraic thinking has not yet generated a framework for systematically characterizing students' algebraic thinking based on the cognitive learning theory such as Biggs and Collis's (1982) SOLO (Structure of the Observing Learning Outcome) model. The SOLO model is only one possibility. In essence, if an algebraic thinking framework is developed, it would provide detailed cognitive models of students' learning that can guide the construction and planning of mathematics instruction and curriculum as Cobb and others (1991 : 3-29) have suggested. Some cognitive frameworks have been developed and used to describe and predict students' thinking within other mathematical domains: addition and subtraction (Carpenter; & Moser. 1984: 179-202), geometry (van Hiele. 1985: 243-252), multidigit number sense (Jones; Thornton; & Putt. 1994: 117-143; Jones; et al. 1996: 310-336), probability (Jones; et al. 1997: 101-125; Tarr; & Jones. 1997: 39-59), statistics (Jones; et al. 2000: 269-307; Mooney. 2002: 23-63) and patterns (Orton; & Orton. 1999: 104-120; Pegg; & Redden. 1990: 386-391). However, only the framework on patterns focuses to a small extent on algebra. There is a need for comprehensive and coherent studies of this kind.

In Thailand, the new mathematics curriculum was developed and implemented in 2003. Although algebraic thinking, a tool for learning algebra, is one of the themes for developing students' understanding of mathematics, there is little suggestion of what should be used in classroom activities. Nor is there a benchmark or research-based knowledge of students' algebraic thinking for teachers to assess students. There is a lack of teachers' understanding in developing students' algebraic thinking. In order to fulfill this need, the researcher seeks to characterize Thai lower secondary school students' algebraic thinking by developing a framework of algebraic thinking. The framework can be used as a benchmark in assessing students' algebraic thinking which will help teachers identify their students' level of algebraic thinking. In learning how their students think when confronted with algebraic content, teachers can use it as a base line to plan, implement, and assess lessons on algebraic thinking. Since the lower secondary curriculum is where algebra is systematically grounded and plays a major role, the researcher believes that a framework of algebraic thinking of students in the lower secondary school will be the most fruitful.

Research Question

This study was conceptualized due to the need to develop students' algebraic thinking in the lower secondary school level and there was no evidence of previous research in constructing a framework of algebraic thinking with lower secondary school students.

The researcher sought to find out "what is a framework that characterizes lower secondary school students' algebraic thinking in three key areas: pattern, representation, and variable?"

Purposes of the Study

The purposes of this study were to

a) formulate the framework for characterizing lower secondary school students' algebraic thinking in the content areas of pattern, representation, and variable;

b) refine and validate an initial framework using students' responses to the assessment protocol and case study analyses.

Significance of the Study

Understanding students' algebraic thinking is essential in providing better instructional practice for developing students' algebraic thinking. In a broader sense, there is a need to assess students' algebraic thinking to improve meaningful instruction and in class monitoring. Thus, this study significantly contributed to the lack of an existing framework for assessing and instructing students' algebraic thinking. It also suggested new generated knowledge on algebraic thinking for future research. In particular, this study provided teachers a cohesive tool to evaluate students' progress in algebraic thinking as a basis for making decisions about appropriate methods of instruction. The resulting framework provided benchmarks for developing curriculum in the future and also helped teacher to design a sequence of teaching episodes that further enhanced students' algebraic thinking from the lowest level of thinking to the highest level in the framework.

Theoretical Framework

In this study, the researcher reviewed the existing studies about algebraic thinking, the SOLO model of Biggs and Collis (1982; 1991: 57-76) and other previous studies which related to classify levels of thinking as background knowledge in developing an initial framework. The theoretical framework of this study is shown in Figure 1.

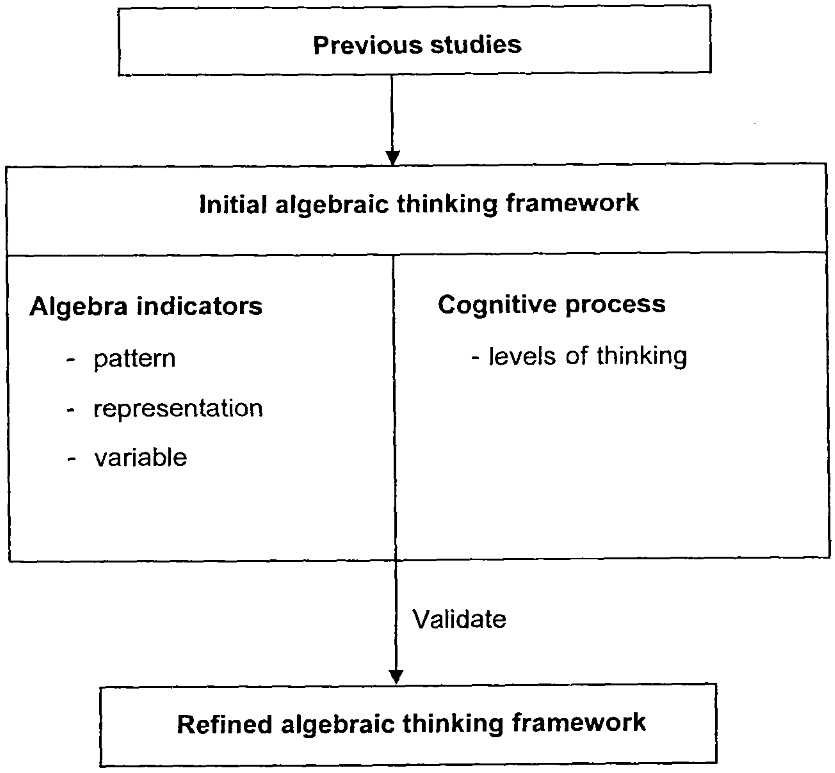


FIGURE 1 THE PROCESS OF DEVELOPING AN ALGEBRAIC THINKING FRAMEWORK

Definition of Terms

1. Algebraic thinking refers to students' abilities in using their thinking skills to understand algebra. In this study, algebra will be limited to three key indicators of algebra: pattern (generalizing pattern), representation (interpreting and transforming) and variable (understanding the role of variable).

1.1 Algebraic thinking on pattern refers to the abilities of students to use their thinking skills to generalize the pattern in symbolic form when they are confronted with algebra problems.

1.2 Algebraic thinking on representation refers to the abilities of students to use their thinking skills to solve algebra problems by interpreting different forms of representations such as tables and graphs. It also transforms among representations.

1.3 Algebraic thinking on variable refers to the abilities of students to use their thinking skills to understand the role of variable as generalized number.

2. **A framework for characterizing algebraic thinking** means a detailed cognitive model describing systematically students' thinking across four levels when they are confronted with algebraic tasks. This framework consists of three components: indicators, levels of thinking, and descriptors.

2.1 An indicator means a critical key for characterizing algebraic thinking in developing a framework. This study focuses on three key indicators of algebra which are considered to represent the characteristics of algebraic thinking that is the generalization of pattern, the use of various forms of representation, and the understanding of variable.

2.2 A level of thinking means an identifiable and distinguishable cluster in students' thinking when developing understanding of each key indicator of algebra. This study identifies four different levels of thinking: Level 1 (prestructural), Level 2 (unistructural), Level 3 (multistructural), and Level 4 (relational) based on the levels of the SOLO model.

2.3 A descriptor means an expected characteristic of students' algebraic thinking for each level of development within each indicator of a framework.

Scope of the Study

Sample

The subjects consisted of twenty-four students from two Thai secondary schools, that is twelve students from each school. They were lower secondary school students from all three grades: Mathayomsuksa one, Mathayomsuksa two and Mathayomsuksa three, four from each grade.

Content area

In general, algebraic thinking is very complex and has different aspects. This study used pattern, representation, and variable as indicators to characterize students' algebraic thinking. Although other indicators of algebraic thinking could have been used, this framework for characterizing algebraic thinking was limited to the three indicators described previously.

CHAPTER 2

REVIEW OF RELATED LITERATURE

In this study, the researcher investigated and characterized lower secondary school students' algebraic thinking according to a cognitive framework. The initial cognitive framework was formulated from previous studies and documents and the researcher validated the cognitive framework empirically. As such, the framework of this study provided research-based knowledge in developing student's algebraic thinking. Thus, in this chapter the researcher studied the documents and research related to such areas.

1. Algebraic Thinking
2. The SOLO Model
3. Related Studies
    3.1 Research Using the SOLO Model
    3.2 Research on Framework other than the SOLO Model
    3.3 Research on Pattern
    3.4 Research on Representation
    3.5 Research on Variable
4. The Tentative Initial Framework on Algebraic Thinking

The chapter is developed as follows:

1. Algebraic Thinking

Algebraic thinking is a part of mathematical thinking which combines mathematical thinking skills and areas of algebraic content, that is using mathematical thinking skills such as reasoning, representation, problem solving, and so on, to understand the ideas of algebraic content. Likewise, Kriegler (2003: Online) suggested that algebraic thinking is organized into two major components. One is the development of mathematical thinking tools including analytical habits of mind, specifically problem solving skills, reasoning skills, and representation skills. The other is fundamental algebraic ideas in which mathematical thinking tools can be developed. Hyde and Hyde (1991: 29) also defined mathematics thinking as the broader term of process involving four key processes, that is reasoning, communication, metacognition, and problem solving. In a related fashion, mathematical thinking involves using mathematically rich thinking skills to understand ideas in mathematics which relate the various branches of mathematics such as arithmetic, algebra, and geometry (O' Daffer; & Thornquist. 1993: 43; Caroll. 1994: 5).

Mathematical reasoning is a part of mathematical thinking that involves forming generalizations and drawing valid conclusions about ideas (O' Daffer; & Thornquist. 1993: 43). Moreover, the developing of algebraic thinking relates to representation, proportional reasoning, balance, meaning of variable, pattern and function, inductive reasoning and deductive reasoning (Greenes; & Findell. 1999: 127-137). Similarly, Arens and Meyer (2000: 6) stated that algebraic thinking is understood as using mathematical processes in which students seek patterns and relationships, sort and classify, formulate rules, and discover strategies for solving problems. By analogy, algebraic reasoning can be considered as an aspect of algebraic thinking.

However, algebraic thinking can be considered as using thinking skills to understand the algebraic ideas as the key word in developing algebraic thinking. It is necessary to understand how algebraic ideas are developed. There are many mathematics educators who have suggested algebraic content in middle schools. It can be divided into three themes. The first theme is an understanding of such algebra concepts as variables, expressions, functions, and equations (Christmas; & Fey. 1999: 14-16; Friedlander; & Hershkowitz; 1997: 442; Lodholz. 1990: 26; National Council of Teachers of Mathematics (NCTM). 1989: 102; 2000: 222-231; Mullis; et al. 2001: 14). Second is the content area of patterns; that is, students should have the ability to generalize situations, patterns and relationships among quantities, to identify and express patterns (Balacheff. 2001: 250; Howden. 1990: 9; NCTM. 1989: 102; 2000: 222-231; Mullis; et al. 2001: 14; Usiskin. 1999: 5-6). The third theme of algebra is the use of multiple representations which students can construct and analyze, such as graphs, tables, and algebraic symbols. The students can also compare and contrast these representations (Driscoll. 1999: 141; Dyke; & Craine. 1999: 215-219; Moses. 2000: 5; NCTM. 1989: 102; 2000: 222-225; Tang; & Ginsburg. 1999: 51-52). Similarly, the current Thai curriculum (B.E. 2544), the Institute for the Promotion of Teaching Science and Technology (IPST) (2001: 126-128) includes patterns, variables, algebraic expressions, equations, inequality, and representations in the algebraic strand.

Although the content areas of algebra in the middle grades incorporate patterns, variables, and representations, some authors described and considered the meaning of algebraic thinking in its different aspects. Herbert and Brown (1999: 123-124) defined algebraic thinking as using mathematical symbols and tools to analyze different situations by 1) extracting information from a situation; 2) representing that information mathematically in words, diagrams, tables, graphs, and equations; and 3) interpreting and applying mathematical findings [such as solving for unknowns, testing conjectures, and identifying functional relationships] to the same situations and to new and related situations. As, Friel and others (2002: 1-5) suggested in Navigating through Algebra in Grade 6-8, "to think algebraically, students should be able to understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; and analyze change in various contexts."

Algebraic thinking also involves the construction and representation of patterns (Austine; & Thompson. 1997: 274; Kaput. 1993: 11-19). Hence researchers concerned about the dimensions of algebraic thinking, have found that it consists of knowledge of structures, use of variables, understanding of functions, symbol facility and flexibility, generalizing, inverting, and reversing operations and relations, and the ability to formalize arithmetic patterns (Wagner; & Kieran. 1999: 368). Moreover, algebraic thinking has been referred to as the ability of students to think about situations or concept areas that relate to algebra. When one talks about algebraic thinking, one refers to the ideas that engage students in illustrating their thinking on algebra. These topics consist of the construction and representation of patterns and regularities, meaning of variable, patterns, functions, generalization, proportional reasoning, balance, inductive reasoning, deductive reasoning, as well as active exploration and conjecture (Carpenter; & Levi. 2000: Online; Greenes; & Findell. 1998: 127-135; Kieran; & Chalouh. 1993: 179-191; Noguera; & Ndunda. 2002: 2-3).

In this study, the term algebraic thinking means students' abilities in using their thinking skills tō understand a common component of algebra limited to pattern (generalizing pattern), representation (interpreting and transforming among representations) and variable (understanding the role of variable).

2. The SOLO Model

Many articles (Biggs; & Collis. 1982; 1991: 57-76; Chick. 1998: 4-8; Collis; Romberg; & Jurdak. 1986: 206-211; Jurdak. 1991: 155-162; Pegg; & Davey. 1998: 109-135) have described the features of the SOLO model which is briefly summarized in the following paragraphs.

The SOLO model (sometimes called the SOLO taxonomy) was initially developed in 1982 by Biggs and Collis as a general model of intellectual development. This model provided a method to analyze and categorize different levels of cognitive performance by considering the structure of the observed learning outcome from students' responses to the degree of complexity of questions posed in a variety of subject/topic areas. There is research using the SOLO model in several areas, such as probability, statistics, history, geography, and poetry.

This model consists of two characteristics: five modes of functioning and five levels of response within each mode of functioning. The SOLO model postulates that all learning occurs in one of five modes of functioning. These five modes closely related to Piaget's stages of cognitive development, increase in sophistication from concrete operations to abstract concepts. However a major modification to Piaget's original work is in placing the early formal stage into the earlier group of stages covered by concrete actions. Biggs and Collis claimed that most children between 13 and 15 years of age are "concrete generalizers", not "formal thinkers". That is, students in this age range are still tied to their own concrete experience.

Although the correspondence between SOLO modes and Piaget's stages does not extend to Piaget's structural tradition, the SOLO classification does not imply that the way students perform in different situations is typical of their stage of cognitive development, nor that this is necessarily related to their age. In this sense, student' s performance is not seen in terms of stages, that is, Biggs and Collis viewed student's performance as a more individual characteristic that is both content and context specific. The amount of information that can be retained in memory, and features specific to the task, are important determining variables. Hence, coding a response is dependent on its abstractness (referred to it as the mode of functioning) and on the individual's ability to handle the task with increased sophistication (referred to it as the level of response).

The five modes of functioning in the SOLO model of Biggs and Collis, progressing from concrete actions to abstract concepts (with an indication of the age at which they generally begin to appear) are described as follows:

1. *Sensorimotor* (from after birth). The infant can only interact with the world in the most concrete way. Learning develops through sensory stimulus.

2. *Ikonic* (from around 2 years). The child learns through internal imagery or pictures. The process of internalizing an action is to imagine it by forming an internal picture or "ikon."

3. *Concrete-symbolic* (from around 6 years). This mode involves a more abstract process of learning and is considered as a significant shift in abstraction, from direct symbolization of the world through oral language, to written, second order, symbol systems that apply to the experienced world. This process is the major task in primary and secondary schooling according to any curriculum theory.

4. *Formal* (from around 15 or 16 years). An advanced abstract system is developed in this mode that can be used to build hypotheses about alternative ways of ordering the world. This formal mode is the level of abstraction usually required in undergraduate study, and some evidence of formal thought in the proposed area of study should be essential for admission to a university.

5. *Postformal* (from around 22 years). Postformal thinking is evident in the questioning of current theories and establishment of new theories and practices. It can be seen in the high level innovations in many fields.

Biggs and Collis suggested that there is no implication which shows that a student who is able to respond in the concrete symbolic mode in one context is able or would wish to respond in the same mode in other contexts. Most students in elementary and secondary school are capable of operating within the concrete symbolic mode, usually the target mode for instruction in primary and secondary school. Teaching techniques are appropriately adapted to learners at this level. Some students may, however, still respond to stimuli in the ikonic mode, whereas others may respond with formal reasoning in some topics. This is really expected as part of the latest model (Biggs; & Collis. 1991: 57-76).

The second characteristic of the model is a series of levels that measures an increasing complexity of students' responses in handling certain tasks within a certain mode. The five levels of response which are applied to each mode of functioning are the following:

1. *Prestructural* (below the target mode). The learner is frequently distracted or misled by irrelevant aspects of the situation and does not engage the task in the mode involved. Typical examples are responses such as, " I do not understand the question"; "Haven't done those yet."

2. *Unistructural* (U). The student focuses on the domain/problem, but uses only

one piece of relevant data. The conclusion will be invalid. For example, when a student is asked for the value of □ that makes the following true, 2 + 3 = 3 + □, one of the most common responses at this level is " 5," that is, 2 +3.

3. *Multistructural* (M). The student uses two or more pieces of data, but with no synthesis of information or without perceiving any relationships between them. No integration occurs. An example of this level of student's response may be seen in the ability to give the correct answer to questions such as, "Which operation, +, -, x, ÷, must * be to make the following statement true, (4 * 2) * 6 = 12?"

4. *Relational* (R). The student can now use all data available, with each piece woven into an overall mosaic of relationships. The whole has become a coherent structure with no inconsistency within the known system. At this level, students respond correctly to items such as, "Which operation, +, -, x, ÷, must ^ and * be to make the following statement true, (3 ^ 4) ^ 1 = 12 * (6 * 2)?"

5. *Extended abstract.* The student goes beyond the data into a new mode of reasoning and can generalize from new and abstract features. The response involves a comprehensive use of the given data together with related hypothetical constructs and abstract principles. Responding at this level is necessary for students to achieve correct answers to the following question: "a, b, c, ..., etc. can be any of the numbers 0, 1, 2, 3, ..., etc., and * is an operation such that a * b = a + 2 x b. Examine each of the following statements and indicate when each statement will be true.

i) a * b = b * a

ii) a * (b * c) = (a * b) * c

iii) a * d = a

iv) a * (b + c) = (a * b) * c

v) a + (b * c) = (b * c) + a."

From the descriptions of the five levels, Chick (1998: 7) used diagrams to illustrate students' response for each level. By actually examining a response and identifying the data used and the relationships between them, these response maps will help determine the SOLO level of the response which is shown in Figure 2.

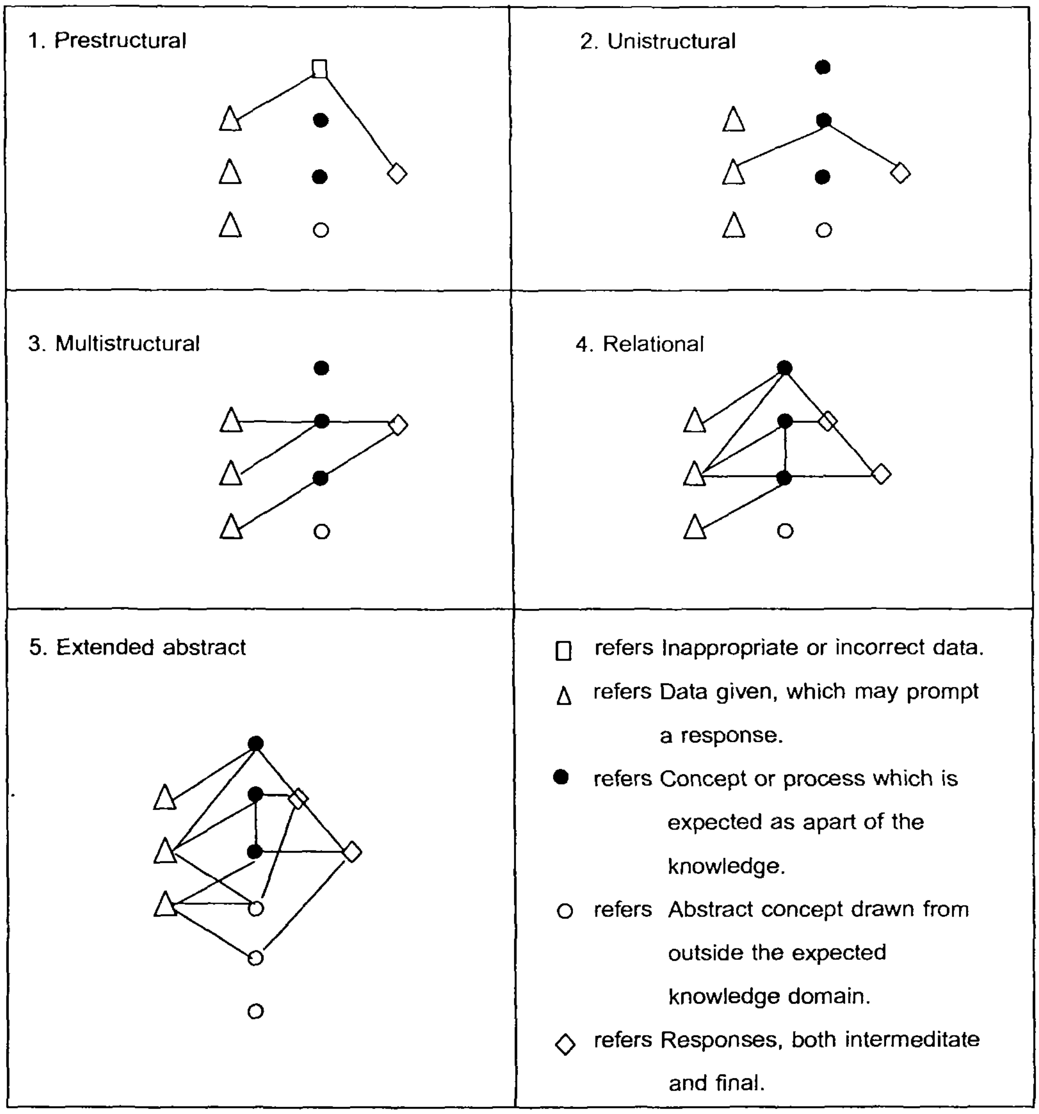


FIGURE 2 RESPONSE STRUCTURE AT DIFFERENT LEVELS OF THE SOLO MODEL
FROM: Helen Chick. (1998). Cognition in the Formal Modes: Research Mathematics and the SOLO Taxonomy. *Mathematics Education Research Journal.* 10(2): 7.

For the SOLO model, the relationship between the levels of response and the modes of functioning is shown in Figure 3. It should be noted that the extended abstract response of one mode becomes a level (possibly the unistructural level) of the next mode, and the prestructural level can sometimes be interpreted as a level (possibly the relational level) of an earlier mode (Pegg; & Davey. 1998: 118). For simplicity both the prestructural level and extended abstract level have been omitted. Figure 3 also highlights several important features of the model. Perhaps the most significant aspect of the model is the cyclical nature of learning identified (as indicated by the levels), which recurs in each mode.

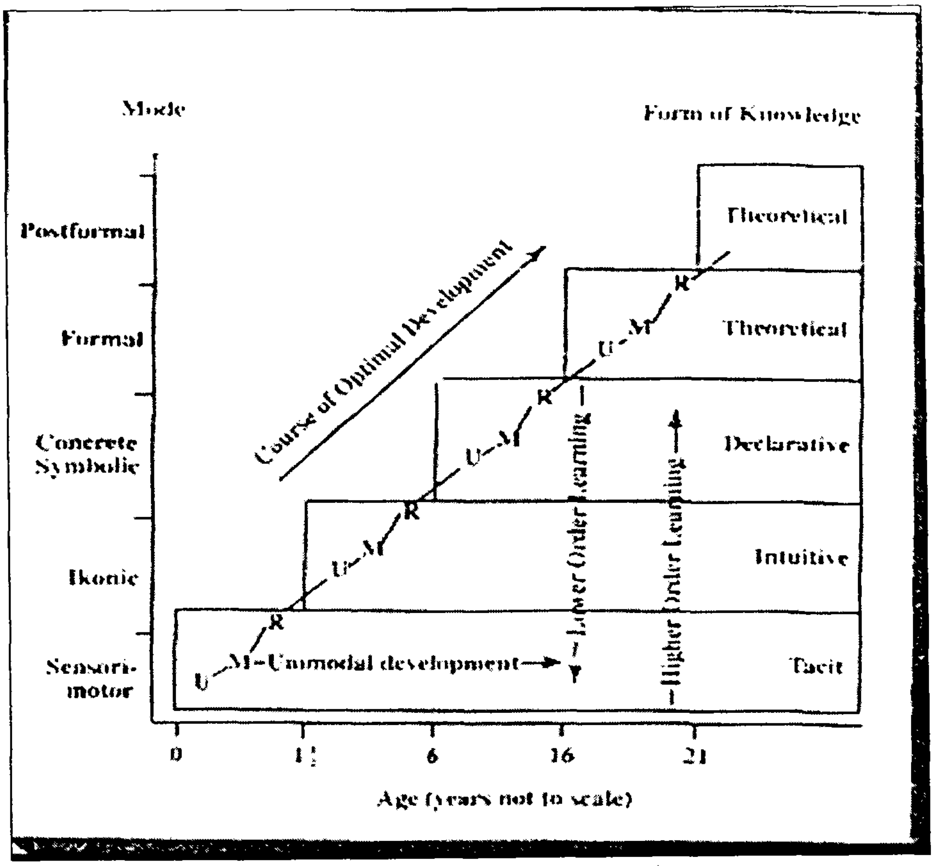


FIGURE 3 THE SOLO MODEL: MODES, LEARNING CYCLES, AND FORMS OF KNOWLEDGE

FROM: John Pegg; & Geoff Davey. (1998). Interpreting Student Understanding in Geometry: A Synthesis of two Models. In *Designing Learning Environments for Developing Understanding of Geometry and Space.* p. 119.

When researchers characterize students' responses, they focus on the target mode. The middle three levels (relational, multistructural, and unistructural) fall within the target mode, the first and last fall outside it. Prestructural responses belong in the previous mode, indicating that learning is at too low a level of abstraction to the task in question. Extended abstract responses are at a level of abstraction that is extended into the next mode, and which become the first, or unistructural level of that next mode (Biggs; & Collis. 1991: 65). The detail of processes shown in Table 1 describes the levels in relation to a given mode, called the target mode.

TABLE 1 MODES AND LEVELS IN THE SOLO MODEL

Mode	Structural Level
Next	5 *Extended abstract*. The learner now generalizes the structure to take in new and more abstract features, representing a new and higher mode of operation.
Target	4 *Relational*. The learner now integrates the parts with each other, so that the whole has coherent structure and meaning. 3 *Multistructural*. The learner picks up more and more relevant or correct features, but does not integrate them. 2 *Unistructural*. The learner focuses on the relevant domain, and picks up one aspect to work with.
Previous	1 *Prestructural*. The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode.

FROM: John B. Biggs; & Kevin F. Collis. (1991). Multimodal learning and intelligent behavior. In *Reconceptualization and Measurement Intelligence.* p. 65.

Although the SOLO model is used in describing and categorizing students' performance in various areas and also other topics in mathematics, Biggs and Collis never attempted to apply this model to geometry or spatial thinking. Interestingly, Jurdak (1991: 155-162), Olive (1991: 90-111) and Pegg and Davey (1998: 109-135) attempted to compare the van Hiele theory and the SOLO model with respect to geometrical thinking. In their work, the van Hiele theory is considered in light of the SOLO model and some insightful conclusions are presented:

The van Hiele model may be more appropriately viewed as a theoretical construct providing some overall global view or average features of the thinking process in geometry, whereas the SOLO model may beter describe the variances of individual behavior.

The SOLO model, although ostensibly very different in form, nature, and purpose from the van Hiele model, shares a great many common features. Indeed, given the necessary qualifications that must be made between the two notions of response and thinking, the van Hiele theory can be interpreted within the structure identified in the SOLO model. In effect, although van Hiele continually focused on thinking, his judgments were made based on the responses students gave.

The SOLO model has some clear advantages over the van Hiele model in describing students' understanding. Because of the overt focus on students' responses, the SOLO model is much more context sensitive than the van Hiele theory. Thus, the SOLO model is sufficiently robust and fine-grained to assist such explorations.

Another major strength of the SOLO model lies in its advocacy of continued development of earlier acquired modes. Instead of being replaced by later developing modes, these earlier modes continue to evolve, complement, and support development in other modes.

The SOLO model also provides the means for the van Hiele model to move to a new phase of development, namely, the exploration and explanation of individuality in geometry education. The challenge for researchers and teachers is to investigate and identify individual paths of development: to seek out variability. It is only in this way, with a move away from a single dimension learning path, that a true understanding of the nature of individual cognitive growth in geometry can be achieved.

In this study, the researcher investigated students' algebraic thinking in the lower secondary school. The SOLO model was chosen as the base theory to characterize students' responses in algebraic content. The target mode of this study was the concrete symbolic mode which Biggs and Collis suggested was most appropriate for students in elementary schools and secondary schools. This concrete symbolic mode contained a series of levels of thinking.

3. Related Studies

3.1 Research Using the SOLO Model

To characterize students' levels of thinking within algebraic content, it is necessary to study previous research which developed theoretical frameworks based on the SOLO model with mathematics content. A theoretical framework can provide a structure to characterize students' level of thinking across major components within a content area. There are many studies which apply the SOLO model to build theoretical frameworks for certain areas of mathematics, i.e. addition and subtraction, multidigit number sense, probability, and statistics. The following section presents the results of such studies.

3.1.1 Number sense

Jones and others (1996: 310-336) extended their earlier framework describing children's thinking in number sense. Based on previous research on number sense, the four major constructs incorporated in this framework were: counting, partitioning, grouping, and number relationships. For each of these constructs, five levels of thinking were identified across all four constructs: Level 1 (pre-place value), Level 2 (initial place value), Level 3 (developing place value), Level 4 (extended place value), and Level 5 (essential place value). Descriptors for each level of a construct were refined and validated after interviews with first-and second-grade students. The details of student thinking are presented below:

Students at the pre-place value level saw numbers in terms of single units instead of composite units such as tens and ones. They did not use a coordinated tens-ones approach to counting. Students at this level did not represent numbers using groups of tens nor could they partition a number into different forms. Students used counting to determine the relationship of numbers (i.e., greater than or less than). However, they could not determine the magnitude of the relationship. For example, one student knew a given number was greater than 5 but could not say by how much.

Students at the initial place value level understood the significance of "tens" as the key indicator in ordering two-digit numbers. Students at this level showed some evidence of using a coordinated tens-ones approach to counting. They identified 10 as a unit in representing two-digit numbers. For comparing the numbers, students could construct numbers in a "tens-ones" form.

At the developing place value level, students were able to use the part-part-whole relationship in solving problems. Students at this level consistently used a coordinated tens-ones approach with counting and partitioning problems and they also could operate or compare two representations at the same time when dealing with grouping and number relationships.

Students at the extended place value level started to expand their number sense to three-digit numbers. While they demonstrated their ability to recognize 100 as a single unit or a group of 100 individual units, they were still unable to see 100 as 10 tens. Place value level students made use of distinct mental solutions and often integrated counting, partitioning, grouping, and ordering strategies when solving problems.

At the essential place value level, students could see 100 as 10 tens. While incorporating the characteristics of the extended place value level, students at this level also demonstrated a preference for mental representations over the use of physical or paper-and-pencil representations. These students had a connected understanding of all four constructs.

3.1.2 Probability

Jones and others (1997: 101-125) developed a framework to describe and predict elementary students' probabilistic thinking. The framework incorporated four key constructs: sample space, probability of an event, probability comparisons, and conditional probability. Descriptors for each construct extended through four levels of development: subjective thinking, transitional thinking, informal quantitative thinking, and numerical reasoning. Jones and others focused on theoretical probability situations in developing the framework.

In order to determine and refine descriptors of the initial framework, a 20 – task interview assessment was administered to 8 third-grade students throughout the school year. For the purpose of the study, an understanding of sample space was based on a student's ability to list an entire set of outcomes in a one-or two-stage experiment. The ability to identify and justify which event, from two or three events, is most or least likely was the basis for determining a student's understanding of the probability of an event. An understanding of probability comparisons was based on a student's ability to determine and justify which probability situation is more likely to generate the target event in a random draw or whether two probability situations offer the same chance for the target event. To show understanding of conditional probability, students had to recognize whether the probability of an event is influenced by the occurrence of another event.

Analyses of students' interview responses indicated students at the same level held the same conceptions and misconceptions in their probabilistic thinking. Students at the subjective level had a limited perspective in relation to probability situations. Judgments were based on subjective beliefs and they could not recognize random phenomena. Students at this level listed an incomplete set of outcomes for the sample space. For situations involving probability of an event, probability comparisons, and conditional probability, students used subjective judgments instead of quantitative ones. For example, on a task for determining the probability of an event, a spinner with 1/2 shaded red, 1/3 blue and 1/6 yellow was used. Instead of choosing the color with the greatest probability, one student chose yellow because she liked yellow. Students at this level reflected thinking parallel to the prestructural level of the learning cycle described by Biggs and Collis (1991: 57-66).

At the transitional level, Jones and others (1997) found that students were beginning to shift from subjective to informal quantitative judgments. Students could provide the complete sample space for a one-stage experiment but only sometimes listed the outcomes for a two-stage experiment. They lacked a general strategy for determining the outcomes. Even with a better ability to list the outcomes when examining probabilities, students at this level often overlooked some outcomes. Transitional level students also had difficulty with conditional probability. Thinking at this level was aligned with Biggs and Collis (1991: 57-66) unistructural level.

At the informal quantitative level, students developed a generative strategy for listing the outcomes of two-stage experiments. They made use of quantitative judgments in determining probabilities and used numbers to compare probabilities. With conditional probability, they began to understand that the probabilities of all events changed in situations with non-replacement. Students' thinking at this level demonstrated characteristics of both the multistructural and relational level of Biggs and Collis (1991: 57-66) learning cycle.

At the numerical level, Jones and others determined that students could systematically list the outcomes for two-and three-stage experiments. These students also made use of the sample space for finding and comparing probabilities. They assigned numerical probabilities to both equally and non-equally likely events and had an increased understanding of conditional probability. Thinking at this level coincided with Biggs and Collis' s relational level of thinking (1991: 57-66).

Later on, Tarr and Jones (1997: 39-59) used the framework developed by Jones and others (1997: 101-125) as a basis for their framework to describe and predict middle school students' probabilistic thinking about conditional probability and independence. Fifteen students, three from each of Grades 4 through 8, were interviewed using a 14-task interview protocol in order to validate and refine the framework. For this study, students' understanding of independence was measured by their ability to determine and justify if one event had influence on another event.

Students at the subjective level believed that previous outcomes affected future outcomes and denied the existence of independence. Using subjective instead of quantitative reasoning, students formed conditional probability judgments by either searching for non-existing patterns or creating their own sense of regularity. Students at this level had no meaningful understanding of independence and conditional probability.

At the transitional level, students used appropriate quantitative information in making conditional probability judgments. However, irrelevant features could distract them. Students at this level tended to rely on subjective judgments when unexpected outcomes occurred. While they recognized that the probabilities of some events changed in "without replacement" situations, the recognition was incomplete, confined to previously occurring events.

Students with informal quantitative thinking were aware of the role quantity plays in conditional probability situations. They often made use of frequencies, ratios or some form of odds in determining and describing conditional probabilities. Students at this level also recognized independent events in "with replacement" situations; however, they had less awareness of independence in dealing with a series of independent trials.

At the numerical level, students understood the importance of sample space in determining conditional probabilities. They recognized dependent and independent events as well as the conditions in which two events were related. Students at this level could make predictions with equally likely outcomes.

From the results of this study, Tarr and Jones determined that the levels of development coincided with the levels of learning described by Biggs and Collis (1991: 57-66).

3.1.3 Statistics

Jones and others (2000: 269-307) developed a framework that characterized elementary school students' statistical thinking. This framework consisted of four constructs: describing data, organizing data, representing data, and analyzing and interpreting data. For each construct, Jones and others hypothesized four levels of statistical thinking in accord with Biggs and Collis's general developmental model (1991: 57-66): idiosyncratic, transitional, quantitative, and analytical. The responses from interviewing 20 target students in Grades 1 through 5 were analyzed to refine and validate each level of a construct.

At Level 1, students were generally unfocused and invariably included irrelevant or incomplete information. They did not recognize that two displays represented the same data because they focused only on some cosmetic features of the displays. They did not usually group the categorical data in a systematic manner, and they were not able to characterize data in terms of typicality (average) or spread. That is, students exhibiting this level did not seem to show any prior knowledge about center and spread. At this level, students did not normally complete or construct a valid graph of the data. They often constructed an idiosyncratic graph that had little or no relation to the original data. When they analyzed and interpreted data, they did not seem to be able to "read between" the data or "read beyond the data" (Curcio. 1987: 382-393).

Students at Level 2 attempted to make sense of the data and showed evidence of using quantitative reasoning. However, their responses were typically hesitant and incomplete, and they were often satisfied with just a single response where multiple responses or perspectives would have been appropriate. In organizing and reducing data, students tended to produce groupings that overlapped or were incomplete. When dealing with measures of center and spread, they used invented measures that were partially valid. They attempted to represent the data but generally produced a display that was not completely valid. Moreover, students normally provided relevant but limited responses to questions involving analysis and interpretation.

At Level 3, students used quantitative reasoning more effectively and were more spontaneous and more complete in their responses. They consistently attempted to make sense of the data and generally achieved this, sometimes using their own valid invented measures. They normally used conventions and labels meaningfully and, as a result, were able to give accurate descriptions of the display. In organizing and reducing data, they were usually able to group the categorical data into consistent and nonoverlapping classes and were able to explain the criteria for their classes. Students at this level tended to use quantitative reasoning that enabled them to approximate one of the centers. Students represented data by completing a graph and using the scale provided in a precise manner. They generally provided multiple and meaningful responses where asked to analyze and interpret data.

The ability to use both analytical and numerical reasoning in explorations of data and to produce responses that were more analytical was the key characteristic of Level 4. These thinking processes were not evident in the previous level. Students at this level recognized the role of conventions and used these conventions to identify relations when describing data displays. Students were able to group complex data sets into consistent and nonoverlapping classes and were able to develop criteria for their groupings. They were expected to demonstrate a conceptual understanding of the mean or the median. Moreover, they showed some conceptual understanding of spread. Students were capable of reorganizing even more complex data when they were also been given the opportunity. They were able to generate alternative interpretations of the data.

In 2002, Mooney (2002: 23-63) developed a framework for characterizing middle school students' statistical thinking. Based on the previous study of Jones and others on statistical thinking, Mooney adopted the four statistical processes: describing data, organizing and reducing data, representing data, and analyzing and interpreting data. He hypothesized five levels of statistical thinking for each of the four statistical processes which describe middle school students' statistical thinking: Level 1 (idiosyncratic), Level 2 (transitional), Level 3 (quantitative), Level 4 (analytical) and Level 5 (extended analytical). He situated these levels within the general developmental model of Biggs and Collis (1991: 57-66). Twelve students in Grades 6 through 8 were individually interviewed to refine and validate the framework. Although the hypothesis classified the characteristics of middle school students' statistical thinking into five levels, analysis of target students' responses indicated only four levels. Thus, Level 5 (extended analytical) descriptors for all four processes were removed.

According to Mooney (2002: 23-63), students at Level 1 (idiosyncratic) did not recognize the same data in different data displays. They were unable to construct a data display for a given data set or constructed a data display that was incomplete and unrepresentative of the given data set. Moreover, they made subjective judgments when conducting statistical applications in that no direct or logical use of the data was employed in their responses. They depended on the subjective or visual aspects of data displays to make decisions or to find information.

The ability to make use of some quantitative information in statistical situations was begun in Level 2. Students at this level constructed data displays with either missing labels or missing data but not both. When calculating measures of center or spread, they used partially valid invented measures. Students at Level 2 focused more on relevant display features when evaluating data displays. They were also able to make at least a partially correct comparison as well as inferences that were based partially on the data.

To move from the transitional level to the quantitative level, students made more use of the quantitative information in statistical application than the previous levels of statistical thinking. They read data from a display and made use of relevant display features in evaluating data displays. Calculations of measures of center or spread were done with valid invented measures or with traditional measures of center and spread, although the students had some flaws in the process of calculating the measures. Data displays were representative of the data and contained the appropriate display features. At this level, students made comparisons that were either local, relating to part of the data presented, or global, relating to all data presented.

At Level 4, students could balance the quantitative and contextual aspects of the data without one or the other overriding their analysis of the data. They generalized data displays and constructed complete, and sometimes unique, displays that were representative of and appropriate for the data given. Students would make both local and global comparisons, that is they made use of both quantitative and contextual aspects of the data when making inferences or comparing data.

3.2 Research on Framework other than the SOLO Model

Other cognitive frameworks were developed to characterize students' thinking, that is addition and subtraction and multidigit number sense. Although the researchers did not use the SOLO model to develop a hierarchy framework, they provided the idea to predict students' levels of thinking within algebraic content. The following section presents the results of such studies.

3.2.1 Addition and Subtraction

Carpenter and Moser (1984: 179-202) studied student performance in solving addition and subtraction problems. The target of this study was students who were studying in Grades 1 through 3. Students' processes were described in five levels as follow:

At Level 0, students could not solve any addition or subtraction problems.

Students at Level 1 were restricted to strategies that could be directly modeled. The action or relationships in the problem were modeled using physical objects or fingers. Students used counting strategies to solve addition problems. Furthermore, they made use of adding on, separating from, and matching strategies to solve subtraction problems. Moreover, they could not solve combined types of problems.

The ability to make a transition from direct modeling strategies to counting strategies was observed in the performance of students at Level 2. Students at this level made use of both direct modeling and counting strategies to solve problems including counting on and counting up from given strategies.

At Level 3, students internalized direct modeling actions and primarily used counting sequences to solve problems. Although, students liked to use their fingers, they used their fingers to keep track of the counting sequence, not to model items being tallied. Students primarily made use of the counting on and counting up/down from strategies to solve addition and subtraction problems.

At Level 4, students used derived or recalled number facts to solve addition and subtraction problems. Children progressed through these levels over time as they built an understanding of basic addition and subtraction.

3.2.2 Multidigit Number Sense

Jones, Thornton and Putt (1994: 117-143) constructed a framework that described and predicted how students thought in multidigit number situations. Four major constructs related to multidigit number sense were identified: counting, partitioning, grouping, and number relationships. For each construct, four different levels of thinking existed: Level 1 (pre-place value), Level 2 (initial place value), Level 3 (extended place value # 1), and Level 4 (extended place value # 2). A case study involving 6 target first grade students was used in order to refine and develop the framework.

Students at Level 1 (pre-place value) saw number as a collection of ones, instead of as a whole made up of parts consisting of tens and units. These students did not partition a number using groups of tens. To group numbers into fives or tens was difficult for them. They had not developed a concept of ten objects as one unit. Students at this level recognized that the given number was greater or less than another number but could not always tell by how much.

At Level 2, students showed some evidence of being able to coordinate counting on by tens and ones and being able to think of a number in terms of 10 as a unit when modeling multidigit numbers. They had captured the significance of the "tens" as the key indicator in ordering two-digit numbers.

At Level 3, students constructed concrete tens-ones representations of two-digit numbers and then used a ten-ones "count all" strategy. They formed two number representations based on tens and ones as well as operated on the two representations at the same time.

Students at Level 4 used a partitioning strategy in adding two-digit numbers and three digit numbers. That is, they counted and constructed multidigit numbers mentally, even into the hundreds but they could not represent the three-digit numbers by tens.

In summary, these previous studies demonstrated that theoretical frameworks which were based on the SOLO model could be used to describe and predict students' thinking in various mathematical content areas. Hence, this study applied the SOLO model of development and the levels of thinking in the area of algebra. This study was conducted with lower secondary school students who were in the concrete-symbolic mode. Although some researchers have hypothesized five levels of thinking to classify students' thinking, students' responses indicated only four levels of thinking, such as in the study of Mooney. The researcher supposed in this study that students' responses would fall within four levels of thinking: prestructural, unistructural, multistructural, and relational levels.

3.3 Research on Pattern

Patterns play an important role in developing algebraic thinking (Ferrini-Mundy; Lappan; & Phillips. 1999: 112). Many studies on learning patterns investigated the capacities of students to generalize a variety of patterns, such as numerical and geometric patterns in general or formula terms.

Pegg and Redden (1990: 386-391) tried to group 700 students according to the level of their abilities to generalize the algebraic rule for the number of tiles around the perimeter of a square swimming pool. These students whose ages ranged from 11 to 13 years were asked to write in words a rule and then rewrite the rule using algebraic symbols. Students' responses were grouped into five types. The first group used drawing or counting or both. A second group moved on from counting, and perhaps used tabulation to summarize the data they had collected relating pool length to number of tiles, but were not able to use letters as variables. A third group generalized the process, but only in terms of a relationship between one pool and the next, such as adding four tiles to the previous pool, perhaps expressed as T = S + 4. A fourth group focused on the perimeter of the pool in producing responses similar to the previous group, but there was little use of drawing or counting. The students in the final group were able to relate the length of the side of the squares to the number of tiles, and most students were able to write a correct algebraic expression, such as T = 4L + 4. The developmental sequence described by Pegg was related to the SOLO model.

Zack (1995: 106-113) studied the use of students' strategies to solve and generalize relationships between the number of diagonals and the number of sides of figures in a pattern of geometric figures problems. Twenty-five students in a Grade 5 elementary school classroom worked both alone and together when asked to find the number of diagonals in other figures with 10 sides, 52 sides and *n* sides. From the results of the investigation, there were four types of students' strategies in solving problems and in expressing the pattern in the general term. These are discussed in the next paragraph.

The first group of students drew all the diagonals and counted them. Sometimes they used different colors to help them distinguish diagonals belonging to each vertex. Students in the second group drew some diagonals and then saw a pattern in the number of diagonals within one polygon (for example for the 10-sided one you would have: 7, 7, 6, 5, 4, 3, 2, 1, 0 that is, the number of diagonals is 35). A third group saw a pattern of differences between the number of diagonals contained in each polygon and then continued the pattern without drawing any diagonals. Students in the last group calculated the number of diagonals emanating from a vertex, multiplying that number by the number of sides and dividing by 2.

Orton and Orton (1999: 104-120) studied students' abilities to generalize number patterns. They expected it would be possible to evaluate the approach to algebra through generalizing from number patterns. When presented with the number pattern 1, 4, 7, 10, ..., students might be expected first to continue the pattern for a few more terms. Then they might be expected to predict the twentieth or fiftieth number. They described the pattern in words and finally predicted what the n^{th} number would be, where *n* was any number. They arranged their study by having 1000 students complete a written test and then selected 30 of these students to interview about their written responses. The result of this study suggested that about one-half of these students managed to give the next number in each sequence, but many arithmetical errors concealed the pattern for determining the twentieth and hundredth numbers. A few students could cope with the n^{th} term. It is interesting that for some problems in this study, students could not use algebraic symbols as a general rule but other students could and more than half of all students omitted the question about a simple rule.

Moreover they tried to categorize students' responses into the four levels they expected. For example, from the number pattern 1, 4, 9, 16, 25, 36, 46, ..., students at Level 1 shown some properties of the numbers with perhaps partial patterns, described them as, "some numbers are odd" and "16 and 36 are in the four times table". Students' responses like "odd even odd even..." and "difference is odd" would be at Level 2, that is they noticed a pattern but this was not described so as to allow the next number to be derived. If students knew how to derive the next number using other patterns observed from the differences, such as "add 3, 5, 7, 9, 11, 13 and so on" and "keep adding the next odd number", they would be at Level 3. Students at Level 4 showed clear evidence of understanding the relationship, though an algebraic formula might not be articulated such as "times the row number by itself" and "(1 x 1), (2 x 2), (3 x 3), ..."

In Bishop's study, Bishop (1997: 1-11; 2000: 107-126) characterized seventh and eighth-grade students' thinking about patterns by conducting task-based interviews in which the students were to generalize the relationships between the number of the figure and its perimeter or area. She found that a few students expressed the relationships of patterns in a general term and also classified the characteristics of the students into four levels of understanding problems. These four levels are described below.

At the most concrete level, students modeled the figures and counted the number of block-sides to find the perimeter or area and to solve situations. When asked to generalize a method for finding perimeter or area, students at this level gave verbal directions for counting the number of the sides of the figure. They struggled to interpret symbolic expressions and proceeded mostly by substituting a number into an expression and comparing the results to a known perimeter. They demonstrated little or no awareness of the relationships between the number of the sides of the figure and its perimeter or area.

At the second level, students demonstrated awareness that there exists a relationship between the number of sides of the figure and its perimeter or area, but they misinterpreted the relationship, expressing it in terms of a single operation. These students frequently attempted to find perimeter or area and solve situations by applying a single operation and reversing it with little success. Their awareness that there must exist a relationship presents a teachable moment.

Students at the third level interpreted the patterns in terms of the relationship between the perimeter or area of consecutive figures. Students solved the situations by skip counting by two or three, but they did not solve equations. However, several students who used the add and skip-count strategies to find perimeter and area later recognized the relationships between the number of sides of the figure and the perimeter or area, and they began to operate at the fourth level of understanding.

Students at the fourth level of understanding recognized the relationships between the number of sides of the figure and its perimeter or area, and expressed the relationships symbolically. In Bishop's study, students at this level were more likely to explain their calculations in terms of the relationships between the number of sides of the figure and the perimeter or area. They solved situations either informally by working backwards or with formal algebraic solution methods.

In summary, students can be classified according to their abilities in generating patterns into levels, since their responses in pattern problems depend on the complexity and difficulty of problems and the maturity of student. In this study, the researcher chose patterns as an indicator to characterize students' algebraic thinking. The researcher hypothesized that students in lower secondary school should generalize about a wide variety of patterns symbolically and explain their generalization of patterns.

3.4 Research on Representation

Representation has been mentioned as the big idea in studying algebra. In developing algebraic thinking, students should understand and use the process of representation to describe and interpret the situation and also translate freely among representations (Booth. 1989: 244; Friel; et al. 2001: 4; Herbert; & Brown. 1999: 123-124;

NCTM. 1993: 11-42; 2000: 222). Moreover the fluency of multiple representations plays an important role in the successful development of algebraic thinking. When the researcher thinks of representation, it is not only as an image (e.g., graph, table, and symbol) but also as a process for illuminating ideas that have usefulness in the learning of mathematics. Specifically, interpretation and translation of representations are tools that can extend students' algebraic thinking (Coulombe; & Berenson. 2001: 166-172). There are many studies on students' abilities in representations.

Dreyfus (1991: 25-41) classified representation in learning processes as consisting of four stages: 1) using a single representation, 2) using more than one representation in parallel, 3) making links between parallel representations, and 4) integrating representations and flexible switching between them. In stage one, processes started from a concrete case, a single representation. In the second stage, several representations of the same mathematical object were used in parallel. The difficult process of transition to the abstract concept depended in an essential manner on the links that were formed between the representations. The establishment of these links constituted the third stage. Strong links allowed students to switch representations, which in turn made them aware of the underlying concept and was thus likely to positively influence abstraction. At the fourth stage, a process of integration between the differentiations was happening.

McCoy (1994: 174-179) examined students' ability in translating the different representations of algebraic situations which consisted of words, tables, graphs, and equations. Eighteen college students and thirteen high school students who had just finished the first semester of Algebra I were given the Algebra Representation assessment. The results showed that students had difficulty in translating graphs and equations to real world situations in words. This result also indicated that students could translate tables to words and graphs and also could translate graphs to words and tables, but they were not able to translate either tables or graphs to equations. Similarly, they were not able to translate equations to any of the other representations.

Chick and Watson (2001: 91-111) classified levels of representing data with twenty-seven Grade 5 and 6 students. Students were confronted with data cards which consisted of the name, age, weight, weekly fast food consumption, favorite activity, and eye color for a fictitious juvenile. Students' responses were analyzed using the SOLO model for each level of representation. This study showed that students were considered to be at a prestructural SOLO level of representation because one student made no contribution at all. While the others tried to construct a table they failed to complete it. They only filled and colored it with patterns. In the unistructural level, students depicted individual aspects of the data such as recording some or all data values in a table with no attempt to aggregate. Students represented the graphs which showed serial classifications of one variable. Their abilities were judged to be at the multistructural level, if they could represent a single variable of all the data. Students produced representations considered to be at a relational level when they could depict a two variable relationship such as drawing the graphs to show an association between two variables.

Coulombe and Berenson (2001: 166-172) argued that traditional instruction in algebra began with posing problems in the form of symbolic representations. Students translated to a tabular form first and then to graphical representations. Students were familiar with a problem like this and so were not stimulated in their ideas. Hence, Coulombe and Berenson suggested the Problem-Based Approach which provided students opportunities to illustrate their algebraic ideas with representations. This approach used a different way of translating representations, such as students were given problems with graphs and then translated them to tables.

Table 2 shows the detail of the traditional approach and the Problem-Based Approach and also the comparison of two approaches used to improve students' understanding.

TABLE 2 THE COMPARISON OF TWO APPROACHES USED TO IMPROVE STUDENTS' UNDERSTANDING

Traditional Approach			Problem-Based Approach		
More from	To	Translation process	More from	To	Interpretive process
Symbols (y = 2x)	Table	Compute values	Graph	Words	Graphical interpretation
Table	Graph	Plot points	Table	Words	Verbal pattern description
			Words	Graph	Qualitative graphical construction
			Graph	Table	Data generation
			Table	Symbols	Pattern finding

FROM: Wendy N. Coulombe; & Sarah B. Berenson. (2001). Representations of Patterns and Functions: Tools for Learning. In *The Roles of Representation in School Mathematics*. p. 167.

In summary, students' ability to represent data can be classified into levels when they are confronted with the task. In this study, the researcher evaluated students' responses in representation by focusing on interpreting and transforming among representations.

3.5 Research on Variable

The concept of variable plays an important role in algebra; it provides the basis for the transition from arithmetic to algebra. In a similar way, many mathematics educators such as Friel and others (2002: 1), Kieran (1988: 91), Wagner and Kieran (1999: 362-363) and Usiskin (1999: 9) defined algebra as "generalized arithmetic". In the middle grades, understanding the different meanings and use of variables and specifically working with variables and equations is a significant part of the curriculum (NCTM. 2000: 223-225). Moreover, Schoenfeld and Arcavi (1999: 150-156) stated that students' understanding of variable should go far beyond simply recognizing that a letter can be used to stand for unknown numbers in equations. This relates to Leitzel' s suggestion (1989: 30) that students will understand when they see a variable and represent it in a large set. Many mathematics educators defined the meaning of variable as "a variable is a quantity which may assume an unlimited number of value" (Philipp. 1999: 157) or "a variable is a letter that stands for one or more numbers" (Chalouh. 1988: 33). Hence, the concept of variable as a general number is difficult and blocks students' success in algebra. There are a lot of studies that have shown student's difficulties in dealing with variables.

Collis (Kieran. 1989: 41; citing Collis. 1975. *The Development of Formal Reasoning*) found that students, prior to the stage of formal reasoning (Piaget' s stages of cognitive development), worked at one of three levels in dealing with the notion of variable—depending on their experience and mathematical maturity. Children at the lowest level tended to substitute one specific number for a letter. For example, in an equation-solving task, if the specific value does not work, they will give up. At the second level, students were willing to try several numbers in a trial-and-error method. By the time they were at the third level, students had extracted a concept of generalized number by which a symbol *x* can be regarded as an entity in its own right, but having the same properties as any number with which they had previous experience.

A large-scale study of some of the various ways in which students used algebraic letters was carried out by Küchemann (1981: 102-119). As part of the Concepts in Secondary Mathematics and Science (CSMS) project, Küchemann administered a 51-item paper-and-pencil test to 3000 British high school students, aged 13, 14 and 15 years. Küchemann classified each item into one of six categories of interpretation of letters. These six categories were used by Küchemann in the analysis of the data. The six categories are described as follows:

1) *letter evaluated* refers to items where students are asked to find a specific value for an unknown, but without first having to operate on the unknown, such as what can you say about m if $m = 3n + 1$ and $n = a$,

2) *letter not used* refers to an item in which students ignore the letter, or acknowledge its existence without giving meaning to the letter, such as if $a + b = 43$ then $a + b + 2 = \ldots\ldots$,

3) *letter as object* refers to students' thinking of the letter as shorthand for an object, all items will ask students to simplify various algebraic expressions, such as $3a - b + a = \ldots.$,

4) *letter as specific unknown* refers to students' thinking of the letter as a specific unknown number and can operate upon it directly, such as multiply $n+5$ by 4,

5) *letter as generalized number* refers to students' thinking of the letter with more than one value, such as what can you say about c if $c+d=10$ and c is less than d?,

6) *letter as variable* refers to students' knowing that the letter represents a range of unspecified values and knowing that the meaning may change within the context, such as which is larger, $2n$ or $n+2$? explain.

Küchemann found that the first three categories indicated a low level of response, and for students who had any real understanding of even the beginnings of algebra they must at least began to see that the letter stands for a specific unknown. Küchemann also found that very few of the secondary school students (age 14+) in the study reached the last category where a letter is interpreted as a variable. Rather most of the students tended to regard the letters as names or labels, ideas they could relate to. Although, Küchemann tried to classify the level of students' responses and the items into 4 levels of understanding, it is difficult to link the algebra levels and Piaget's stages of cognitive development since Piaget major study focused on adolescents in science rather than mathematics tasks. Küchemann's classification is shown as follows:

Level 1 students could solve the items which were extremely easy.

Level 2 students could cope with more complex items than Level 1 but still could not consistently cope with specific unknowns, generalized numbers or variables.

Level 3 students' major advance was that they could use letters as specific unknowns, though only when the item-structure was simple.

Level 4 students could cope with more complex items in specific unknowns and also they could solve the items involving generalized numbers and or even variables.

Based on the study of Küchemann from the Concepts in Secondary Mathematics and Science (CSMS) in 1981, Booth (1984; 1988: 20-32) investigated in depth some of the common errors in algebra. This is one part of the results of the Strategies and Errors in the Secondary Mathematics Project (SESM). The investigation focused on generalized arithmetic such as the use of letters for numbers and the writing of general statements which represented given arithmetical rules and operations. She studied 27 children aged from 13 to 15 years. These children were interviewed and a total of 3550 children of the same age range were tested. The results showed that a small percentage of children successfully made a correct answer and a large number of children made incorrect answers. In some cases, over 40 or 50 percent of the children tested committed errors. Children might interpret letters in a variety or sometimes, unexpected ways. They did not use the standard formal mathematical methods taught in the classroom, but rather used informal methods of their own.

In 2001, Jirawat Meeluksana (2001: 1-124) investigated students' understanding of variables with lower secondary school students in Bangkok Province. She adapted Küchemann's test as the instrument. She found that almost all students' responses were similar to those in Kuchemann's investigation. Students were very good at understating "letter evaluated" and "letter ignored" categories. They were good on the "letter as object" category and were average on the "letter as specific unknown" category. A few students were able to answer questions on the "letter as generalized numbers" and the "letter as variables" categories.

Similar to Booth and Küchemann's studies, Chuan and Fong (2002: 155-161) used the test items of the Küchemann test with 40 students in a secondary school in Singapore. They found that students' responses were similar to students' responses in Booth and Küchemann' s studies, but 80% of students who were familiar with items (although they were more complex) were more successful in achieving a high level of understanding. However, students who lacked familiarity with items preformed at a lower level.

Kieran (1992: 390 -418) tried to describe a progression of levels of understanding for the concept of variable. His levels are as follows:

Level 0: using ordinary language to solve problem involving unknowns;

Level 1: using letters as specific unknown numbers;

Level 2: using letters as generalized numbers - (independent) variables that can take on a set of values;

Level 3: using letters as both independent and dependent variables.

In summary, students can be classified into levels according to their abilities in understanding the role of variable in equations, expressions and inequality problems. In this study, the researcher chose variable as an indicator to characterize students' algebraic thinking. The researcher hypothesized that students in lower secondary school should understand the role of variable as a generalized number.

4. Tentative Initial Framework on Algebraic Thinking

To develop the framework on student's algebraic thinking, there is a need to understand the notion of algebraic thinking and the key algebraic indicators that constitute algebraic thinking.

Based on the review of literature from previous research on secondary school algebraic thinking and the teaching and learning of algebra, this study developed a framework for describing students' algebraic thinking. The researcher started by constructing the initial lower secondary school algebraic thinking framework. This initial framework comprised three key indicators, four levels of thinking for each indicator and descriptors for each level of thinking within each indicator.

4.1 Indicator

In building this initial algebraic framework, the researcher used three key indicators to represent lower secondary school students' algebraic thinking. These key indicators are pattern (generalizing pattern), representation (interpreting and transforming) and variable (understanding the role of variable).

4.2 Level of Thinking

Based on the SOLO model suggested by Biggs and Collis (1991), there are five modes of functioning: sensorimotor, ikonic, concrete-symbolic, formal, and postformal. Students who are studying in lower secondary schools are able to respond within the concrete symbolic mode. Hence, the researcher chose the concrete symbolic mode as the target mode.

Each mode of functioning for the SOLO model (Biggs; & Collis. 1991) consists of five levels of thinking: *prestructural*, *unistructural*, *multistructural*, *relational*, and *extended abstract*. The prestructural level may locate in a level (possibly the relational level) of an earlier mode, whereas the extended abstract level may be interpreted as a level (possibly the unistructural level) of the next mode. Some studies indicated that students responded at only four levels of thinking. They could not go beyond the relational level. In this study, the researcher hypothesized that students would demonstrate four levels of algebraic thinking in the concrete symbolic mode: Level 1 (prestructural), Level 2 (unistructural), Level 3 (multistructural), and Level 4 (relational).

4.3 Descriptor

In order to design the initial lower secondary school algebraic thinking framework, the researcher summarized descriptors from all recommendations made by researchers and educators on pattern, representation and variable which related to algebra thinking and algebra in lower secondary school students. These descriptors were added to the appropriate levels and indicators so as to provide a basis for describing lower secondary school students' algebraic thinking.

CHAPTER 3

METHODOLOGY

The aim of this study is to formulate and validate a cognitive framework of lower secondary school students' algebraic thinking. The researcher began by analyzing previous research which investigated students' algebraic thinking. This review together with the SOLO model of Biggs and Collis (1991) was used to formulate an initial framework that illustrated students' algebraic thinking. Three key indicators were used to identify the four cognitive levels of algebraic thinking. These indicators were pattern, representation, and variable. The method and procedures employed in this study were divided into two phases as follows:

Phase 1: Formulate the Framework of Algebraic Thinking Process

The researcher studied documents, books and studies which related to the SOLO model, algebraic content, algebraic thinking, levels of thinking in the mathematical domain and methods of developing frameworks. All information was used to formulate the initial algebraic thinking framework.

Phase 2: Validate the Framework of Algebraic Thinking Process

In this phase, the researcher selected lower secondary school students as the sample, constructed instruments, collected data, analyzed the data and validated the framework.

The two phases are shown in Figure 4 below.

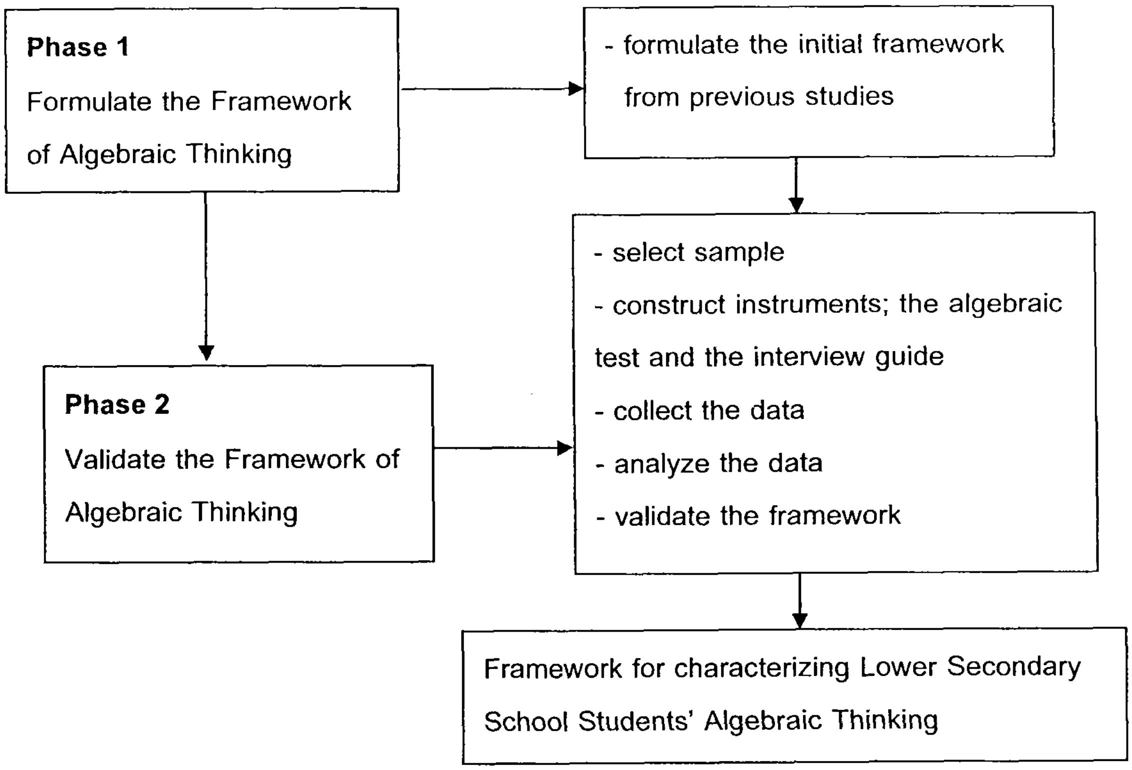


FIGURE 4 THE PROCEDURE IN DEVELOPING THE ALGEBRAIC THINKING FRAMEWORK

The detail of developing a framework for characterizing the lower secondary school students' algebraic thinking is described in each phase as follows:

Phase 1: Formulate the Framework of Algebraic Thinking Process

In formulating an initial framework of algebraic thinking, the researcher generated the documents, books and studies which related to the SOLO model, algebraic content, algebraic thinking, levels of thinking in the mathematical area and methods of developing the framework. Since this study was conducted with lower secondary school students, the researcher focused on the concrete symbolic mode of the SOLO model.

Students in this mode were capable of linking concrete operations with symbols which were the abstract concepts. For students in the concrete symbolic mode of the SOLO model the researcher hypothesized four levels of algebraic thinking: Level 1 (prestructural), Level 2 (unistructural), Level 3 (multistructural), and Level 4 (relational). The framework was a synthesis of previous research on algebraic thinking (Bishop. 2000: 107-126; Orton; & Orton. 1999: 104-120; Pegg; & Redden. 1990: 386-391; Zack. 1995: 106-113).

After the initial framework on algebraic thinking was constructed, the researcher applied the general framework to three indicators of algebraic thinking: pattern, representation, and variable. Within the context of this study, algebraic thinking on pattern refers to the abilities of students to use their thinking skills to generalize the patterns in symbolic form. Algebraic thinking on representation refers to the abilities of students to use their thinking skills to solve algebra problems by representing the given data in the form of tables, graphs, and by interpreting them. Algebraic thinking on variable refers to the abilities of students to use their thinking skills to understand the role of variable as generalized number. Descriptions of four levels of students' algebraic thinking were incorporated into each indicator in the first draft of the framework.

In order to confirm the expected Thai students' thinking based on previous research, the researcher conducted preliminary algebraic tests and some interviews in order to build the idea of students' responses. Ninety- eight students from Samutsakhonburana School in Samutsakhon Province were involved. The results of this preliminary study were used to indicate the descriptors of three indicators in the first draft of the initial framework. After an initial framework was completed, the researcher had three committees act as experts to recommend and adjust the indicators of each level. The experts' recommendation was about the appropriateness of the words used for the descriptor and some overlap of the descriptors.

The researcher adjusted the descriptors in the framework according to the suggestions of the experts. This framework was used as the initial framework of algebraic thinking for this study (see Table 3).

TABLE 3 INITIAL LOWER SECONDARY SCHOOL STUDENTS' ALGEBRAIC THINKING FRAMEWORK

Level / Indicator	**Prestructural (Level 1)**	**Unistructural (Level 2)**	**Multistructural (Level 3)**	**Relational (Level 4)**
General characteristic	- makes no attempt to answer the question - hardly understands the task or fails to engage the task - hardly remembers the question or cannot keep the question in mind while answering makes no logical connections; does not answer the question, guesses the answer, gives an irrelevant answer or restates the question	- uses only one relevant aspect or concept of the information to find the answer - keeps the question in mind and tries to relate to the question; deals with at least one logical idea - gives a correct answer but may not be consistent	- uses two or more aspects or concepts of the data to find the answer - fails to link aspects with each other	- applies most or all aspects of the data - shows an integrated understanding of the relationships among various aspects of the data

TABLE 3 (continued)

Level / Indicator	Prestructural (Level 1)	Unistructural (Level 2)	Multistructural (Level 3)	Relational (Level 4)
Pattern	(**PA11**) does not understand or is confused about the pattern (**PA12**) knows that the given data is a pattern, but does not know how to use this data to identify the characteristics of the pattern	(**PA21**) can find the value of the next term of the given pattern (**PA22**) uses concrete objects to find the value of a specific term of the given pattern rather than providing a general pattern description (**PA23**) tries to generalize from only one given term or only one aspect of the given pattern, but the generalization is not correct	(**PA31**) sees the given pattern as an ongoing process; that is, describes how the value of a term moves from one term in a sequence to the next (**PA32**) does not relate the value of a term to the term number in the given pattern; for example $a_n = 6+2(n-1)$	(**PA41**) understands the relationship between the value of a term and the term number in the given pattern (**PA42**) explains correctly the relationship in the given pattern verbally (**PA43**) generalizes the relationship of the pattern symbolically or provides a formula giving the value of a general term; for example $a_n = 6+2(n-1)$

TABLE 3 (continued)

Level / Indicator	Prestructural (Level 1)	Unistructural (Level 2)	Multistructural (Level 3)	Relational (Level 4)
Pattern		(**PA24**) identifies one characteristic of a pattern	(**PA33**) identifies two characteristics of a pattern	(**PA44**) identifies all characteristics of a pattern and explains the relationships between them
Representation	(**RE11**) does not understand the question or is confused by the question; avoids the question or answers the question based on irrelevant data (**RE12**) understands the question but does not know how to answer it	(**RE21**) represents the given data inappropriately by using only one aspect of the information (**RE22**) interprets the graph by reading the graph on one axis only	(**RE31**) represents some data correctly, but omits data or leaves out some conditions (**RE32**) compares data from the graph with respect to one axis only	(**RE41**) represents all data correctly and recognizes all conditions on the data (**RE42**) interprets the graph with respect to both axes

TABLE 3 (continued)

Level / Indicator	**Prestructural (level 1)**	**Unistructural (level 2)**	**Multistructural (level 3)**	**Relational (level 4)**
Representation	(**RE13**) cannot interpret the points on a graph (**RE14**) does not transform the given representation to another representation			
Variable	(**VA11**) hardly understands the task or fails to engage the task (**VA12**) answers the question based on irrelevant data or guesses the answer	(**VA21**) does not understand the role of a variable as a generalized number, uses only one specific number to make a general conclusion	(**VA31**) does not understand the role of a variable as a generalized number, uses several specific numbers to make a general conclusion	(**VA41**) understands the role of a variable as a generalized number, and gives a well reasoned general conclusion

Phase 2: Validate the Framework of Algebraic Thinking Process

In this phase, the researcher selected a sample for gathering data, then administered the algebraic test in an interview setting used this data to validate an initial framework.

1. Sample

The researcher used the purposive sampling technique to choose two schools from 86 schools, one among the best schools and another from average schools. Students participating in this study were students of lower secondary school (Mathayomsuksa one through Mathayomsuksa three) from two Thai secondary schools. Both schools are located in Surin Province. The first school is Sirithorn School which is the highest achieving school and the other is Thatsrinakorn School.

Forty-eight students were selected from these two schools. For each school, twenty-four students were selected, eight students from each grade: Mathayomsuksa one, Mathayomsuksa two, and Mathayomsuksa three. The 8 students comprised two high achievers in mathematics (mathematical achievement grade point of 4), four average achievers in mathematics (mathematical achievement grade point of 3 or 2) and two low achievers in mathematics (mathematical achievement grade point of 1 or 0) based on their mathematical achievement grade of the previous semester. The algebraic test was administered to all forty-eight students. According to the results of all students' papers, the researcher selected 24 students who had interesting paper work as the subjects of this study to be interviewed. In selecting students to be interviewed, the researcher maintained a balance between the two schools, grade level and mathematical achievement. A brief description is shown in Table 4.

TABLE 4 NUMBER OF SUBJECTS IN THIS STUDY

Name of School	Grade Level (Mathayomsuksa)	Number of Students			
		High Achievement	Average Achievement	Low Achievement	Total
School 1	One	2(1)*	4(2)	2(1)	8(4)
School 2	One	2(1)	4(2)	2(1)	8(4)
School 1	Two	2(1)	4(2)	2(1)	8(4)
School 2	Two	2(1)	4(2)	2(1)	8(4)
School 1	Three	2(1)	4(2)	2(1)	8(4)
School 2	Three	2(1)	4(2)	2(1)	8(4)
Total		12(6)	24(12)	12(6)	48(24)

NOTE: The symbol *a(b)*, in each cell for number of students, means that *b* students were selected from the *a* students who took the written test.

2. Instruments

The instruments in this study consisted of the initial framework explained in phase 1, the algebraic test, and the interview guide. The initial framework was used as a guideline to construct the algebraic test and the interview guide.

2.1 The algebraic Test

In order to construct the algebraic test, the researcher used the initial framework to guide the design and selection of problems and questions. The algebraic test consisted of three sessions covering the three key indicators of algebraic thinking: pattern, representation, and variable.

The first session focused on the four pattern problems. Students were asked to find the next term, the specific higher term and then the n^{th} term. Along with each problem, a series of questions were constructed which enabled the researcher to investigate students' algebraic thinking across four levels of thinking in the generalization of patterns.

The first three patterns focused on a semi-concrete pattern and the last pattern was on number sequence.

A second session of the test served as the instrument to collect data on thinking of representation and the transformation among representations. This session involved three problems and each problem had a listed of question to stimulate students. The problems were developed to see if students could represent the data (i.e. distance and time of a journey) in the form of a graph or they could interpret the given graph.

Finally, the last session on variable comprised a series of 7 questions in a problem. This task was used to investigate students' understanding of letters as a generalized number.

The algebraic test was tried out with three students, one from each grade to determine the appropriateness (language and difficulty) of the test and to study the way students responded to questions. After that, the test was revised. Three experts, a secondary school mathematics teacher and two university mathematics educators evaluated the appropriateness and commented on ways to improve the questions (see Appendix A, pp.160-161).

The test was adjusted according to the experts' recommendations. The pilot test was conducted with 9 students who were studying in lower secondary school at Rattanaburi School in Surin Province and were not selected for the main sample; three students from each grade. After that, the problems and questions were revised again for the final form (see Appendix B, pp.162-171). The test took approximately two hours.

2.2 The interview Guide

The researcher constructed the tentative interview questions in order to find what students were thinking on each test question. The semi-structured interviews were used. The interview questions could be adjusted depending on the students' paper works. The interview questions aimed to motivate students to explain their thinking orally. Materials (pencils, papers, pens etc.) were provided in case they wanted to use them for their explanation. The interview guide was tried out with three students of a pilot group. The results of the pilot study were transcribed and coded by the researcher and another two coders who are doctoral students in mathematics education. Coders were trained by the researcher before they did the coding.

3. Data Collection

The algebraic test was administered to 48 students in two schools but all of them did not take the test at the same time. The interview was administered right after the paper test. Four students were tested each day. In the morning, the researcher tested two students who were in the same grade and the same level of mathematical achievement and then chose one student to be interviewed. In the afternoon, the researcher used the same method with another pair of students. Each student was interviewed for 45 minutes to one hour. The data in algebraic thinking were collected from all papers and interviews of twenty-four students who were selected as subjects for this study. Audio tapes and video tapes were used for recording the interview. The data from interviews was later transcribed. Such data collection was done during June, July and August of the year 2004.

4. Data Analysis

In coding the data, the researcher applied a double coding procedure described by Miles and Huberman (1994: 64) and Boyatzis (1998: 150-159) who suggested that two or more coders independently code data and then verify their differences until consensus is reached. In this study, there were three persons who did the coding; the researcher and another two coders (doctoral students in mathematics education). All three coders coded the data independently. Descriptors were used to determine the level of thinking for each indicator: pattern, representation, and variable. Each coder and the researcher used the descriptors in the initial algebraic thinking framework as criteria to code by level of thinking on each problem. The code to be used in this study ranged from Level 1 to 4 for algebraic thinking.

In order to justify each student's level of thinking, the researcher had to determine each student's level of thinking on pattern, representation and variable. Since there are four problems on the pattern test and three problems for representation, the mean value of all coding for each indicator was used to determine the level of algebraic thinking.

If a student has a mean value of 1.0 to 1.5, he or she will be assigned as a Level 1 algebraic thinker.

A student who has a mean value of greater than 1.5 and less than or equal to 2.5 will be assigned as a Level 2 algebraic thinker.

A student who has a mean value of greater than 2.5 and less than or equal to 3.5 will be assigned as a Level 3 algebraic thinker.

A student who has a mean value of greater than 3.5 will be assigned as a Level 4 algebraic thinker.

After all responses of each indicator were coded from descriptors, the researcher looked for the reliability among coders which would indicate the consistency of coding. This study applied the reliability formula of Miles and Huberman (1994: 64) and Boyatzis (1998: 154). This formula is shown below.

$$\text{Reliability} = \frac{\text{number of times which all coders agreed}}{\text{number of times which all coders agreed} + \text{disagreements}}$$

Typically, the excepted reliability in coding is 70% or higher (Boyatzis. 1998: 156; Krippendorf. 1980: 147-148). The reliability among coders in this study was 88%. When three coders coded differently, they discussed their reasoning and reconsidered the level of thinking given to the students' responses. The discussion continued until consensus on the same level of thinking was reached. In this study, the researcher used pseudonym for all 24 students.

5. Validating the Framework

Coded data from 24 students' paper works and interviews were used to refine the descriptors in the initial framework. This data also determined the consistency of students' algebraic thinking and clarified the characteristics of algebraic thinking across four levels.

In refining the framework, students at the same level were grouped together for one indicator at a time. Then a matching of the descriptors of such levels of the indicator in the framework and students' responses were done. If the initial descriptor in the framework matched with the response of students in that level then that descriptor was maintained in the framework. On the other hand, if the descriptor in the framework was not found in students' responses, such a descriptor was removed from the framework and was replaced with a descriptor consistent with the students' responses. If there were some indicators accrued from students' responses, such descriptors were added. If such a descriptor was not clear, the researcher and two coders examined the pattern of responses to determine the descriptor.

CHAPTER 4

FINDINGS

The results of this study are presented into two phases. The first phase presents the formulating of the algebraic thinking framework and the second phase presents the results of validating the algebraic thinking framework.

Phase 1: Formulating the Algebraic Thinking Framework

After the researcher studied documents, books and related studies which related to the SOLO model, algebraic content, algebraic thinking, levels of thinking in mathematical domains and methods of developing frameworks, the initial algebraic thinking framework was formulated (see Table 3, pp. 39-42 in chapter 3). This framework consisted of four levels of thinking: prestructural (Level 1), unistructural (Level 2), multistructural (Level 3), and relational (Level 4). The general descriptors for each level were developed along with the descriptors of three specific indicators: pattern, representation and variable.

At the *prestructural* (Level 1), students hardly understand the problem or may even fail to engage in the problem. They make no attempt to produce an answer and also show no logical connections. Students at this level do not answer the question or guess the answer. They give an irrelevant answer, or restate the question. Sometimes, they cannot keep the question in mind when they are answering. When students at this level are confronted with the *pattern* problems, they do not understand the given pattern and are also confused about the pattern question. Sometimes they know that the given pattern is called a pattern, but they do not know how to use this given pattern to identify the characteristic of the complex pattern. With respect to *representation* problems, when they are asked to represent the given data in the form of graph or table, Level 1 students do not understand the given data and are confused by the question. Level 1 students like to avoid the question or answer the question based on irrelevant data. Sometimes they understand the question but do not know how to answer it. Students do not interpret the given graph and also cannot transform the given representation to another. When asked questions on *variable*s, students exhibiting Level 1 thinking fail to engage the question. They may say that "I don't know." or "I try to do this question, but I cannot." These students also make little attempt to answer the question.

At the *unistructural* (Level 2), students focus on only one relevant case or aspect of the information. They give the correct answer but not consistently. They try to relate the question and answer using at least one logical process. For the *pattern* problems, students can find the value of the next term of the given pattern. If they generalize the given pattern, they focus on one term of the pattern. Sometimes, they recognize the given pattern as a concrete object and use it to find the value of a specific term of the given pattern. Students at this level often identify one characteristic of a complex pattern. In *representation*, these students try to represent the situation, but they represent the given data inappropriately by using only one aspect of the information. When these students interpret the graph, they read the graph on one axis only. When students exhibiting unistructural thinking solve the problem which relates to *variable*, they use only a specific unknown number or one condition to make a conclusion.

At the *multistructural* (Level 3), students use two or more relevant concepts or aspects of the data to find the answer, but they cannot link them. With the given *pattern*, they see it as a successive process. They can describe how the values move from one term in a sequence to the next term but they cannot relate the value of a term to the term number. Students at this level also identify characteristics of the patterns with two aspects. In *representation*, students show some data correctly, but some data is left out. When interpreting the given graph, they read the graph on one axis and make a comparison between the given points on the graph using one variable only. Also in understanding of *variable*, students at this level use some numbers in solving a variable problem but do not cover all values of the determined variable.

At the *relational* (Level 4), students show the use of all relevant data in the problem and demonstrate an integrated understanding of the relationships among the data. When conducting the *pattern* problem, students understand the relationship between the value of the term and the term number in the given pattern and can make a generalization in symbolic form or explain with a verbal statement. Students at this level also identify all aspects of the characteristics of patterns and explain the relationship between them. When asked to *represent* the data for the given situation, they represent all data correctly and recognize all conditions of the problems. Students interpret the graph recognizing the relationship between two axes. In confronting questions on *variable*, students solve and complete the problem by seeing the variables as generalized numbers.

Phase 2: Validating the Algebraic Thinking Framework

This result is presented in three parts. The first part explains the refinement of the initial algebraic thinking framework after the researcher read and analyzed all 24 students' responses. The second part describes students' algebraic thinking for each level across the three key indicators and then in the final part presents a summary of the descriptors of the revised framework.

Part I: Refinement of the Framework

In order to adjust and refine the initial framework and eliminate discrepancies between the descriptors in the initial framework and target students' responses, all 24 student assessments and interviews were analyzed. Table 5 shows the descriptors of the refined algebraic thinking framework across three key indicators. Underlined statements are those modified in the initial framework so as to represent more closely students' responses in algebraic thinking. Boldfaced statements indicate descriptors of students' algebraic thinking that were found in student responses but were not contained in the initial framework.

TABLE 5 REFINED LOWER SECONDARY SCHOOL STUDENTS' ALGEBRAIC THINKING FRAMEWORK

Level / Indicator	Prestructural (level 1)	Unistructural (level 2)	Multistructural (level 3)	Relational (level 4)
General characteristic	- makes no attempt to answer the question - hardly understands the task or fails to engage the task - hardly remembers the question or cannot keep the question in mind while answering - makes no logical connections; does not answer the question, guesses the answer, gives an irrelevant answer or restates the question	- uses only one relevant aspect or concept of the information to find the answer - keeps the question in mind and tries to relate to the question; deals with at least one logical idea - gives a correct answer but may not be consistent	- uses two or more aspects or concepts of the data to find the answer - fails to link aspects with each other	- applies most or all aspects of the data - shows an integrated understanding of the relationships among various aspects of the data

TABLE 5 (continued)

Level / Indicator	**Prestructural (level 1)**	**Unistructural (level 2)**	**Multistructural (level 3)**	**Relational (level 4)**
Pattern	**(P11)** does not understand or is confused about the pattern **(P12)** does not find the value of the next term in the given pattern **(P13)** guesses the answer, gives an irrelevant answer, restates the problem, or duplicates the given term of the pattern when finding a specific term **(P14)** uses every given term of the pattern but does not make logical connections	**(P21)** can find the value of the next term of the given pattern, but when finding the value of a higher term, the response is irrelevant **(P22)** recognizes only the relationship between the values of the given terms but does not use the relationship to find the value of a higher term	**(P31)** recognizes only the relationship between the values of the given terms and uses this relationship to find the value of a higher term **(P32)** uses the relationship between the values of the given terms to generate a systematic method to find the value of any specific term of the given pattern, even the higher terms, without finding every previous term	**(P41)** recognizes the relationship between the value of a term and its term number in the given pattern and is able to explain the relationship verbally **(P42)** generalizes the relationship of the pattern symbolically or provides a formula giving the value of a general term; for example $a_n = 6+2(n-1)$

TABLE 5 (continued)

Level / Indicator	**Prestructural (level 1)**	**Unistructural (level 2)**	**Multistructural (level 3)**	**Relational (level 4)**
Pattern	**(P15)** uses only one given term of the pattern to find a specific term, but is not consistent in finding other terms **(P16)** makes no attempt to capture and find the answer **(P17)** knows that the given data is a pattern, but does not know how to use the given pattern to answer the question	**(P23)** uses the relationship between the values of the given terms to find the value of a specific term by drawing, counting or computing every previous term of the given pattern **(P24)** tries to generalize from only one given term or only one aspect of the given pattern but the generalization is not correct	**(P33)** does not relate the value of a term to its term number in the given pattern; for example, cannot find a relationship like $a_n = 6 + 2(n-1)$	
Representation	**(R11)** does not understand the question or is confused by the question; avoids the question or	**(R21)** understands the question but does not know how to construct a graph or	**(R31)** represents some data correctly and clearly in the form of a graph or	**(R41)** represents correctly and systematically all of the given data in the form

TABLE 5 (continued)

Level / Indicator	Prestructural (level 1)	Unistructural (level 2)	Multistructural (level 3)	Relational (level 4)
Representation	answers the question based on irrelevant data **(R12)** does not understand the conditions on the problem **(R13)** does not answer the question; answers the question based on irrelevant data, guesses the answer, or restates the problem **(R14)** cannot interpret the points on a graph	a table to represent the given data **(R22)** understands the question but constructs an invalid graph because the given data is interpreted incorrectly **(R23)** constructs other displays which are not required, such as a map or wave and calls them a graph **(R24)** understands the question but represents only	table but leaves out some data or conditions **(R32)** does not recognize all conditions of the problem when constructing a graph **(R33)** interprets the graph with respect to both axes	of a graph or table which recognizes all conditions and relationships in the data **(R42)** represents data in the table which is extrapolated from the given data **(R43)** compares data from the graph with respect to both axes and explains the relationships in the graph

TABLE 5 (continued)

Level / Indicator	**Prestructural (level 1)**	**Unistructural (level 2)**	**Multistructural (level 3)**	**Relational (level 4)**
Representation		some of the data accurately **(R25)** constructs a table which does not organize the data; the table is difficult to read **(R26)** interprets the graph by reading the graph on one axis only, compares data from the graph with respect to one axis only		
Variable	**(V11)** hardly understands the task or fails to engage the task	**(V21)** does not understand the role of a variable as a generalized number, uses only one specific number to	**(V31)** does not fully understand the role of a variable as a generalized number, uses several	**(V41)** understands the role of a variable as a generalized number, and gives a well reasoned

TABLE 5 (continued)

Level / Indicator	**Prestructural (level 1)**	**Unistructural (level 2)**	**Multistructural (level 3)**	**Relational (level 4)**
Variable	**(V12)** answers the question based on irrelevant data or guesses the answer **(V13)** does not know the meaning of variable **(V14)** holds misconceptions about operations in equations, expressions and inequalities	make a general conclusion	specific numbers to make a general conclusion **(V32)** does not recognize any conditions on the variable that were not explicitly stated	general conclusion **(V42)** recognizes all conditions on the variable even these were not explicitly stated

Pattern refinements

The researcher reorganized, adjusted and refined the descriptors in four levels of this indicator. For pattern, the analysis of students' responses indicated that the descriptors PA24, PA33 and PA44 for identifying the characteristics of a pattern for all four levels in the initial framework (see Table 3, pp. 40-43 in chapter 3) did not match students' responses. Thus, these descriptors were removed from the framework. A number of other changes are described below according to the level of the framework to which they belong.

At Level 1, the researcher added the descriptor "but does not know how to use the given pattern to answer the question" to accommodate students' responses that they knew that the given data was a pattern, but they did not know how to use this data to answer the question. This change was made by modifying the descriptor PA12 and inserting the descriptor P17 in the refined framework.

Students' responses also indicated that they could not find the value of the next term in the given pattern. Sometimes they were not interested in the problem. They also made no attempt to find the answer. They liked to guess the answer or gave an irrelevant answer. In addition, students at this level restated the pattern problem and also duplicated the given term of the pattern to find a specific term. Therefore, the descriptors P12, P13, and P16 were added into the refined algebraic thinking framework. Similarly, the descriptors P14 and P15 were inserted into this framework too because when students were confronted with pattern problems, they used only one given term of the pattern to find a specific term, but was not consistent in finding other terms. Sometimes, they used every given term of the pattern to find answer but they did not make logical connections.

At Level 2, when students were asked to find the value of the next term of the given pattern, they could find it, but their answer was irrelevant for a higher term. Therefore, the descriptor PA21 in the initial algebraic thinking framework was adjusted with the addition "but when finding the value of a higher term, the response is irrelevant" and this amended descriptor P21 was inserted in the refined framework. Although students at this level recognized only the relationships between the values of the given terms without relating to the term number, they did not use this relationship to find the value of a higher term. Thus, the descriptor P22 was added to the refined framework. In addition, when finding the value of a specific term of the pattern, students drew each picture or counted or computed every previous term of the given pattern. For example, when they had determined 5 terms of the given pattern, students found the value of the 20^{th} term by finding the value of the 6^{th}, 7^{th}, 8^{th} , ... terms until they got the value of the 20^{th} term. They did not skip any previous terms to find the value of the 20^{th} term. Therefore, the descriptor PA22 in the initial framework "use concrete objects to find a specific term of the given pattern rather than a general pattern description" had to be extended and changed to the new descriptor P23 in the refined framework (see Table 3, pp. 40-43 in chapter 3).

Students at Level 3 recognized only the relationship between the values of the given terms similar to Level 2 students. However, Level 3 students used that relationship to find the value of the higher term. Thus, the descriptor PA31 was modified and changed to the new descriptor P31 in the refined framework. Students at this level tried to construct a new idea to find the value of a specific term of the given pattern when confronted with a pattern, although they did not relate the value of a term to its term number. They used the relationship between the values of the given terms to generate a systematic method to find the value of any specific term of the given pattern, even higher terms, without finding all previous terms. Therefore, the descriptor P32 was added in the refined framework. It can be concluded that students at this level were able to find both the value of the next term and the value of higher terms by skipping some previous terms of the given pattern.

At Level 4, the descriptor PA41 "understands the relationship between the value of terms and the number of terms in the given pattern" and the descriptor PA42 in the initial framework "explains correctly the relationship in the given pattern verbally" were combined in order to make clear the students' responses. The new descriptor P41 accommodated this change in the refined framework.

Representation refinements

For representation, refinements were made to descriptors for all four levels of thinking in order to make the descriptors fit with students' responses. When asked to represent the given data according to certain conditions, Level 1 students did not understand the conditions on the problem. They did not answer the question, answered the question based on irrelevant data, guessed the answer or restated the problem. Thus, the descriptors R12 and R13 were added to the refined framework. For this indicator, there were no students' responses concerning transforming the given display to another. This descriptor RE14 in the initial framework was eliminated.

At Level 2, students showed that they understood the question. The descriptor RE12 "understands the question but does not know how to answer" was moved to Level 2 thinking as the descriptor R21 because it was seen to be more characteristic of Level 2 thinkers than Level 1 thinkers. That is, students understood the question but they did not know how to construct a graph or table to represent the given data (see Table 3, pp. 40-43 in chapter 3). The descriptor RE21 "represents the given data inappropriately by using only one aspect of the information" was modified to resemble more closely to students' responses and labeled in the refined framework as the descriptor R23. Sometimes Level 2 students constructed an invalid graph because the given data were interpreted incorrectly. To accommodate this change, the descriptor R22 was added to the refined framework.

In constructing a table, students at this level understood the question but they represented only some of the data accurately. Their table was difficult to read and the data in the table was not organized. Therefore, the descriptors R24 and R25 were added into the refined framework. When presented with a graph, students exhibiting Level 2 interpreted the graph by reading it on only one axis; they also compared data from the graph with respect to one axis. Thus, the descriptor RE32 of Level 3 thinking was moved to Level 2 thinking and combined with descriptor RE22 in the initial framework and then labeled in the refined framework as the descriptor R26.

At Level 3, the descriptor RE31 in the initial framework was rewritten as "represents some data correctly and clearly in the form of a graph or table but leaves out some data or conditions" and was labeled as the descriptor R31 in the refined framework. When students were asked to construct a graph to represent the given data, Level 3 students did not recognize all conditions of the problem. Thus the descriptor R32 was added into the refined framework. In interpreting the graph, students at this level could only interpret the graph with respect to both axes; they did not compare the data. Thus the descriptor RE42 of Level 4 thinking in the initial framework was moved down to descriptor R33 of the Level 3 thinking in the refined framework to suit students' abilities.

At Level 4, the descriptor RE41 in the initial framework was modified as the descriptor R41 the refined framework, that is students represented correctly and systematically all of the given data in the form of a graph or table which recognized all conditions and relationships in the data. Students at this level showed their ability to represent data in the table which were extrapolated from the given data. Therefore the descriptor R42 was added into the refined framework. Furthermore, in interpreting the graph, Level 4 students were able to compare data from the graph with respect to two axes and some students could explain the relationship in the graph. Thus, the descriptor R43 was added.

Variable refinements

Most refinements were made to descriptors at Levels 1, 2 and 3. Students at Level 1 did not know the meaning of variable and also hold misconceptions about the operations in equations, expressions and inequalities. Thus the descriptors V13 and V14 were added (see Table 3, pp. 40-43 in chapter 3).

Students exhibiting Level 3 thinking did not recognize any conditions on the variable that were not explicitly stated; however students exhibiting Level 4 recognized all conditions on the variable even these not explicitly stated. Therefore the descriptors V32 and V42 were added into Level 3 and Level 4 of the refined framework respectively.

Part II: Students' levels of Algebraic Thinking

Based on the results of coded responses, a student was assigned a level of thinking for each indicator according to the agreement of coders and the researcher. Figure 5 shows profiles of 24 students' level in algebraic thinking for all three indicators.

The results are arranged by grade levels. From these profiles, it is noted that most students are at Level 2 in thinking for each indicator. There is a strong consistency in students' algebraic thinking across indicators. Ten students (41.67%) had the same levels of thinking across all three indicators. Twelve students (50%) had levels of thinking that were the same for two indicators with the third indicator one level higher or one level lower. Two students ($8.\bar{3}3$%) had levels of thinking that were the same for two indicators with the third indicator two level lowers. Students' algebraic thinking within each indicator is discussed below with examples of students' responses for each level of thinking.

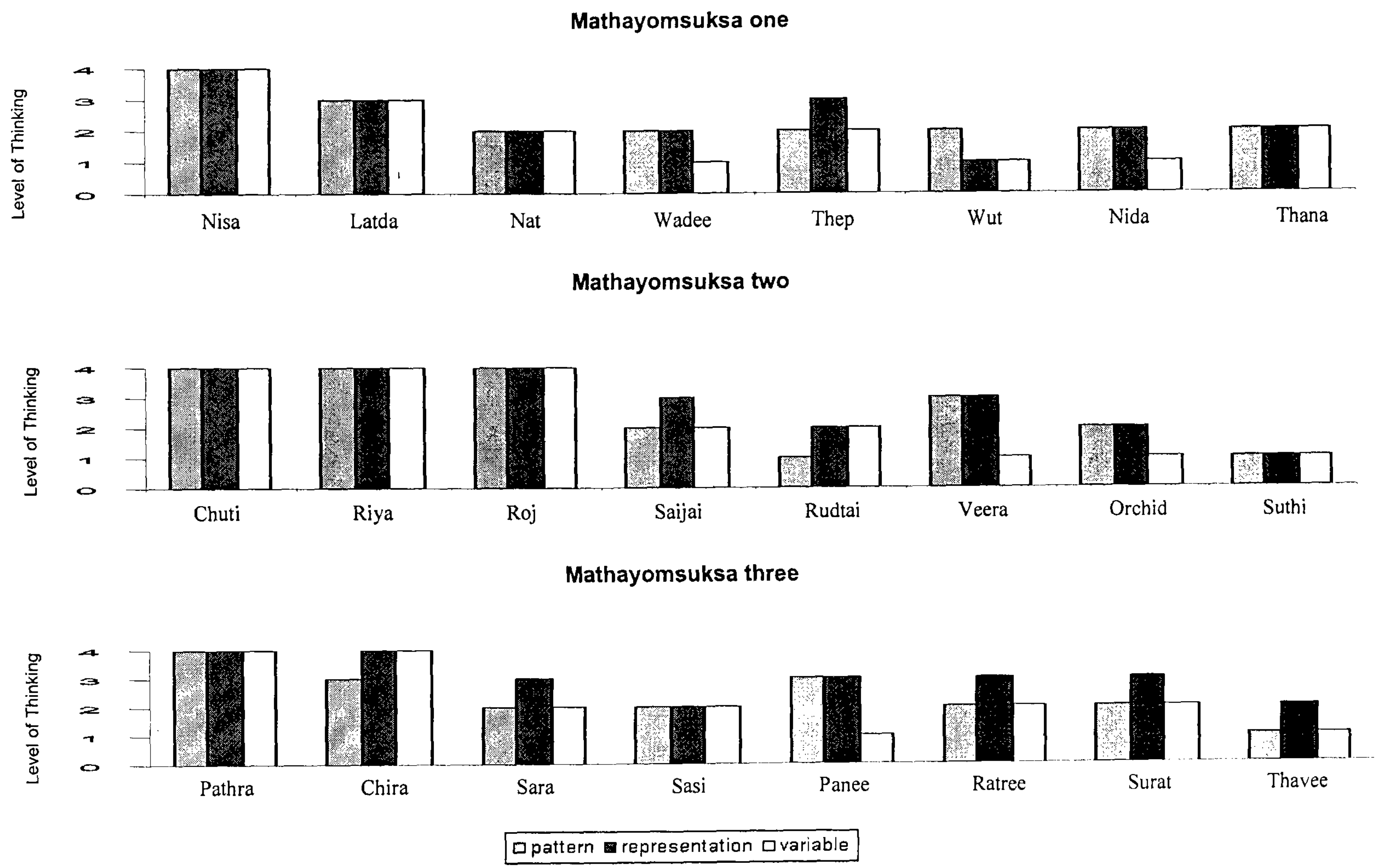


FIGURE 5 PROFILES OF STUDENTS' LEVELS OF ALGEBRAIC THINKING

These profiles can illustrate the number of students in each level of thinking across three indicators as shown in Table 6.

TABLE 6 THE NUMBER OF STUDENTS IN EACH LEVEL OF THINKING ACROSS THREE INDICATORS

Grade level	Number of students											
	Pattern				Representation				Variable			
	L1	L2	L3	L4	L1	L2	L3	L4	L1	L2	L3	L4
Mathayomsuksa one	0	6	1	1	1	4	2	1	3	3	1	1
Mathayomsuksa two	2	2	1	3	1	2	2	3	3	2	0	3
Mathayomsuksa three	1	4	2	1	0	2	4	2	2	4	0	2

NOTE: The symbol L1, L2, L3, and L4 stand for Level 1, Level 2, Level 3, and level 4 of thinking, respectively.

Students' algebraic thinking within each indicator is discussed below with examples of students' responses for each level of thinking.

Students' Thinking on Pattern

Students' algebraic thinking on pattern reflects their abilities to use their thinking skills to generalize the given pattern in symbolic form. Thus, the descriptors for pattern reflect students' abilities to a) find a specific term of the given pattern and b) generalize the given pattern. Figure 6 shows students' level of thinking on pattern.

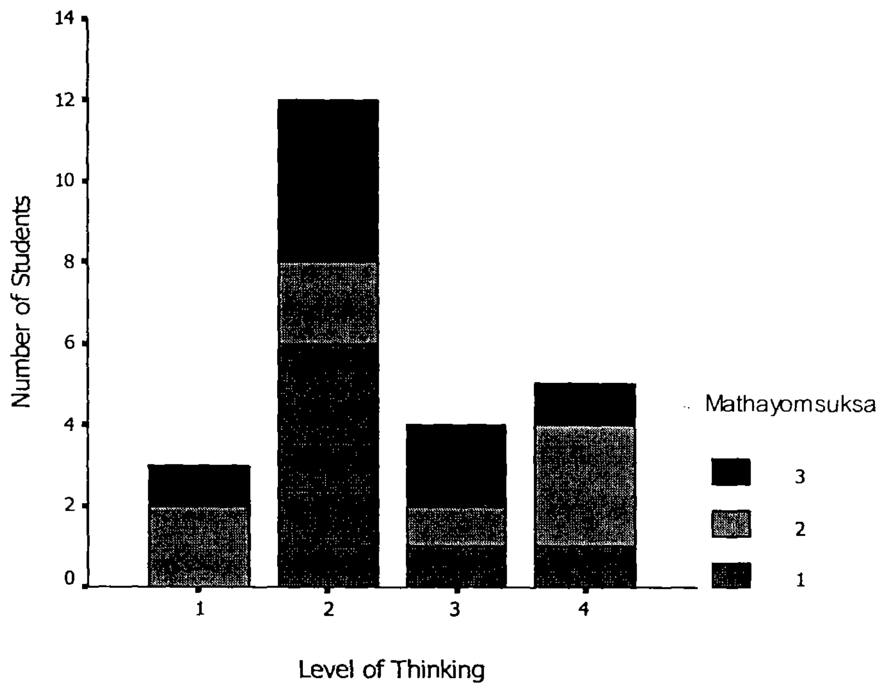


FIGURE 6 STUDENTS' LEVEL OF THINKING ON PATTERN

The largest group of students, 12 out of 24, exhibited Level 2 thinking on the pattern indicator where 4 students were in Mathayomsuksa three, 2 students were in Mathayomsuksa two and 6 students were in Mathayomsuksa one. Five students demonstrated Level 4 thinking, 1 student was in Mathayomsuksa three, 3 students were in Mathayomsuksa two and 1 student was in Mathayomsuksa one. The thinking of 4 students was considered to be at Level 3, that is 2 students were in Mathayomsuksa three, 1 student was in Mathayomsuksa two and 1 student was in Mathayomsuksa one. Only 3 students demonstrated Level 1 thinking, 1 student was in Mathayomsuksa three and 2 students were in Mathayomsuksa two. The result illustrated that Mathayomsuksa one students spread over Level 2, 3, and 4, while none was in Level 1, whereas Mathayomsuksa two and three students demonstrated all 4 levels of thinking. It is noticeable that students' thinking on pattern was not much different between grade levels.

To access students' thinking on pattern, 24 students were presented with four pattern problems and asked to find a specific term of the given pattern. They were also asked to generalize the pattern. The first three patterns were presented in semi-concrete form while the last pattern was a sequence of numbers.

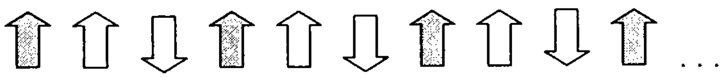
The first problem was the pattern of pictures of arrows (Problem 1: Arrow pattern), which consisted of 10 arrows in the given pattern. It was a change pattern (repeating) with three units of change. Students were given a pattern with pictures of 10 arrows and were asked to find the next term of the given pattern and the 63rd term of the pattern.

The second problem was the pattern with pictures of tables and chairs (Problem 2: Table pattern). Students were asked to find the number of chairs when 54 tables were put next to each other. This problem was presented by giving 3 terms.

The third problem was presented with dots (Problem 3: Dot pattern). The problem was presented by giving 4 terms. Students were asked to find the number of dots for the 5th, 21st, and nth term.

The final problem was a number sequence (Problem 4: Number sequence), which consisted of the term (T) and the number of that term (S). In this problem, students were asked to find the next term, the 70th term and the nth term of the given pattern. These problems are shown in Figure 7 below.

1. *Arrow pattern*: From pattern below:

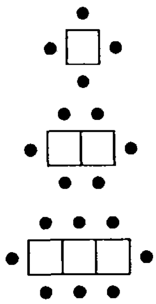


Find (1) What is the next arrow? How do you get it?

(2) What is the 63rd arrow? How do you get it?

FIGURE 7 THE PATTERN PROBLEMS

2. *Table pattern*: In the conference room, the tables and chairs are arranged systematically as follows;



1 table with 4 chairs

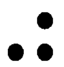
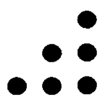
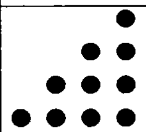
2 tables with 6 chairs

3 tables with 8 chairs

so on

If we want to arrange tables and chairs according to above pattern by using 54 tables put next to each other to be a long table. How many chairs do we need? Explain how you think along with reason?

3. *Dot pattern*: From the table below, find the total number of dots in the 5^{th} , 21^{st} and n^{th} position and explain how you get the answer? (n is any positive integer)

Position	1	2	3	4	5	...	21	. . .	n
Picture						...		...	
# of dots	1	3	6	10	□		□		□
					(1)		(2)		(3)

4. *Number sequence*: From the table below, fill your answer in □ and also explain your thinking along with reason.

S	1	2	3	4	5	...	70	...	m
T	4	7	10	13	□	. . .	□	...	□
					(1)		(2)		(3)

When m is any positive integer.

FIGURE 7 (continued)

Four levels of students' responses on all 4 problems are shown as follows:

Level 1

Students at this level could not understand the pattern or were confused about the pattern question. Some did not know that it was supposed to be continued. Also, they were unable to analyze the given pattern. When they were asked to find the value of the next term of the given pattern, some of them did not even try to understand or find the answer as was the case for Wut who was studying in Mathayomsuksa one. Wut was assigned Level 1 thinking for Problem 1 (Arrow pattern), although in an overall sense, he was a Level 2 thinker on pattern.

Interviewer:	What is the next term of this pattern?
Wut:	I do not know.
Interviewer:	How many arrows are there?
Wut:	10 arrows.
Interviewer:	Can you find the 11^{th} arrow?
Wut:	I do not know because my teacher hasn't taught me about this.

Although the interviewer tried to make him think by asking him to identify from the 1^{st} arrow to the 10^{th} arrow and then asking him about the value of the next term (11^{th} term), Wut just told what he saw next to the 10^{th} term by saying it was "dot."

Although Level 1 thinkers were confronted with the semi-concrete pattern, they also were unable to capture the given pattern. They used only one given term to find the value of the required term of the given pattern, but could not consistently find other terms. Ratree's response was an example of this method. She was a Mathayomsuksa three student and was assigned Level 1 thinking for Problem 2 (Table pattern), although in an overall sense, she was a Level 2 thinker on pattern. When Ratree was presented with three terms of table pattern,she used the given data, but just used the first picture. That is she used only one table from the first term to find the answer to the question of how many chairs there would be if there were 54 tables put together. According to the first picture of one table with 4 chairs, she found the number of chairs to be 54 x 4. However, when asked to find the number of chairs with 5 tables, she used the 2nd and 3rd terms and then combined 2 tables for 6 chairs and 3 tables for 8 chairs, thus she got 14 chairs. It shows that she did not apply the pattern consistently. She worked on it but she did not recognize how the number of chairs changed when the number of tables was changed.

Suthi, a Mathayomsuksa two student, used a similar method to Ratree, but he sketched the table picture to help his counting. He worked on a set of 3 tables which composed of 8 chairs and he drew 18 sets of 3 tables, which is equal to 54 tables. Then he counted the total number of chairs for 54 tables by combining 8 chairs in each set 18 times (8x18 = 144). However, he miscalculated, his answer was 155 chairs. He used only the last given term of the pattern to find the answer (see Figure 8).

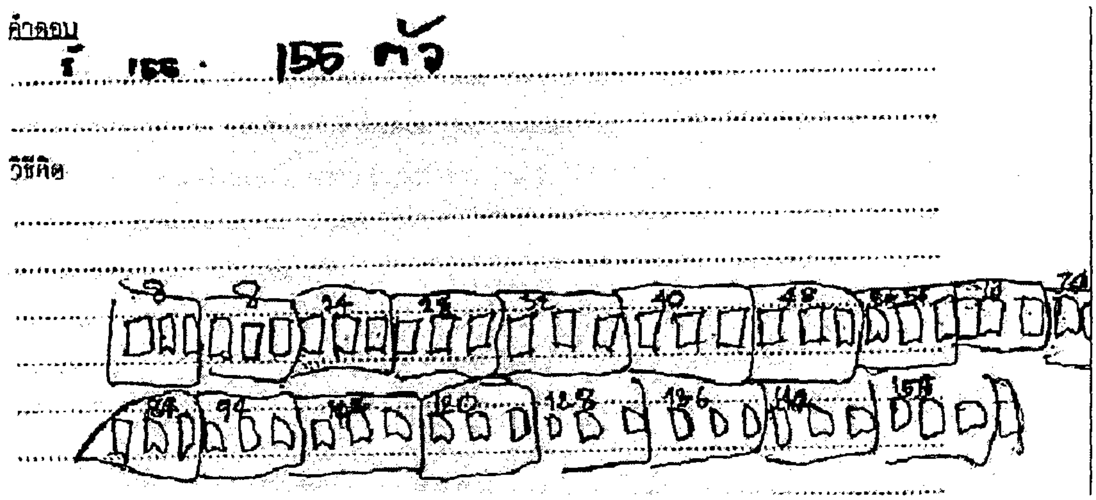
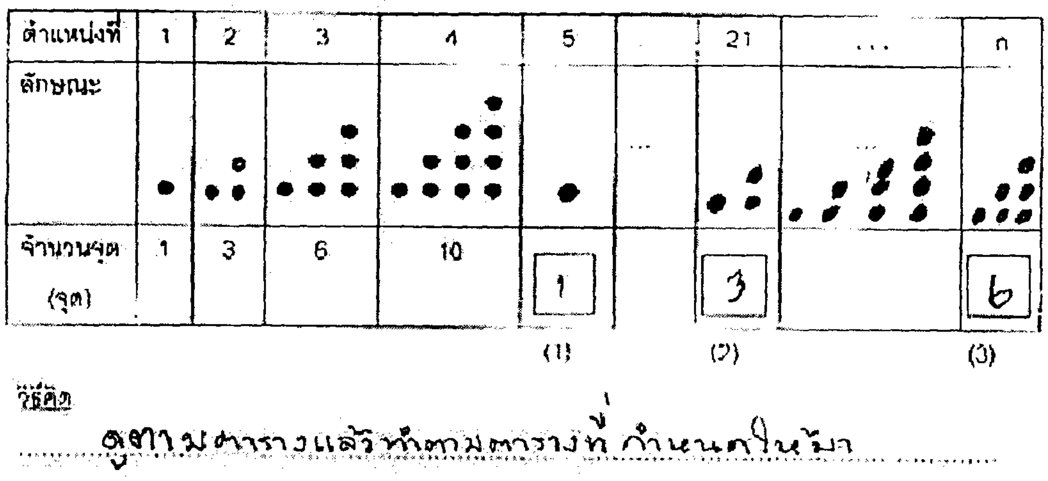


FIGURE 8 SUTHI'S RESPONSE ON TABLE PATTERN

In order to answer the question (Dot pattern), students at this level restated the Question or duplicated the given terms of the pattern without any understanding. Thavee, a Mathayomsuksa three student, used this approach as shown in Figure 9.



ตำแหน่งที่	1	2	3	4	5	...	21	...	n
ลักษณะ						...		...	
จำนวนจุด (จุด)	1	3	6	10	$\boxed{1}$		$\boxed{3}$		$\boxed{6}$
					(1)		(2)		(3)

วิธีคิด

ดูตามตารางแล้วทำตามตารางที่กำหนดให้มา

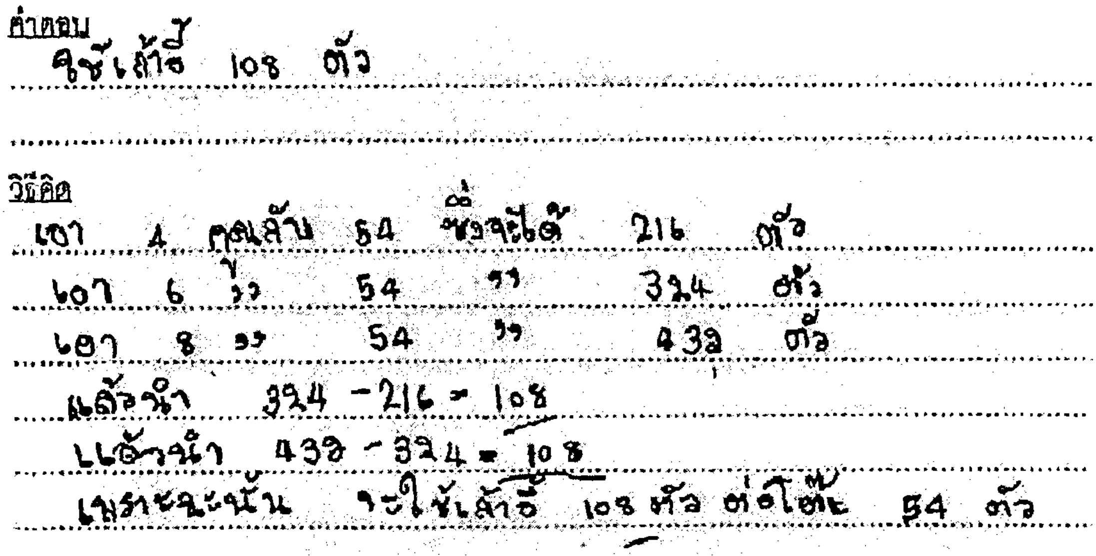
FIGURE 9 THAVEE'S RESPONSE ON DOT PATTERN

When asked to find the next term, some students, such as Surat, Mathayomsuksa three student, guessed the answer. Although in an overall sense, Surat was a Level 2 thinker on pattern, he was assigned Level 1 thinking for Problem 3 (Dot pattern). He found the number of dots in position 5 which was the next term of the dot pattern, by counting the total number of dots in the positions 1, 2, 3 and 4. For him, the total number of dots in position 5 was 20, which is the sum of 1, 3, 6 and 10. He was interested in the numbers, but did not focus on the given pattern. He also did not know how to analyze the given data.

Other students at this level solved the given pattern question based on irrelevant data. They used every given term of pattern to make an answer, even though they did not make logical connections and could not find the next term of the given pattern. Sometimes, they knew that the given data was a pattern, but they did not know how to use the given pattern to answer the question. Examples of students for this case are illustrated below:

Nat, a Mathayomsuksa one student, was assigned Level 1 thinking for Problem 2 (Table pattern), although in an overall sense she was a Level 2 thinker on pattern. Nat tried to find a way to solve the number of chairs for 54 tables by just using every term of the table pattern. However, she did not produce logical connections because she did not know how to analyze the given pattern. First, she used the first term (4 chairs for each table), thus she multiplied 4 (number of chairs) by 54 (number of tables), and got 216 chairs.

216 chairs. Second, she focused on the 2nd term (6 chairs for 2 tables), then she multiplied 6 (number of chairs) by 54 (number of tables) and got 324 chairs. Finally, she used the 3rd term of table pattern (8 chairs for 3 tables), thus she multiplied 8 (number of chairs) by 54 (number of tables) and got 432 chairs. In addition, she found that 324 minus 216 equaled 108 and also 432 minus 324 equaled 108. Therefore she determined that the total number of chairs for 54 tables was 108 chairs (see Figure 10).


คำตอบ
จะใช้เก้าอี้ 108 ตัว

วิธีคิด
เอา 4 คูณด้วย 54 ซึ่งจะได้ 216 ตัว
เอา 6 " 54 " 324 ตัว
เอา 8 " 54 " 432 ตัว
แล้วนำ 324 - 216 = 108
แล้วนำ 432 - 324 = 108
เพราะฉะนั้น จะใช้เก้าอี้ 108 ตัว ต่อโต๊ะ 54 ตัว

FIGURE 10 NAT'S RESPONSE ON TABLE PATTERN

Rudtai who was studying in Mathayomsuksa two guessed the answer, when she was asked to find the number of chairs for 54 tables. She gave an irrelevant answer by including every given chair and table from the given pattern together, which the result that equaled 24 (4 + 6 + 8 chairs, 1+2+3 tables). And then she added 24 to 54 tables, the answer was 78 chairs (24+54).

Thep who was studying in Mathayomsuksa one was assigned Level 1 thinking for Problem 3 (Dot pattern), although in an overall sense he was a Level 2 thinker on pattern. Thep separated the number of tables from the number of chairs, he did not exactly understand the given pattern. He calculated the total number of tables and chairs from the given pattern, which was 6 tables and 18 chairs. He multiplied both 6 and 18 by 2 and the result was 12 tables and 36 chairs. Then 12 and 36 multiplied by 2 and the result was 24 tables and 42 chairs which was miscalculated. He calculated the total number of the first and second set of tables and chairs that was 3 tables and 10 chairs, and then 24 and 42 added to 3 and 10 respectively, the result was 27 and 52. In order to complete the arrangement of 54 tables, he added 27 to 27 that equaled 54 and added 52 to 52 with the result being 104 chairs. He did not apply the pattern consistently in his work. (see Figure 11).

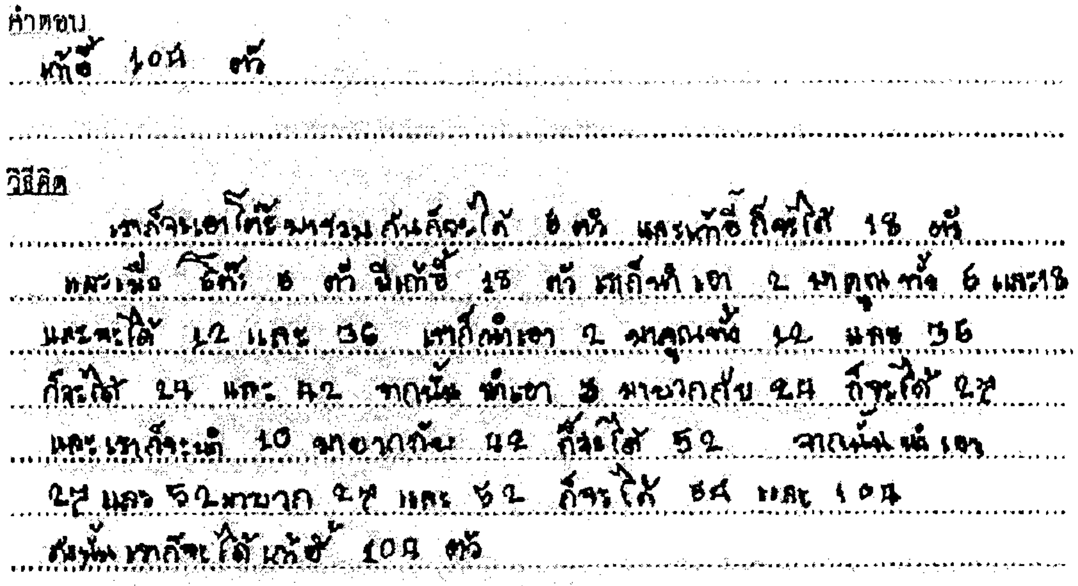


FIGURE 11 THEP'S RESPONSE ON TABLE PATTERN

Another student at this level was Nida who was studying in Mathayomsuksa one. In an overall sense, Nida was a Level 2 thinker on pattern, but she was assigned Level 1 thinking for Problem 2 (Table pattern). She calculated the total number of chairs in the given question which was 18 chairs. Then she multiplied 18 (number of chairs) by 54 (number of tables) to get the total number of chairs for the arrangement of 54 tables. She obtained 942 chairs. She did not use any of the logical connections in the given pattern.

In summary, Level 1 students are unable to understand the pattern. They are confused about the pattern. They make no attempt to capture and find the answer. Students at this level restate the questions and also sometimes duplicate the given terms of pattern as an answer without understanding. Although, they try to find a way to solve the given pattern such as using only one given term, two given terms or every given term, they do not make logical connections systematically and consistently find other terms. Students at this level guess the answer or solve the given pattern question based on irrelevant data. They sometimes know that the given data is a pattern, but they do not know how to use the given pattern to answer the question.

Level 2

Students exhibiting at this level of thinking understood the given pattern and could find the value of the next term of the given pattern, but they focused on only one aspect or condition. Sometimes they did not focus on the conditions associated with the pattern and could not find the value of the higher term. They did not apply a suitable method to solve the pattern like Thep who was Mathayomsuksa one student. In Problem 1 (Arrow pattern), he counted the given arrows one by one by including the 1^{st} arrow to be the 11^{th} arrow. Although the 10^{th} arrow was similar to the 1^{st} arrow, he did not skip the first term of the given pattern when counting the 11^{th} term, hence his answer was "white down arrow". He focused on stasis seeing a pattern as it was, that is as a group of 10. The interview dialogue was as follows:

Interviewer:	What is the arrow at the 63^{rd} position?
Thep:	The "white down arrow".
Interviewer:	Why did you think it's white down arrow?
Thep:	Because the total arrows are 10 and I performed cycle counting 10 arrows until the 63^{rd} position and then I have found white down arrow.
Interviewer:	Could you show me how to find the 63^{rd} position?
Thep:	Yes, I can 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (the student pointed at the 10^{th} arrow), 11(The student pointed at the 1^{st} arrow), 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44,

45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.

Interviewer: Here is the 63^{rd} position (pointed at the 3^{rd} position).

Thep: Yes. That's right.

N at who was studying in Mathayomsuksa one used a similar method to Thep, but she recognized the condition of the given pattern, that is, there was a repeating sequence on a set of conditions imposed on the arrow pattern. When asked to find the arrow at the 63^{rd} position, she counted each arrow until the 10^{th} arrow and then skipped to count the 11^{th} arrow at the 2^{nd} given arrow. After she had counted the given 10 arrows, she counted the second one in the given arrow as the 11^{th} arrow. Then she counted on one by one always starting with the second arrow. It seemed that she recognized the given pattern as a unit of change with 3 arrows and found the 63^{rd} term of pattern by counting each term of pattern from the pictures of arrows systematically up to the 63 as shown in the interview dialogue.

Interviewer: What is the 63^{rd} term of this problem?

Nat: White up arrow

Interviewer: How did you get this answer?

Nat: I counted each arrow until I got the 63^{rd} arrow.

Interviewer: How did you count it? Can you show me?

Nat: I counted from the 1^{st} to 10^{th} arrows and then started to count the 2^{nd} arrow to be the 11^{th} arrow because the 1^{st} arrow is the same as the 10^{th} arrow.

Interviewer: How about another round, did you count the first arrow?

Nat: No, I counted only from the 2^{nd} to 10^{th} arrows. I did not use the 1^{st} arrow again.

Another method used by Level 2 thinkers was to apply the condition of the given pattern using counting. Saijai, a Mathayomsuksa two student, used the same method as Nat, but she counted only the first three terms of the given pattern to find the 63^{rd} term because she recognized 3 terms of the pattern as a unit of change.

Si milarly for Problem 2 (Table pattern), students at Level 2 noticed that when the number of tables increased by one the number of chairs increased by two. So, the students kept counting by two from the last picture of 3 tables. Orchid who was studying in Mathayomsuksa two exemplified this approach. He counted the number of chairs increasing by 2 for every table until 54 tables were reached. The result was 102 chairs which was an incorrect answer. When asked to find the number of chairs with 4 tables, he got the correct answer.

Sara (Mathayomsuksa three) recognized the table pattern in a similar way to Orchid, but she had difficulty with the operation. She observed the table pattern and thought that it was composed of 2 chairs for one table and 2 chairs for the ends of the tables in line. However, she calculated the total number of chairs of each side of the table by 54 plus 54 (the result was 108 chairs), but she multiplied that sum with 54 (number of tables). The total number of chairs for both sides of the tables was 5,832 chairs and then added 2 chairs for the ends of the tables. The total number of chairs was then 5,834 for the arrangement of 54 tables which was an incorrect answer (see Figure 12).

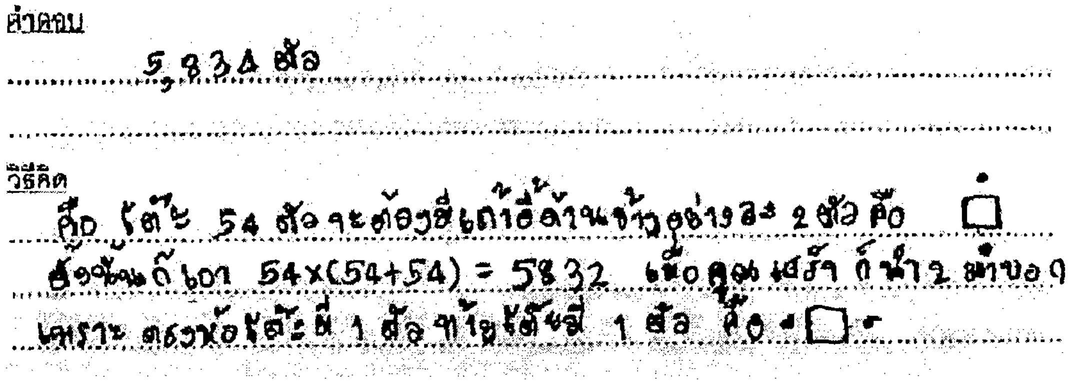


FIGURE 12 SARA'S RESPONSE ON TABLE PATTERN

Some students used concrete objects to find a specific term of the given pattern by drawing the picture until they found the required term of the pattern. That is, this method required students to know every previous term of the given pattern in order to find a certain term.

Wadee who was studying in Mathayomsuksa one had drawn the arrow sign in the same given pattern from the 11^{th} arrow to find the 63^{rd} arrow in Problem 1 (Arrow pattern) as Figure 13.

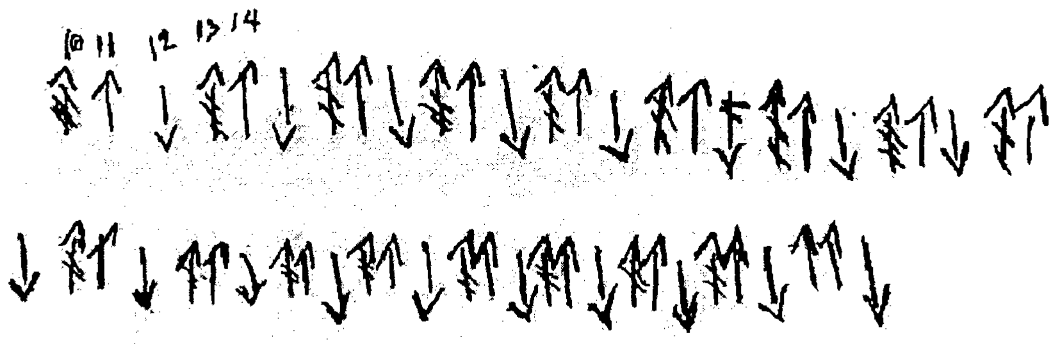


FIGURE 13 WADEE'S RESPONSE ON ARROW PATTERN

At this level, students sometimes recognized the relationship between the value of term and the term number. They tried to generalize from only one given term or only one aspect of the given pattern and used this method to find the specific term of the pattern. But their method was not correct for the given pattern. Examples of student responses are presented below:

L atda (Mathayomsuksa one) was assigned Level 2 thinking for Problem 3 (Dot pattern), although in an overall sense she was a Level 3 thinker on pattern. She found that the number of dots of the 5^{th} term was 12. She recognized the relationship of the 4^{th} term and the number of term by noticing that the 4^{th} term could be found by multiplying the number of term by 2, that is 2 x 4 and adding 2 (that is (2 x 4) + 2 = 10). Thus the 5^{th} term could be found by multiplying 5 by 2 and adding 2 (that is (2 x 5) + 2 = 12). She applied the rule from only one term to make a generalization (see Figure 14).

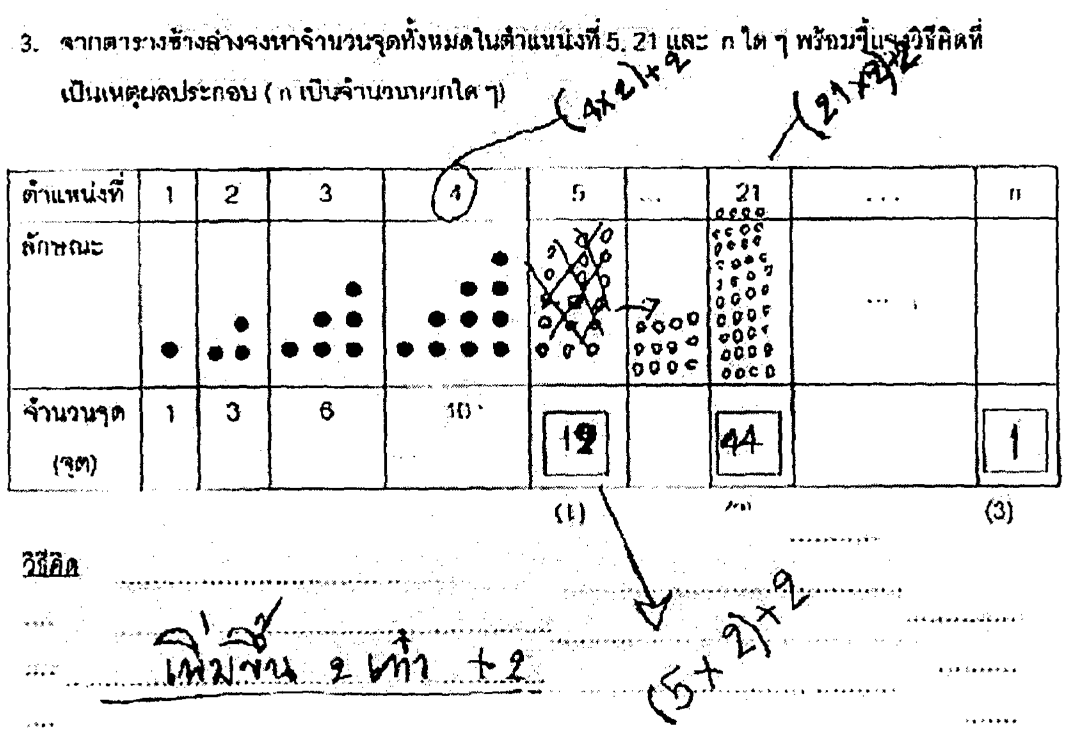


FIGURE 14 LATDA 'S RESPONSE ON DOT PATTERN

With this s ame pattern, in order to find the 5th term, Wut who was studying in Mathayomsuksa one used the difference between previous terms. He answered that the number of dots in 5th term was 13 because he noticed that the difference of the number of dots between the 2nd and 3rd term was 3. So, the number of dots in the 5th term should be different from the number of dots in the 4th term by 3, that is there would be 13 dots in the 5th term. He used this rule to find some other terms also (see Figure 15). But to find the number of dots in the 21st term, he counted by increasing with 3 from the 6th term till the 21st term.

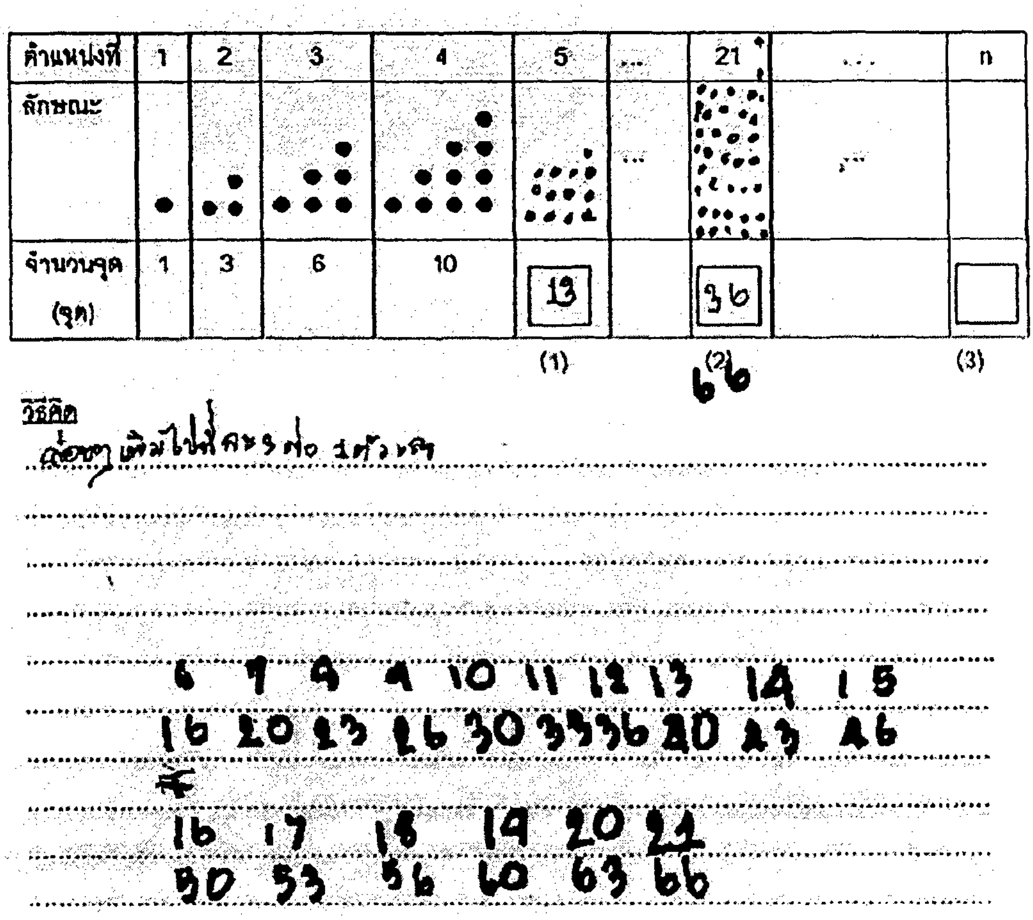


FIGURE 15 WUT'S RESPONSE ON DOT PATTERN

R atree, a Mathayomsuksa three student, recognized that the difference of term between the 1st , 2nd , 3rd and 4th were 2, 3 and 4, respectively. So, the difference between the 4th and 5th term should be 5, thus the 5th term should be 10 + 5 which is 15. This indicated that she could find the value of the next term. However, when asked for the 21st term, she failed to used what she noticed and she gave the answer as 105 which she calculated from 5 x 21 (see Figure 16).

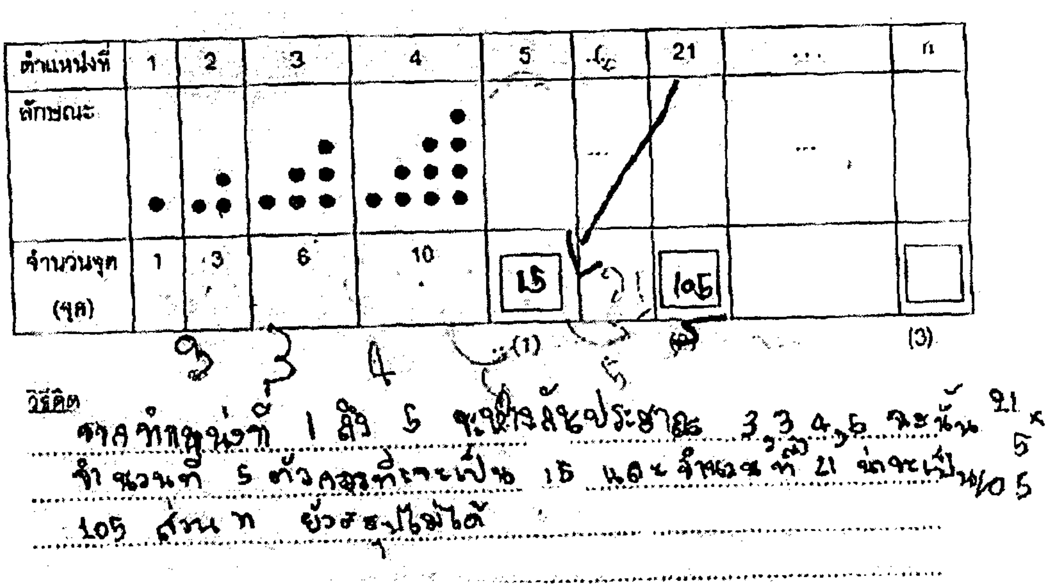


FIGURE 16 RATREE'S RESPONSE ON DOT PATTERN

Thep who was studying in Mathayomsuksa one recognized Problem 3 (Dot pattern) the same as Ratree. But he found the value of the 21st term by multiplying 15 by 4 and then adding 1 which was 61 because he noticed that the 5th term multiplied by 4 should be the 20th term and then added the 1st term to be the 21st term. However, both students could find the value of the next term, for the higher term, their answer was incorrect because they did not use what they recognized (see Figure 17).

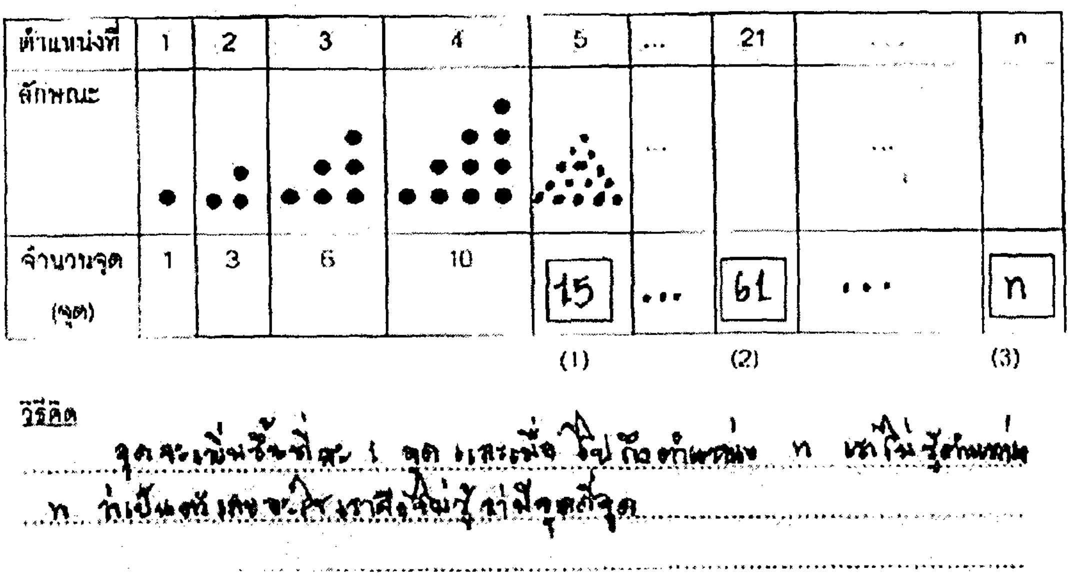


FIGURE 17 THEP'S RESPONSE ON DOT PATTERN

Students at this level still needed to write every term to find a specific term, although they could recognize the relationship of the given pattern from the value of one term in a sequence to the value of the next term by ignoring the term number of the given pattern. They did not use relationship between the value of the term and the term number to find the value of the higher term. That is, they had to compute every previous term to find the value of the required term.

For Problem 3 (Dot pattern), Nida, a Mathayomsuksa one student, recognized that the value of the terms would increase from the previous term by the number of that term such as the value of the 4^{th} term was 10 which was increased from the 3^{rd} term by 4. When she looked for the value of the 5^{th} term, she added 5 to the value of the 4^{th} term which was 10; hence the value of the 5^{th} term should be 15. Thus, if she wanted to find the value of the 21^{st} term, she had to know the value of the 20^{th} term and then she added 21 to the value of the 20^{th} term. However, Nida did some mistake on her calculation on finding the value of term as shown in Figure 18.

ตำแหน่งที่	1	2	3	4	5	...	21	...	n
ลักษณะ						...		...	
จำนวนจุด (จุด)	1	3	6	10	$\boxed{15}$		$\boxed{35}$		$\boxed{}$
					(1)		(2)		(3)

วิธีคิด

เพราะจำนวนเพิ่มที่ละ 1, 2, 3, 4, 5 ไปเรื่อยๆ

5 6 7 8 9 10 11 12 13 14

15 21 28 36 45 55 56 78 91 104

15 16 17 18 19 20 21

119 135 147 160 174 194 215

FIGURE 18 NIDA'S RESPONSE ON DOT PATTERN

For Problem 4 (Number sequence), students at this level could recognize and find the next term of the given pattern as in Problem 3 (Dot pattern). They also needed to know every previous term of the given pattern such as Sasi who was studying in Mathayomsuksa three. She noticed that the difference between T values was 3, thus she counted by 3 from S=4 until S=70 to find the value of T when S was 70. However, Sasi did some mistake on her calculation on finding the value of T (see Figure 19).

FIGURE 19 SASI'S RESPONSE ON NUMBER SEQUENCE

Some students at this level avoided finding every previous term of the pattern, although they could recognize the relationship between the values of the given terms and be able to find the value of the next term. But in finding the value of the higher term, their responses were irrelevant as was the case for Nat (Mathayomsuksa one). When she was confronted with Problem 4 (Number sequence), she noticed that the value of each term of T was increased by 3. However, when the value of S equaled 70, she stated that T was 210 because she multiplied 70 by 3.

Some students noted that the value of each term of T was increased by 3, but failed to find the next term such as Thavee who was studying in Mathayomsuksa three. Although in an overall sense Thavee was a Level 1 thinker on pattern, he was assigned Level 2 thinking for Problem 4 (Number sequence). He found that the value of T was 15 when S was 5 (the next term of the given pattern) which was not correct. He explained that he counted to find the value of T, but he was not consistent in counting as shown in Thavee's interview dialogue.

Interviewer:	What is the value of T when S equals 5?
Thavee:	15
Interviewer:	How did you get 15?
Thavee:	I counted continually.
Interviewer:	Can you tell me, how did you count?
Thavee:	5 6 7 (stop) 8 9 10 (stop) 11 12 13 (stop) 14 15.

In summary, Level 2 students can find the value of the next term of the given pattern but when finding the value of a higher term their response is irrelevant. Students at this level recognize only the relationship between the values of the given terms but they do not use this relationship to find the value of a higher term. Sometimes they use the relationship between the values of the given terms to find the value of a specific term by drawing, counting or computing every previous term of the given pattern. Students at this level also try to generalize from only one given term or only one aspect of the given pattern but the generalization is not correct.

Level 3

At this level, students were able to find a specific term of the given pattern even a higher term by recognizing only the relationship between the values of the given terms of the pattern. They were able to use a systematic method to find the specific term without producing a general formula to find every term of the given pattern. The students at this level did not need to find every other previous term of the given pattern, such as when asked to find the 63^{rd} term of the pattern. Typical responses are shown below.

For Problem 1 (Arrow pattern), Ratda who was studying in Mathayomsuksa one added 2 more arrow signs to the given pattern to make 3 complete units of change so that she could count by 12. She multiplied 5 by 12 in order to find the 60^{th} arrow and then counted the first arrow as the 61^{st} and to the third arrow which became the 63^{rd} arrow. The interview dialogue is shown as follows:

Interviewer: How do you find the 63^{rd} arrow?

Ratda: I counted from the first to the last arrow and the total arrows are 12 (the student has drawn arrows in positions 11 and 12). Then I multiplied 12 by 5. I got 60 and then I counted three more. So, here is my answer (for the 63^{rd} term).

Sara, a Mathayomsuksa three student, was assigned Level 3 thinking for Problem 1 (Arrow pattern), although in an overall sense she was a Level 2 thinker on pattern. She noticed that each set of arrows was composed of 3 arrows. In order to find the 63^{rd} arrow, she used 3 sets of arrows (9 arrows) multiplied by any number to be equal to 63. That is 9 multiplied by 7 equals 63 but she multiplied 9 by 6 and then counted until the 63^{rd} term so, the 63^{rd} arrow was in the third term. The interview dialogue is shown as follows:

Interviewer: How did you find the 63^{rd} arrow?

Sara: There are 3 arrows in each set and total arrows are 9. I seek for the number that multiplied by 9 equals 63. So, I multiplied 9 by 5, 6, 7 until the result equals 63. I counted from 54 (the student pointed to the last arrow of the third set). Here is 54, 55, 56, 57, 58, 59, 60, 61, 62, 63 .. ....63. This is the third position.

The interviewer asked Level 3 students to find the n^{th} term of the given pattern. They said that they did not know the n^{th} term. They could find the required term if they knew the specified position of that term and used the same method as finding the 63^{rd} term.

In Problem 2 (Table pattern), Sasi who was studying in Mathayomsuksa three was assigned Level 3 thinking, although in an overall sense she was a Level 2 thinker on pattern. She observed that from the given pattern, there were 2 chairs for each table and 1 chair on each end. So, she calculated the total number of chairs for 54 tables by adding 54, 54, 1 and 1. Since 54 + 54 + 1 + 1 = 110, she came up with 110 chairs (see Figure 20).

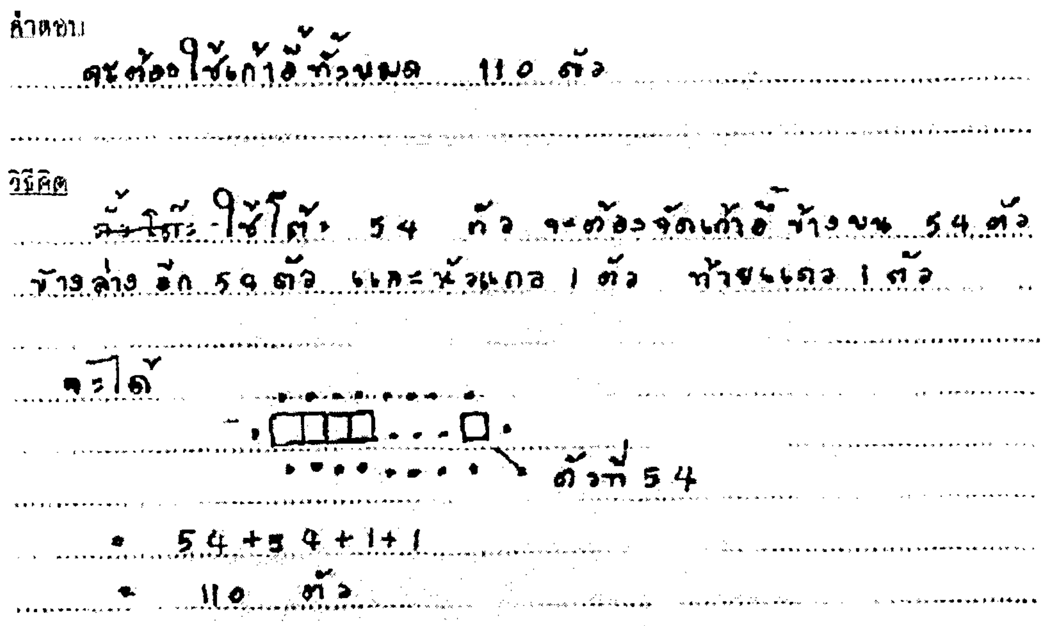


FIGURE 20 SASI'S RESPONSE ON TABLE PATTERN

For Problem 3 (Dot pattern), Veera who was studying in Mathayomsuksa two noticed that the number of dots on the right side related to the term number. For example, in the 4^{th} term, he noticed that there were 4 dots on the right side and when looked towards the left side, there were 3, 2, 1 dots, respectively. So, the number of dots in the 5^{th} term was composed of 5, 4, 3, 2, and 1 dots. Thus he found the number of dots in the 5^{th} term by 5 + 4 + 3 + 2 + 1 which was 15 (see Figure 21).

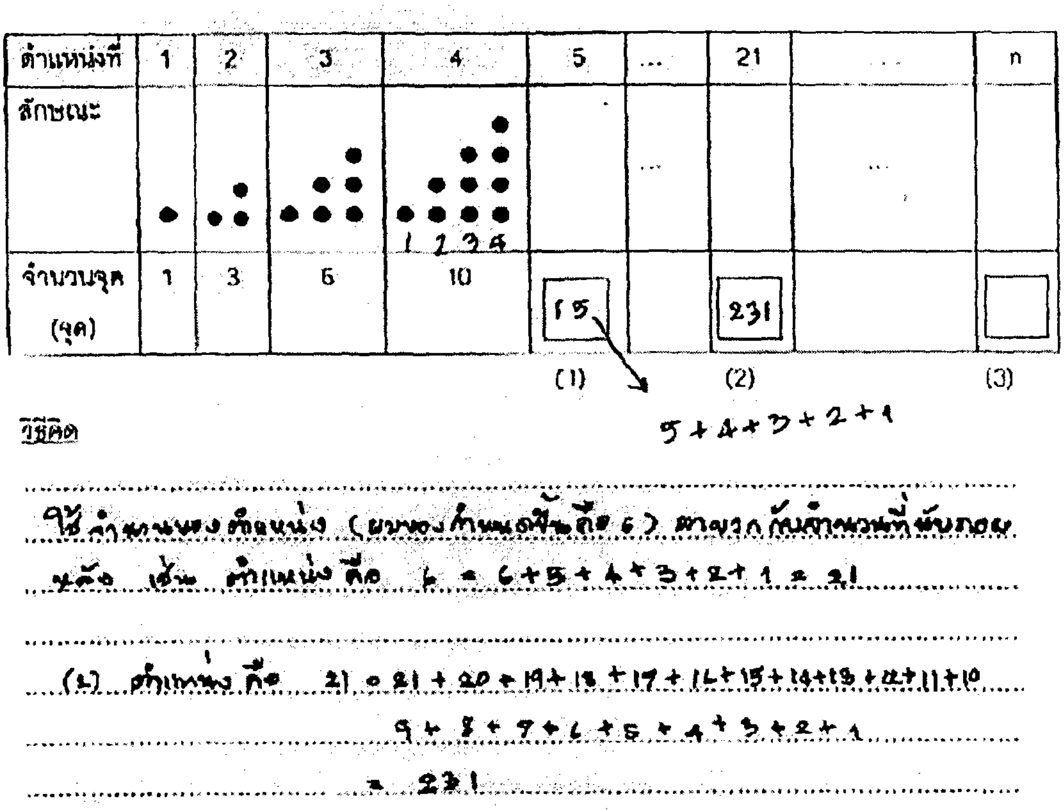


FIGURE 21 VEERA 'S RESPONSE ON DOT PATTERN

With Problem 4 (Number sequence), other students at this level applied two or more conditions of the given pattern to solve the question, but did not recognize the relationship between the value of the term and the term number in the given pattern. Panee who was studying in Mathayomsuksa three exemplified such a strategy. In the table for the sequence, the values of S were 1, 2, 3, and 4 and the values of T were 4, 7, 10, and 13, respectively. Panee focused only on the numbers for T and recognized that the difference between the T numbers was 3. Thus, she gave the answer for T when S equaled 5 as 16 because she added 3 to 13. When asked for a higher term, such as when S equaled 70, she noticed that the first term of T started with 4. She had to add 1 to the result of 70 multiplied by 3 that is (70 x 3) + 1 = 211. Although her process was suitable for this problem, for another pattern she would need to recognize the term number in the pattern to produce a more general formula. Figure 22 shows her response on paper.

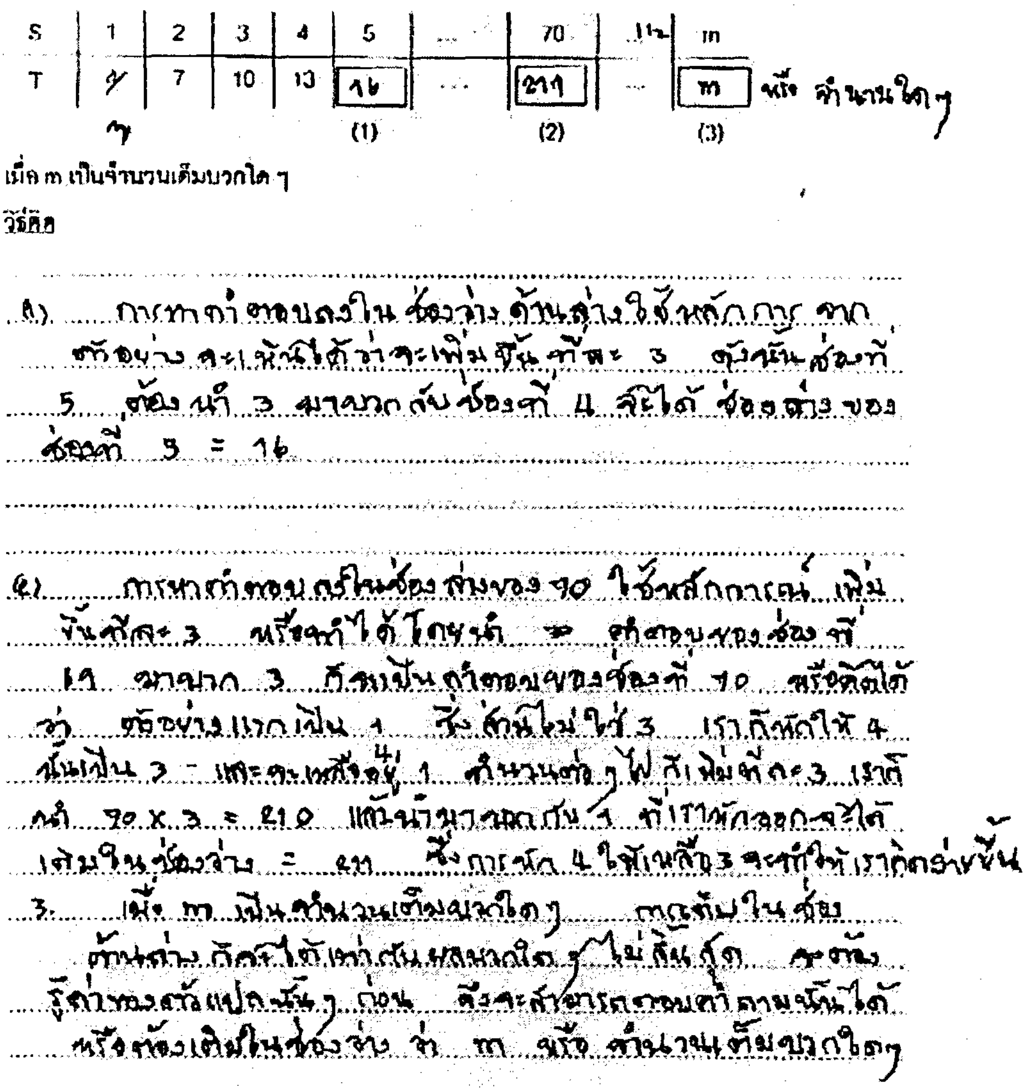


FIGURE 22 PANEE'S RESPONSE ON NUMBER SEQUENCE

In summary, Level 3 students find a specified term of the given pattern even a higher term in the sequence by recognizing only the relationship between the values of the given terms and using this relationship to find the answer. They also use the relationship between the values of the given terms to generate a systematic method to find the value of any specific term of the given pattern, even higher terms, without finding every previous term. Although, their strategy helps them to skip some previous terms which they do not need, students using this method do not relate the value of a term and its term number. They cannot find the nth term of the given pattern either.

Level 4

At this level, the students understood the relationship between the value of a term and the term number and could generalize the relationship of the pattern. Their responses could be classified into two groups. The first group was students who could explain the correct relationship between the value of a term and the term number. The second group was students who could generalize the relationship of the pattern symbolically or as a formula. Examples of students' responses in each group are shown below.

The first group of Level 4 students is liked the students at Level 3 that is they did not know the meaning of the n^{th} term. But they could generalize their method verbally to find any term of the given pattern by focusing on the relationship between the value of the term and the term number in the given pattern. Notice how the students' thinking presented below incorporates their method verbally.

For Problem 1 (Arrow pattern), Roj who was studying in Mathayomsuksa two found the 63^{rd} term of the given pattern by dividing the 63^{rd} term by 3. He noted that 63 was divisible by 3 and gave the answer as the white down arrow. When asked to find the n^{th} term, he said he did not know the n^{th} term, but he could explain how to find the value of every term in the given pattern by dividing the number of the required term by 3. If the required term was divided by 3, the arrow was the same as the arrow in the third term. If it was divisible by 3 with remainder 1 or 2, the arrow was the same as the first or second term in the pattern, respectively.

Moreover, with Problem 3 (Dot pattern) Roj analyzed the relationship between the number of dots and the term number in the given pattern. He concluded that the number of dots in the n^{th} term could be calculated from n divided by 2 and multiplied the number by the next term (see Figure 23). He responded a transition to the group of Level 4 students who used symbols as their dominant way of describing pattern.

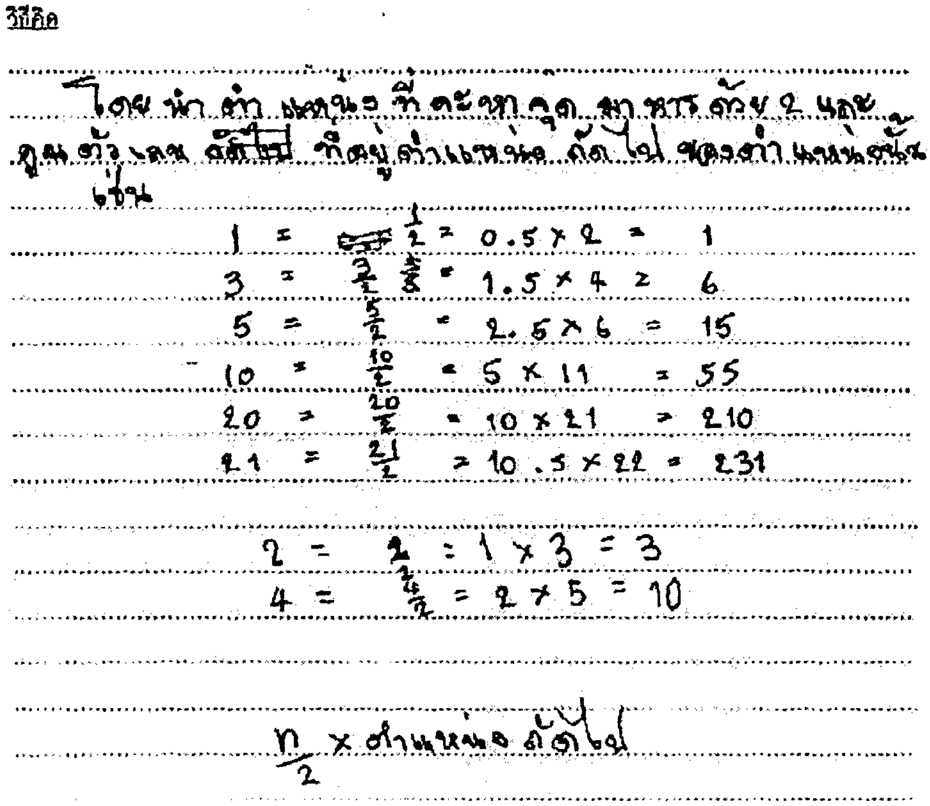


FIGURE 23 ROJ' S RESPONSE ON DOT PATTERN

The second group of Level 4 students analyzed the given pattern and generalized a formula to solve the problem. For example in the picture of the arrow pattern, these students said that they would divide the n^{th} term by 3. If it was divisible by 3, the arrow was the same as the arrow in the third term. If it was divided by 3 with remainder 1 or 2, the arrow was the same as the arrow in the first or second term respectively. Pathra (Mathayomsuksa three) wrote down the general rule to find the n^{th} term of the given pattern as shown in Figure 24.

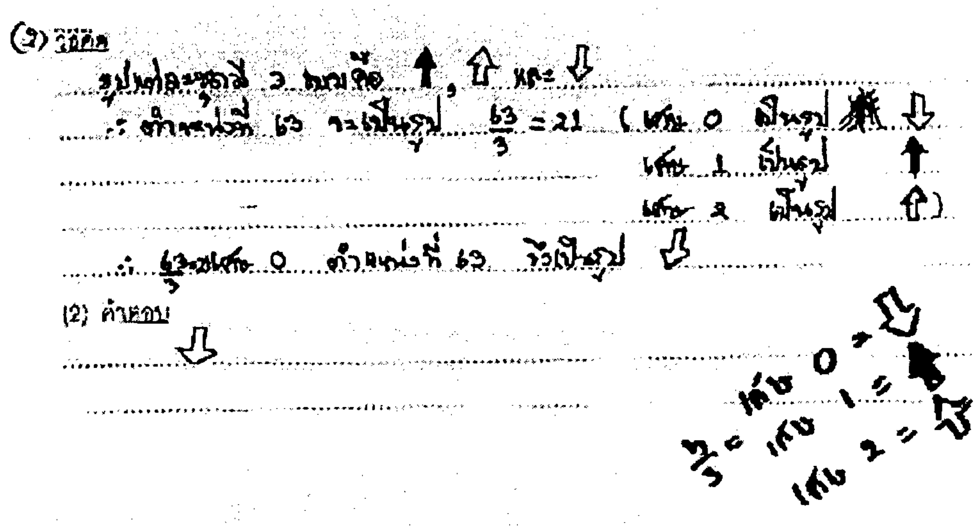


FIGURE 24 PATHRA' S RESPONE ON ARROW PATTERN

For Problem 2 (Table pattern), Chira who was studying in Mathayomsuksa three was assigned Level 4 thinking, although in an overall sense she was a Level 3 thinker on pattern. She noticed that there were two chairs for each side of the table and 2 chairs at both ends of the table line. When asked to find the number of chairs with n tables, she was able to write drown the formula as $2n + 2$.

Riya (Mathayomsuksa two) was also able to find the formula of the pattern, but her formula was different from Chira. Her formula was $[(n-2)\times 2]+6$. She separated 2 tables at each end of n tables which was 6 chairs and the rest of the tables were $n-2$ and then she multiplied the rest of the tables by 2 that is $(n-2)\times 2$ chairs.

For Problem 3 (Dot pattern), Chuti who was studying in Mathayomsuksa two recognized the relationship between the value of the term and the term number. She figured the formula to find the number of dots at the n^{th} term, that is, $\frac{(1+n)\times n}{2}$ (see Figure 25).

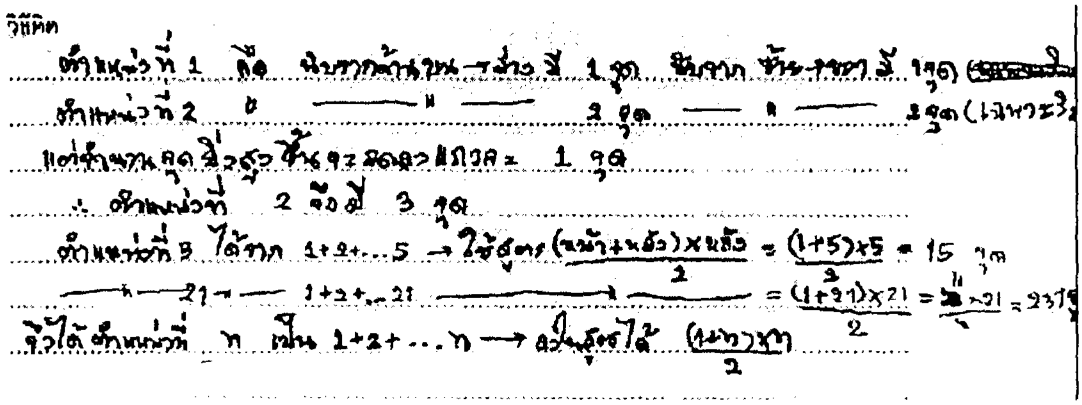


FIGURE 25 CHUTI' S RESPONSE ON DOT PATTERN

For Problem 4 (Number sequence), Nisa who was studying in Mathayomsuksa one generalized a relationship between S and T and constructed the formula to find the value of T. her formula for T when S = m is $(m\times 2)+(m+1)$ as shown in Figure 26.

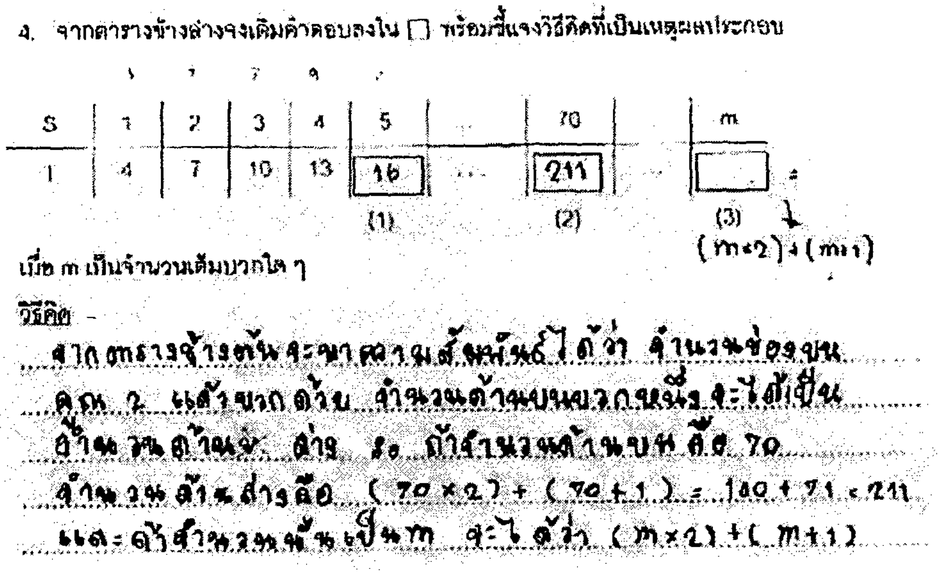


FIGURE 26 NISA' S RESPONSE ON NUMBER SEQUENCE

Other students at this level could analyze the number sequence similar to Nisa's strategy, but Riya constructed a different formula. Riya (Mathayomsuksa two) indicated that when S equaled 5, the value of T was 16 because she multiplied 5 by 3 and added 1. The reason was that the difference between 1 and 4 increased by 3. Then she multiplied 3 to the term number and added 1. She used this method to calculate the value of T for S = 70 and gave the answer correctly as 211. If S equaled m, she informed the researcher that the value of T might be $3m+1$ (see Figure 27).

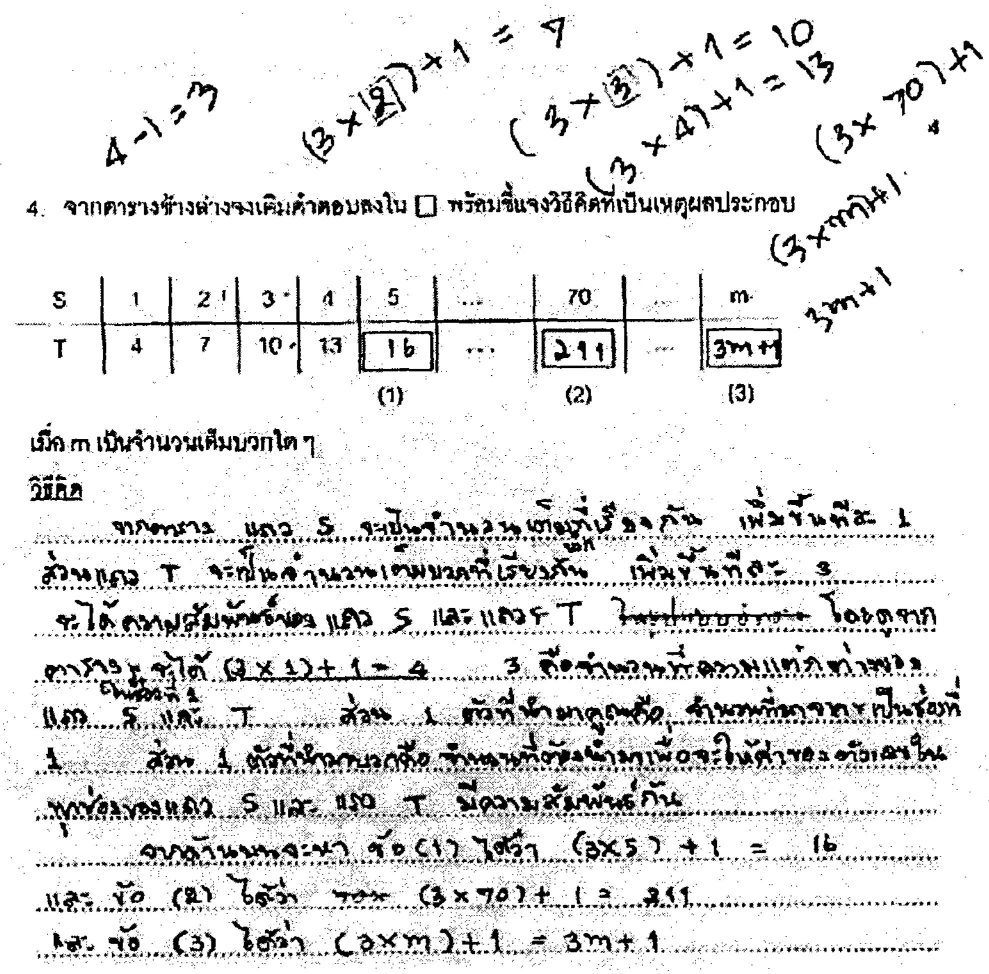


FIGURE 27 RIYA' S RESPONSE ON NUMBER SEQUENCE

In summary, Level 4 students can recognize the relationship between the value of a term and its term number in the given pattern. They either explain this relationship verbally or generalize a formula for the n^{th} term.

Students' Thinking on Representation

In order to learn about students' thinking on representation, the researcher looked at the students' abilities to transform data from one display to another and to interpret the data displayed. The students were asked to represent the data in the form of a graph and a table. Moreover, interpretation of the graph was another area used to assess students' thinking on representation. Hence representation as an indicator of algebraic thinking was assessed through the following categories: a) representing data in the form of a graph, b) constructing a table to represent data, and c) interpreting and comparing between the data on a graph. Figure 28 shows students' levels of thinking on representation.

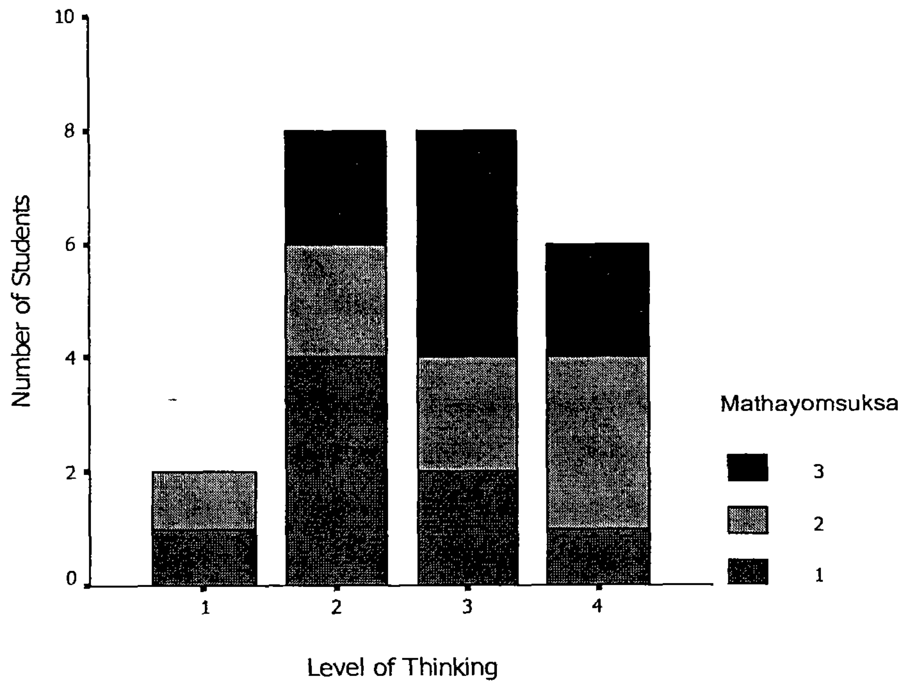


FIGURE 28 STUDENTS' LEVEL OF THINKING ON REPRESENTATION

Most of the students were at Level 2 and 3 with 8 students on each level. With respect to Level 2, there were 2 students in Mathayomsuksa three, 2 students were in Mathayomsuksa two and 4 students were in Mathayomsuksa one. In Level 3, there were 4 students in Mathayomsuksa three, 2 students were in Mathayomsuksa two and 2 students were in Mathayomsuksa one. Six students demonstrated Level 4 thinking, where 2 students were in Mathayomsuksa three, 3 students were in Mathayomsuksa two and only 1 student was in Mathayomsuksa one. Two students were considered to be at Level 1 of thinking, 1 student was in Mathayomsuksa two and 1 student was in Mathayomsuksa one. The result indicated that Mathayomsuksa three students did not demonstrate Level 1 of thinking, while Mathayomsuksa one and two students spread all 4 levels of thinking. It is remarkable that Mathayomsuksa three students could use their thinking to solve algebra problems on representation better than Mathayomsuksa one and two students. There was one student in Mathayomsuksa one illustrated her thinking at Level 4.

Representing the data in a Graph

In assessing students' ability to represent the data in the form of a graph, students were given information that required them to draw a graph to present the relationship between time and distance as shown in Figure 29.

5. *Mr. Big's journey*: Mr. Big has appointment with his friends at school. He lefl home for school by walking. After 10 minutes he realizes that he is going to be late for the appointment which will be in the next 10 minutes. Draw graph showing time and distance of Mr. Big's journey and also explain your thinking along with reasons.

FIGURE 29 PROBLEM 5 (Mr. Big's journey)

The following samples of students' responses illustrate the four levels of thinking on representation with respect to constructing a graph.

Level 1

Level 1 students did not understand the conditions of the given problem and shown little ability in representing the data on a graph. Students' responses at this level are as follows.

Wut who was studying in Mathayomsuksa one tried to capture the problem but he could not understand it. When he was interviewed, he just read the entire problem and did not explain it in his own words. Sometimes, he guessed the answer. The interview dialogue is shown below:

Interviewer:	What does the problem require?
Wut:	(The student kept reading through the problem but did not respond to the question).

While some students partially understood the question and the condition in the problem, they could not represent what the question asked. Wadee exemplified this strategy. She was studying in Mathayomsuksa one. Although in an overall sense Wadee was a Level 2 thinker on representation, she was assigned Level 1 thinking for this problem. She partially understood what the problem asked and also the conditions of the problem, but she was not exactly sure what she was asked to do. She attempted to represent information in the problem by drawing a bar chart, which was not suitable for representing the relationship between time and distance of Mr. Big's journey (see Figure 30). The following was her interview dialogue indicated that she did not completely understand the problem:

Interviewer:	In this problem, what were you asked to do?
Wadee:	......Draw a graph to show time and distance of Mr. Big's journey or....ask me to explain the problem..... I don't understand.
Interviewer:	Which part of the problem did you not understand?
Wadee:	When stated ....Mr. Big made the appointment to work on group assignment with his friend at school which wasn't far from his home. So, he started to walk when 10 minutes

Wadee: passed by, he started to run because he was afraid that he was going to be late. ......I don't know what "in the next 10 minutes" means...... I don't understand.

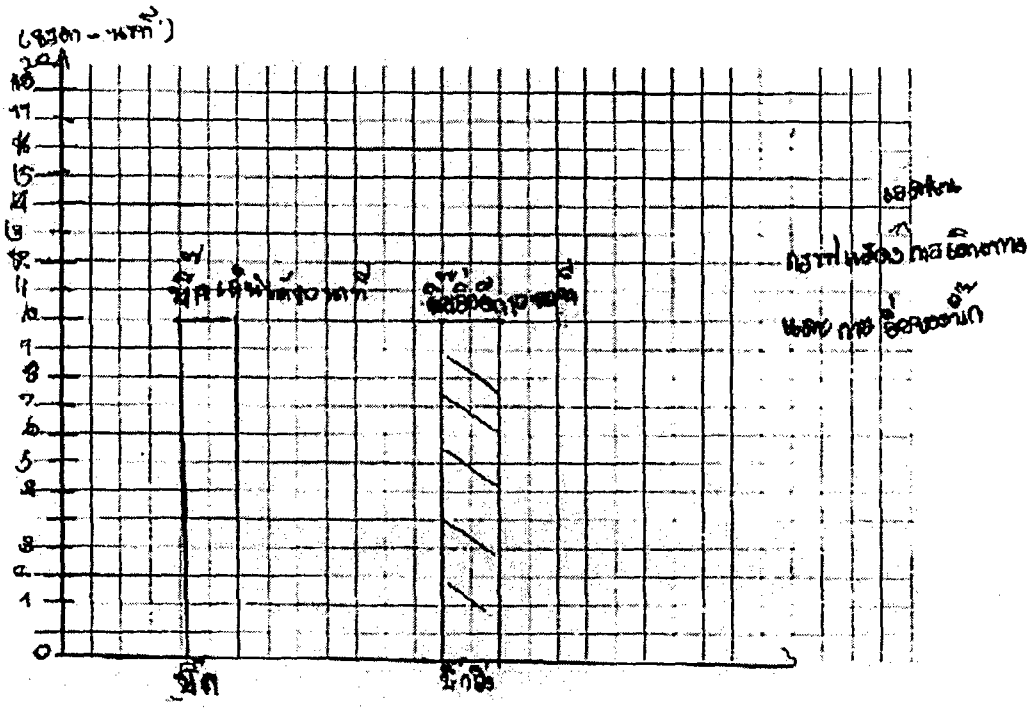


FIGURE 30 WADEE' S RESPONSE ON MR. BIG'S JOURNEY PROBLEM

Similary, Level 1 students could engage in answering the problem but did not completely understand the condition of problem as Veera did. He was studying in Mathayomsuksa two. Although in an overall sense Veera was a Level 3 thinker on representation, he was assigned Level 1 thinking for this problem. He understood that the problem required him to represent a graph to show time and distance of Mr. Big's journey but he did not understand the given data in the problem. So, he could not draw the graph. The interview dialogue is shown below:

Interviewer: In this problem, what were you asked to do?

Veera: Draw a graph to show the distance and time of Mr. Big's

	journey.
Interviewer:	Can you do it?
Veera:	I cannot.
Interviewer:	Why?
Veera:	I do not understand.
Interviewer:	You do not understand. Which part of the problem you do not understand.
Veera:	It looks like the problem is not complete.
Interviewer:	It means that you need more information. What do you need more?
Veera:	Distance and time from home to school.

In summary, it can be concluded that Level 1 students cannot understand what the problem asked or the condition of the problem. They cannot construct the graph. Students at this level do not answer the question. Sometimes, they answer the question based on irrelevant data and guess the answer.

Level 2

Students at this level seemed like they understood the question and the condition of the given data in the problem, but they did not know how to construct the graph. Sometimes, they constructed an incorrect display. However, they tried to represent their thinking in other forms of representation. The following are samples of the students' representations.

Nat (Mathayomsuksa one) understood that the problem asked her to construct a graph, but her graph did not show the relationship between time and distance. She drew a graph which looked like waves to represent for frequency of steps in walking and running along with the words "walking" and "running" (see Figure 31). However, she tried to put the given information in her graph. She explained her thinking as follows:

Interviewer:	Can you explain your representation?
Nat:	Mr. Big had an appointment with his friend here (pointing at the beginning of wave). This was the starting point which

Nat: Mr. Big left home for school by walking, he walked normally. Then he realized that he was going to be late, so he started out running, running ...to...here.... and the graph occured frequently.

Interviewer: Here is walking (pointed at the wide steps) and this frequent graph is ....

Nat: Frequently of steps.

Interviewer: If this is more frequent, it shows........

Nat: Running faster.

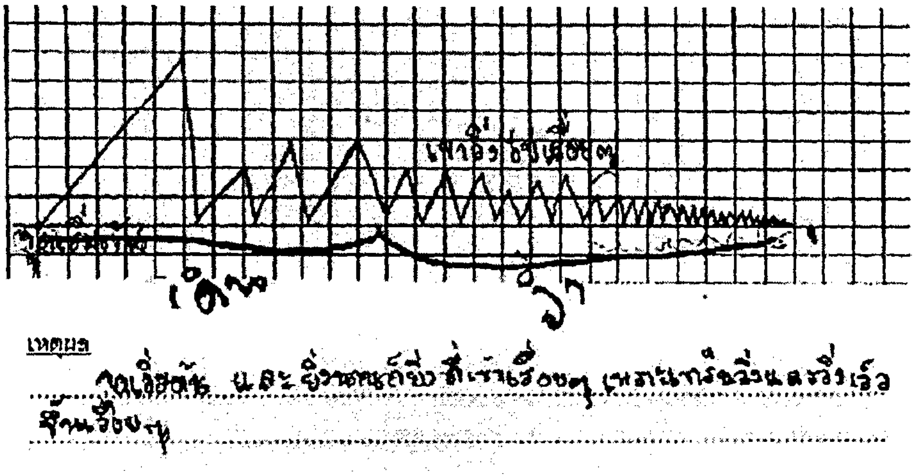


FIGURE 31 NAT'S DISPLAY

Ratree (Mathayomsuksa three) was assigned Level 2 thinking for this problem, although in an overall sense she was a Level 3 thinker on representation. She understood that the question asked her to construct a graph showing time and distance of Mr. Big's journey. However she drew a map or diagram showing the way from home to school and indicated which part Mr. Big was running (see Figure 32). However, Ratree misrepresented because the distance of her running is shorter than the distance of her walking which is not right when the amount of time is the same.

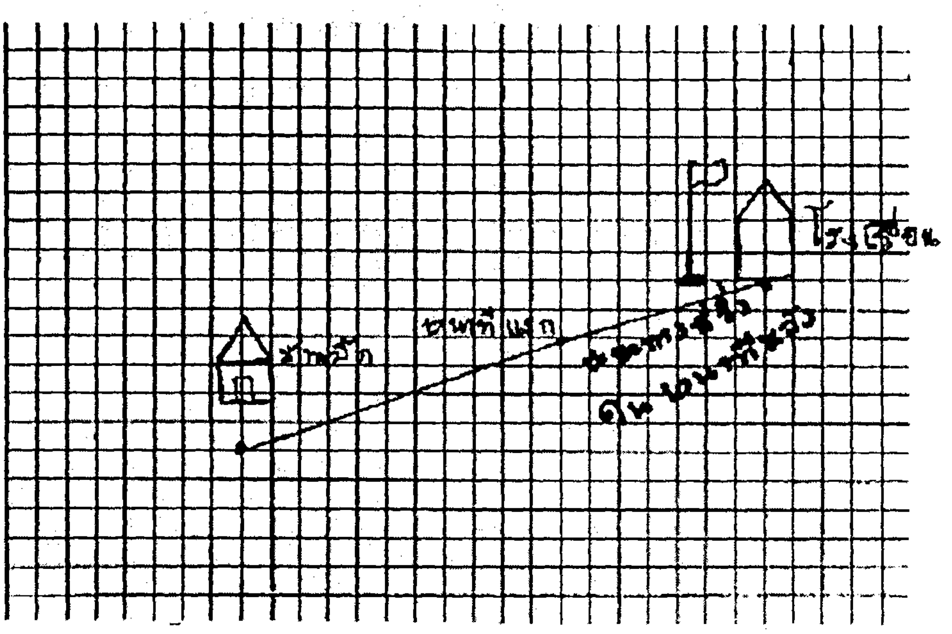


FIGURE 32 RATREE'S DISPLAY

Other students exhibiting Level 2 thinking knew what the question asked and the way to construct a graph, but they interpreted the condition of the problem incorrectly. Nida who was studying in Mathayomsuksa one was such a case. She constructed the graph to represent Mr. Big's journey. The vertical axis represented time and the horizontal axis represented distance. From her graph, she informed the researcher that the beginning of the graph which showed 10 minutes of time, was Mr. Big's house and the end point of the graph was the house of Mr. Big's friend where the meeting occurred as shown in Figure 33.

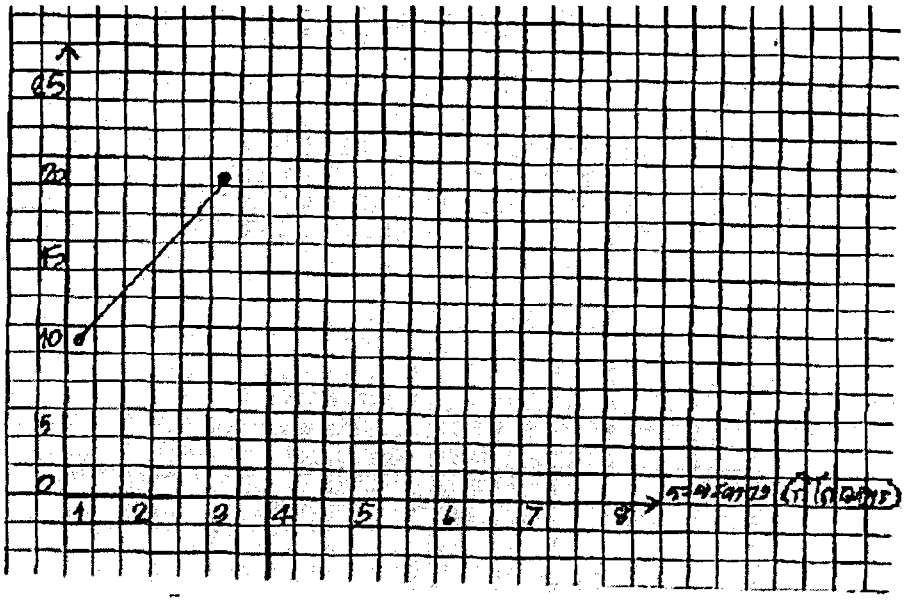


FIGURE 33 NIDA'S GRAPH

While, Thana's graph illustrated a related difficulty (see Figure 34). He was studying in Mathayomsuksa one.

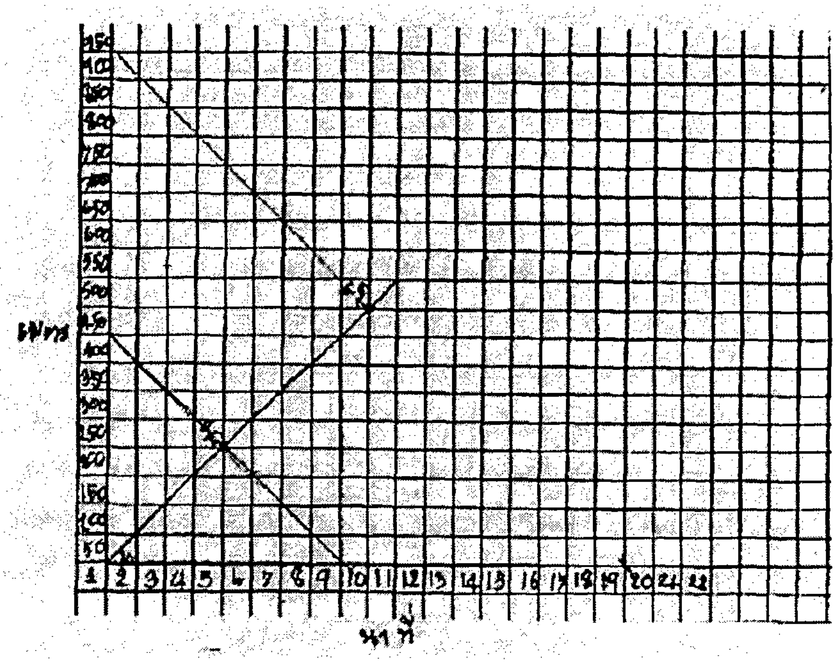


FIGURE 34 THANA'S GRAPH

Although Thana constructed a graph that was almost complete, he had interpreted his graph incorrectly as his interview dialogue reflects. He also made up distance from home to school to be 950 meters.

Interviewer :	Can you explain your graph for me?
Thana :	This is the house. (the beginning of the graph) and this is the minutes (the horizontal axis). This is distance (the vertical axis). It is a meter.
Interviewer :	And then, what?
Thana :	This is Mr. Big's house (the starting point of graph). He travels from the house by walking out 10 minutes which is equal to 450 meters. Only half of the way left. (Thana drew a segment from the vertical axis at 450 meter mark to the 10 minutes mark on the horizontal axis).
Interviewer :	Only half way left. What is he doing?
Thana :	So, he ran from here to school in 10 minutes. The total is equal to 20 minutes.
Interviewer :	This is 20 minutes (the ending point). How about the distance?
Thana :	It is 950 meters. (Thana constructed a line from the 950 meter mark on the vertical axis that was perpendicular to his graph line).
Interviewer :	The distance from here to here. (the distance of walking) And from here to here(the distance of running), is it equal?
Thana :	It's equal.

Thavee who was studying in Mathayomsuksa three, another student exhibiting Level 2 thinking focused on only one aspect of the data and constructed his graph with both axes representing time. The vertical axis represented time for walking and the horizontal axis represented time for running as shown in Figure 35.

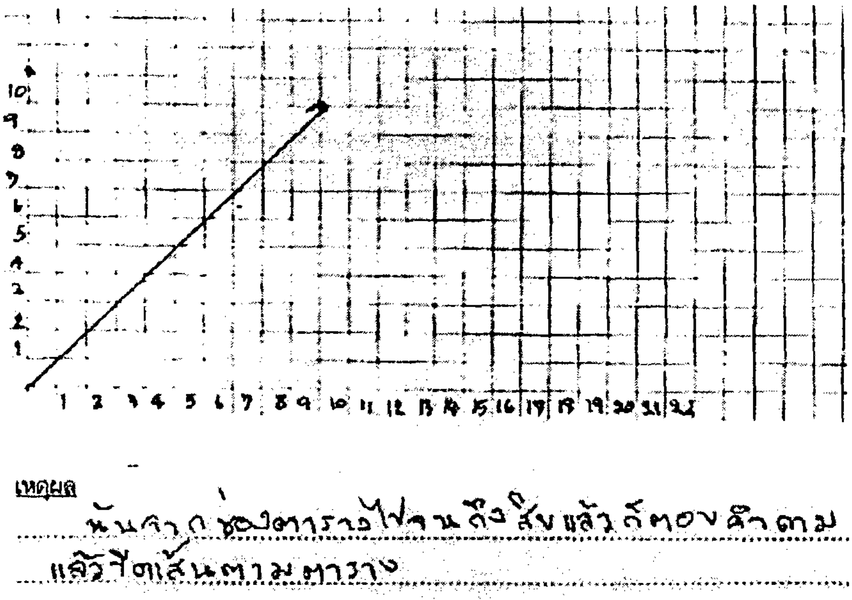


FIGURE 35 THAVEE'S GRAPH

In summary, the analysis reveals that students at Level 2 understand the question in the problem and part of the conditions on data, but they did not know how to construct a graph. Some try to construct their own displays that were generally invalid because the given data is interpreted incorrectly. Students at this level construct other displays which are not required, such as a map or wave and call them a graph.

Level 3

Students at this level understood the problem completely, but they represented some data correctly and omitted other data. For example, Sara who was studying in Mathayomsuksa three plotted some points on her graph correctly but it was incomplete as shown in Figure 36.

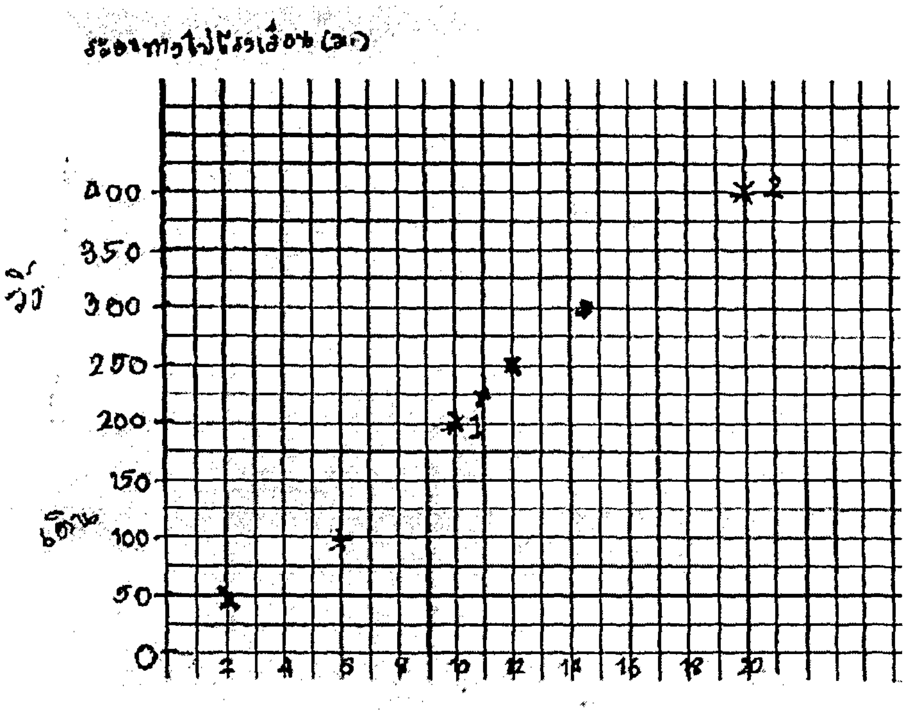


FIGURE 36 SARA' S GRAPH

Thep (Mathayomsuksa one) constructed his graph in the form of a straight line (see Figure 37). He did not recognize that the distance was different between walking and running even though the times were equal.

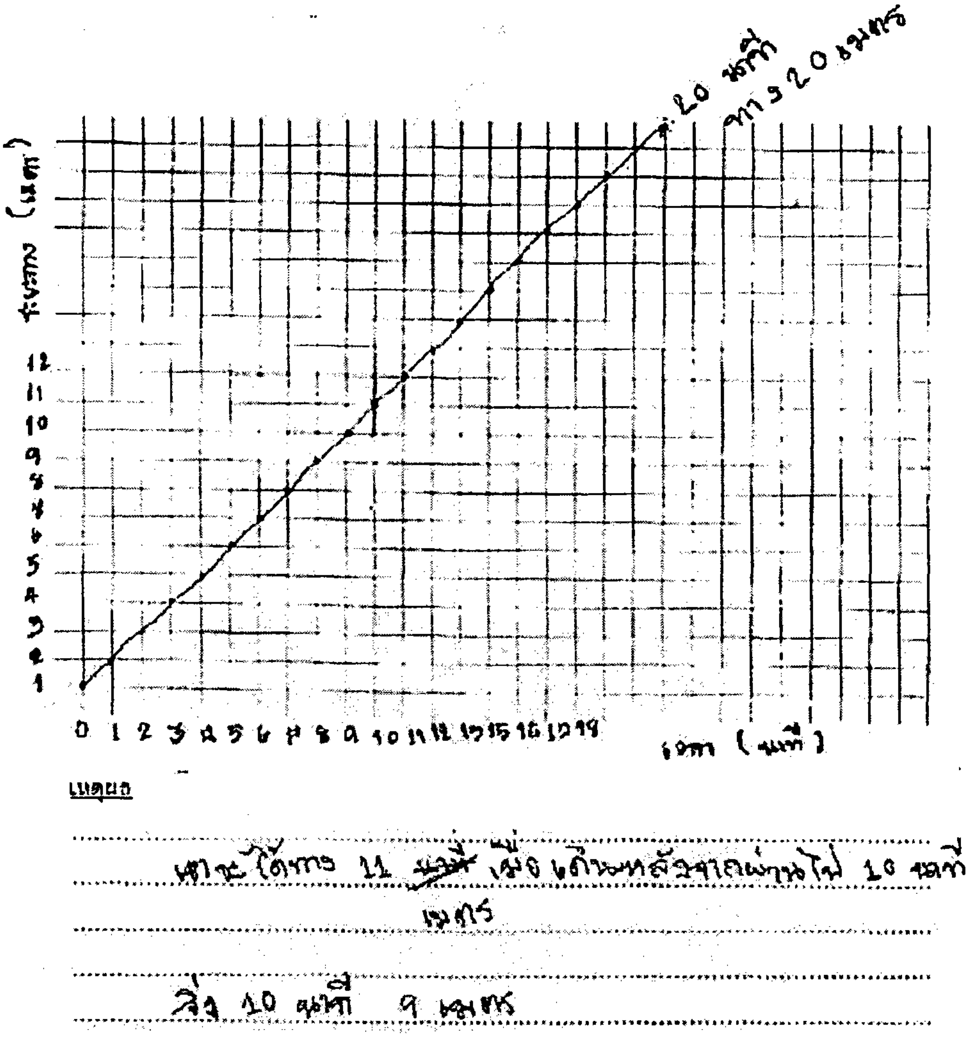


FIGURE 37 THEP'S GRAPH

In summary, Level 3 students can engage the problem and know how to construct the graph. They represent some data correctly and clearly in the form of a graph but leave out some data or conditions, but they represent only part of the data correctly. In addition, some students at this level do not recognize all conditions of the problem when they construct the graph.

Level 4

The graphs which Level 4 students constructed were complete and representative of the data. They realized that the distance for running was greater than the distance for walking when the time was the same. The displays of students at this level are shown below.

Nisa who was studying in Mathayomsuksa one constructed a graph as shown in Figure 38. From Nisa's graph, it can be seen that she did not determine the numbers on the horizontal axis which represented the distance. She realized that the distance for running was greater than the distance for walking when the time was the same. On her graph, she just estimated the possible distance for walking and running.

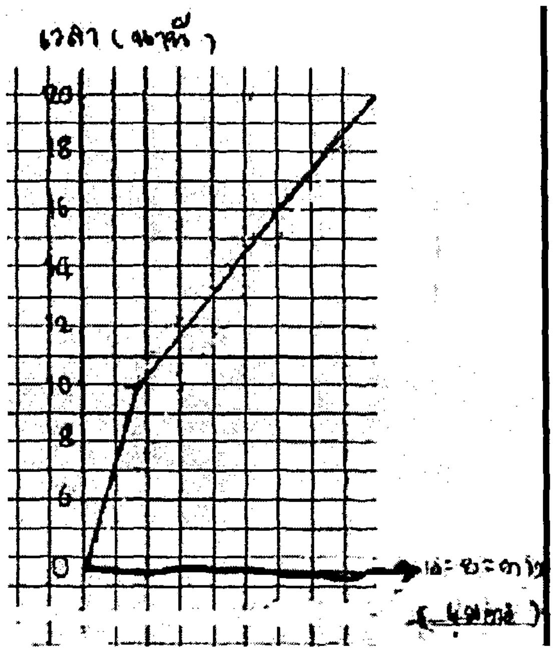


FIGURE 38 NISA' S GRAPH

Chuti (Mathayomsuksa two) constructed her graph to represent the relationship between time and distance of Mr. Big's journey. The vertical axis represented the distance and the horizontal axis represented time as Figure 39. The end point of her graph was stopped 15 minutes early because she explained that Mr. Big might arrive at his school 15 minutes early.

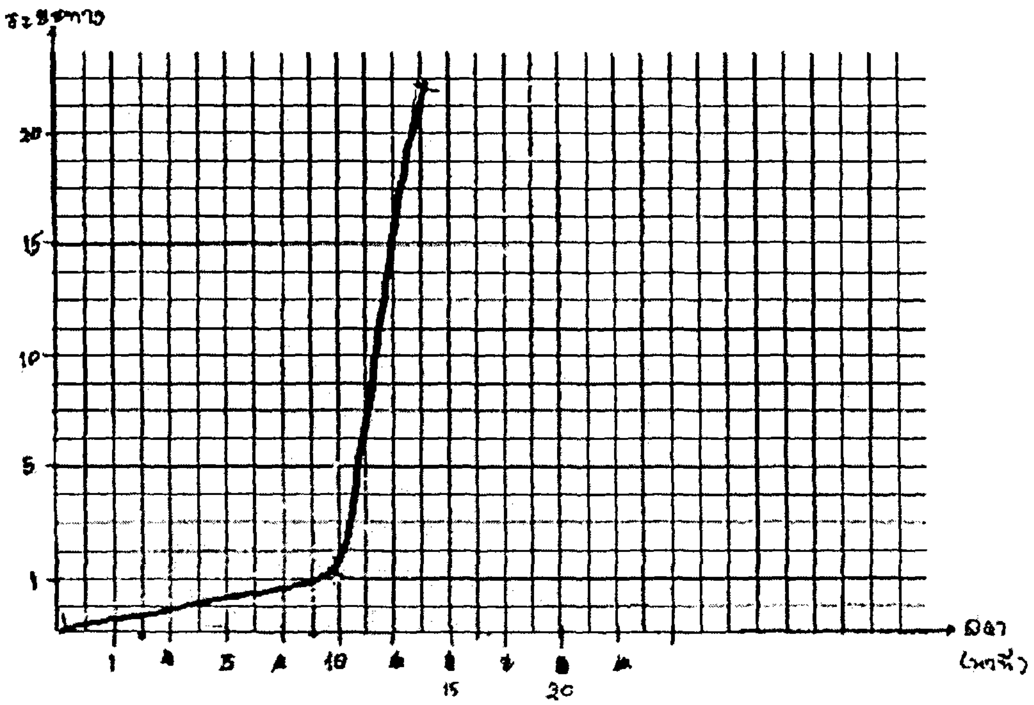


FIGURE 39 CHUTI'S GRAPH

In summary, Level 4 students can represent correctly and systematically all of the given data in the form of a graph. They also recognize all conditions and relationships in the data.

Representing the data in a Table

In assessing students to represent the given data in the form of a table, students were given the total number of the audience, the number of men and the number of women in each session of a cinema over 2 days with 5 sessions per day (see Figure 40).

6. *Cinema problem*: The data shows the numbers of audience in the theater over 2 days. This theater is composed of 120 seats and shows 5 sessions of the movie in each day.

The first day

11: 45	12 men and 18 women
14: 15	27 audience and 19 men
16: 45	35 men and 30 women
19: 15	118 audience and 48 women
21: 45	72 men and 43 women

The second day

11: 45	tickets were sold out and 80 men
14: 15	64 men and 54 women
16: 45	tickets were sold out and 79 women
19: 15	87 men and 33 women
21: 45	91 men and 29 women

Represent the data above in the form of table and explain what you learn from your table.

FIGURE 40 PROBLEM 6 (Cinema problem)

The following results of students' responses were characterized into four levels of thinking.

Level 1

Students at this level did not understand the problem. They also could not construct the table to represent the given data. Sometimes, they constructed other displays which were not related to what the question asked. Samples of student responses are illustrated below.

Suthi (Mathayomsuksa two) did not present the given data or write anything on his paper work. When he was asked what the question in the problem required, he read the entire problem to the interviewer. He did not explain in his dialogue what he understood.

Interviewer:	What does the question ask?
Suthi:	(He read the entire question)
Interviewer:	This means.........What will you have to do?
Suthi:	(come back to read problem again and answer) "present data."
Interviewer:	Do you have to present the data in what form?
Suthi:	Table.
Interviewer:	How?...... Try to show me.
Suthi:	I cannot.

Than a (Mathayomsuksa one) was assigned Level 1 thinking for this problem, although in an overall sense he was a Level 2 thinker on representation. He answered the question incorrectly. The question wanted the student to present the given data in the form of a table. However, he misunderstood and believed that the question wanted him to find the number of women in the first session and the number of men in the session of 16:45 (see Figure 41).

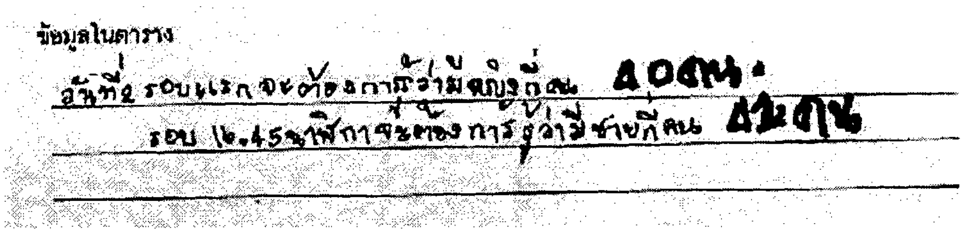


FIGURE 41 THANA' S RESPONSE ON CINEMA PROBLEM

Interviewer:	Explain your answer in this paper, please.
Thana:	The first session of the second day, the question wants to know how many women there were.

Interviewer:	And then, what?
Thana:	In session 16.45,.... want to know how many men.
Interviewer:	From the question you read, what do you have to do first?
Thana:	Find the value.
Interviewer:	What's value?
Thana:	Size of the audience.

In summary, the students at this level can not engage in the problem. They also cannot construct a table. When asked to explain what the question requires, they just read the entire problem. Sometimes, it seems that they know what the question asks, but they give an irrelevant answer which does not relate to what the question asked.

Level 2

Level 2 students knew what the question asked, but they did not know how to organize the data in a meaningful way to represent the given data in the form of a table. Some students at this level could engage the question in the problem and they captured some given data, but they left other data out. The tables were difficult to read as the following illustration shows:

Nat who was studying in Mathayomsuksa one und erstood the question and constructed the table for presenting the data on the people who went to the theater according to the number of men and the number of women. She did not present the total number of the audience in each session and the array was difficult to understand as shown in Figure 42.

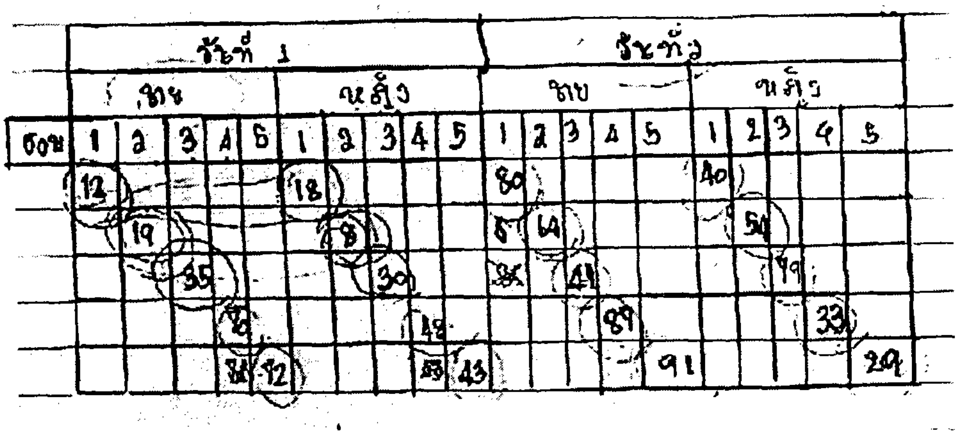


FIGURE 42 NAT'S RESPONSE ON CINEMA PROBLEM

Sasi (Mathayomsuksa three) presented the given data using a bar chart which was not correct according to the question asked. The question wanted the student to present the data in the form of a table; however she presented it in the form of 2 bar charts. The first chart showed the total number of the audience at each session on the first day and the second chart displayed the number of the audience at each session on the second day (see Figure 43).

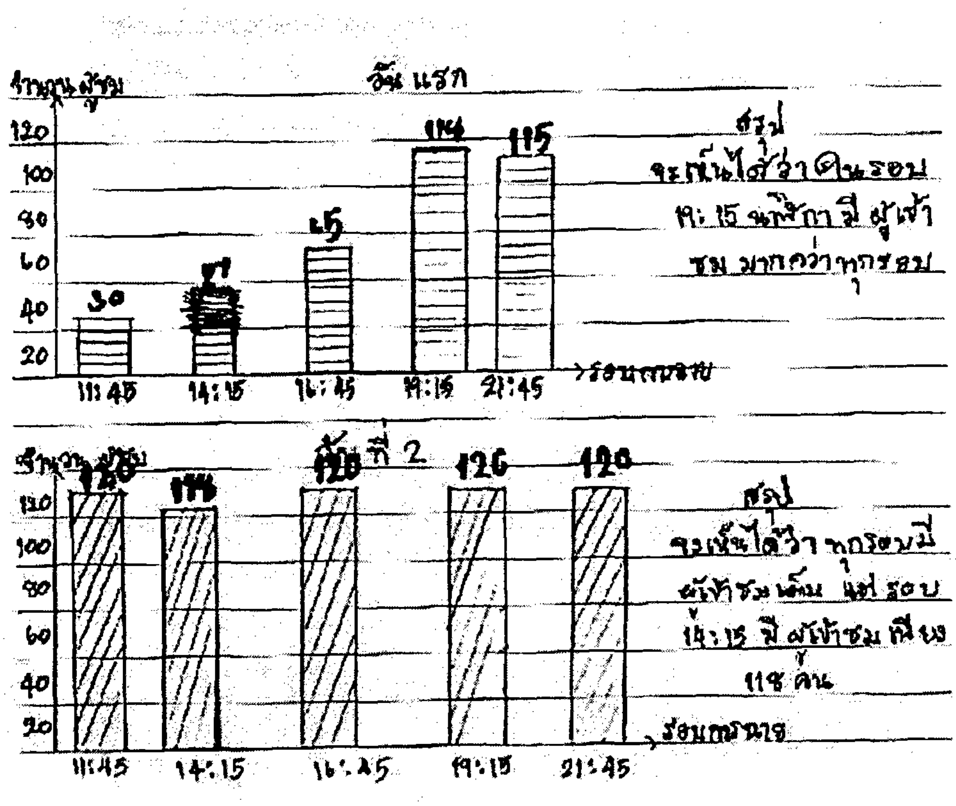


FIGURE 43 SASI'S RESPONSE ON CINEMA PROBLEM

From the interview and the dialogue below it can be seen that Sasi was not confident in answering the question.

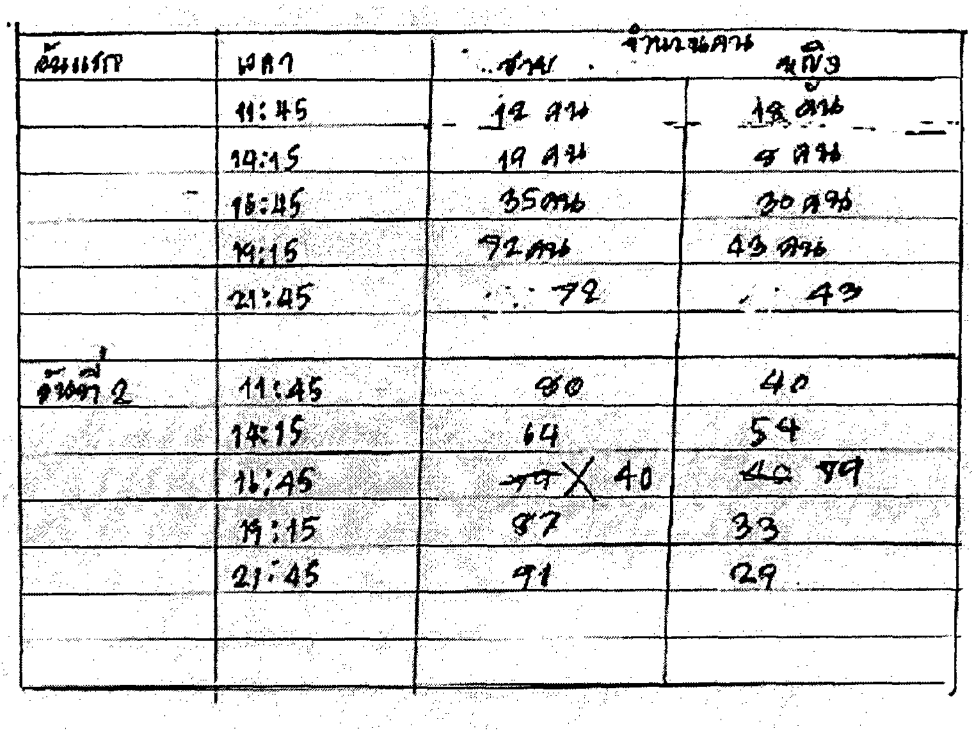
Interviewer:	What does the question require you to do?
Sasi:	Make a graph.
Interviewer:	Pardon.....
Sasi:	To construct the table on the survey data about people who went to the theater.
Interviewer:	Well, then how did you do?
Sasi:	Make a table for both days, that first day and the second day.
Interviewer:	This is the table which you get from that given data, isn't it?
Sasi:	Yes.

In summary, it can be concluded that these students of Level 2 are able to understand the question in the problem, but are not able to represent the given data in the form of a table. Although, they try to construct a table they focus on only some of the given data and leave other data out. Their tables are not organized and are difficult to read.

Level 3

The students at the third level were able to present the data in a table accurately and clearly but did not include the total number of the audience who attended each session.

Nida who was studying in Mathayomsuksa one was assigned Level 3 thinking for this problem, although in an overall sense she was a Level 2 thinker on representation. Nida tried to create the table for presenting data which was specifically the number of men and the number of women in every session as shown in Figure 44, but the total number of the audience in every session was not apparent.



วันแรก	เวลา	จำนวนคน ชาย	หญิง
	11:45	12 คน	18 คน
	14:15	19 คน	[illegible] คน
	16:45	35 คน	30 คน
	19:15	72 คน	43 คน
	21:45	72	43
วันที่ 2	11:45	80	40
	14:15	64	54
	16:45	~~79~~ 40	~~40~~ 79
	19:15	87	33
	21:45	91	29

FIGURE 44 NIDA'S RESPONSE ON CINEMA PROBLEM

Some students presented the data in the table form in a way that covered the total number of the audience in each session. However, as exemplified by Thep, they were not sure of their representation. Thep (Mathayomsuksa one) tried to represent the number of men, the number of women and also the total number of the audience, but he discarded the total number of the audience at the end (see Figure 45).

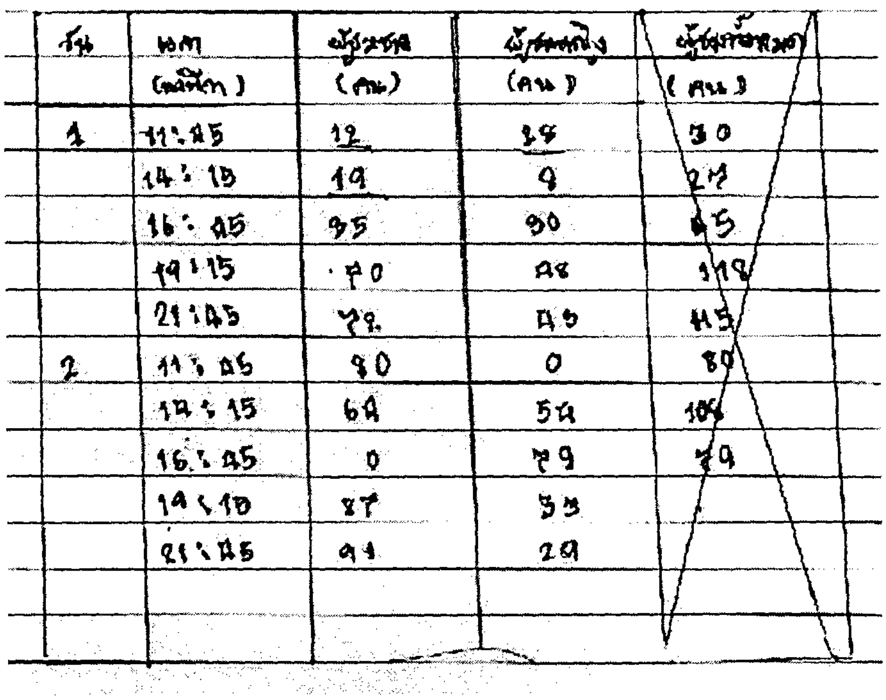


FIGURE 45 THEP' S RESPONSE ON CINEMA PROBLEM

Interviewer:	What does the question require?
Thep:	Present a table.
Interviewer:	In the table that you did, how many columns of the table did you use?
Thep:	Four columns.
Interviewer:	What are they?
Thep:	First column is the first day. The second is the time. The third is the number of men and the forth is the number of women.
Interviewer:	And what is this? (column 5) You wrote that the total

Interviewer: number of the audience, why did you cross it out?

Thep: I thought that it is not correct.

In summary, Level 3 students represent some data correctly and clearly in the form of a table but leave out some data or conditions.

Level 4

The students who were at the fourth level presented all the given data in the form of a table that was arranged systematically and clearly. Moreover, some students represented the data which was not defined in the problem. Their tables provided more information.

Students constructed the table to represent the number of men, the number of women and the total number of the audience in each session clearly and systematically. For example, Nisa who was studying in Mathayomsuksa one identified all the columns in her table; the first referred to the total number of the audience in each session while the second and third referred to the number of men and women, respectively. All columns in the table were easily identified (see Figure 46).

วัน	เวลา	ผู้ชมทั้งหมด(คน)	เป็นชาย (คน)	เป็นหญิง (คน)
แรก	11.45 น.	30	12	18
	14.15 น.	27	19	8
	16.45 น.	65	35	30
	19.15 น.	118	70	48
	21.45 น.	115	72	43
ที่ 2	11.45 น.	120	80	40
	14.15 น.	118	64	54
	16.45 น.	120	41	79
	19.15 น.	120	87	33
	21.45 น.	120	91	29

FIGURE 46 NISA'S RESPONSE ON CINEMA PROBLEM

Roj, however, represented the number of the men first and the number of the women and also the total number of the audience, respectively (see Figure 47). He was studying in Mathayomsuksa two.

ตารางการเข้าชมภาพยนตร์เรื่องโหมโรง

วันที่	รอบ	ชาย (คน)	หญิง (คน)	รวม (คน)
1	11 : 45 น.	12	18	30
	14 : 15 น.	19	8	27
	16 : 45 น.	35	30	65
	19 : 15 น.	70	48	118
	21 : 45 น.	72	43	115
2	11 : 45 น.	80	40	120
	14 : 15 น.	64	54	118
	16 : 45 น.	41	79	120
	19 : 15 น.	87	33	120
	21 : 45 น.	91	29	120
รวมยอดคนที่เข้าชม		571	382	953

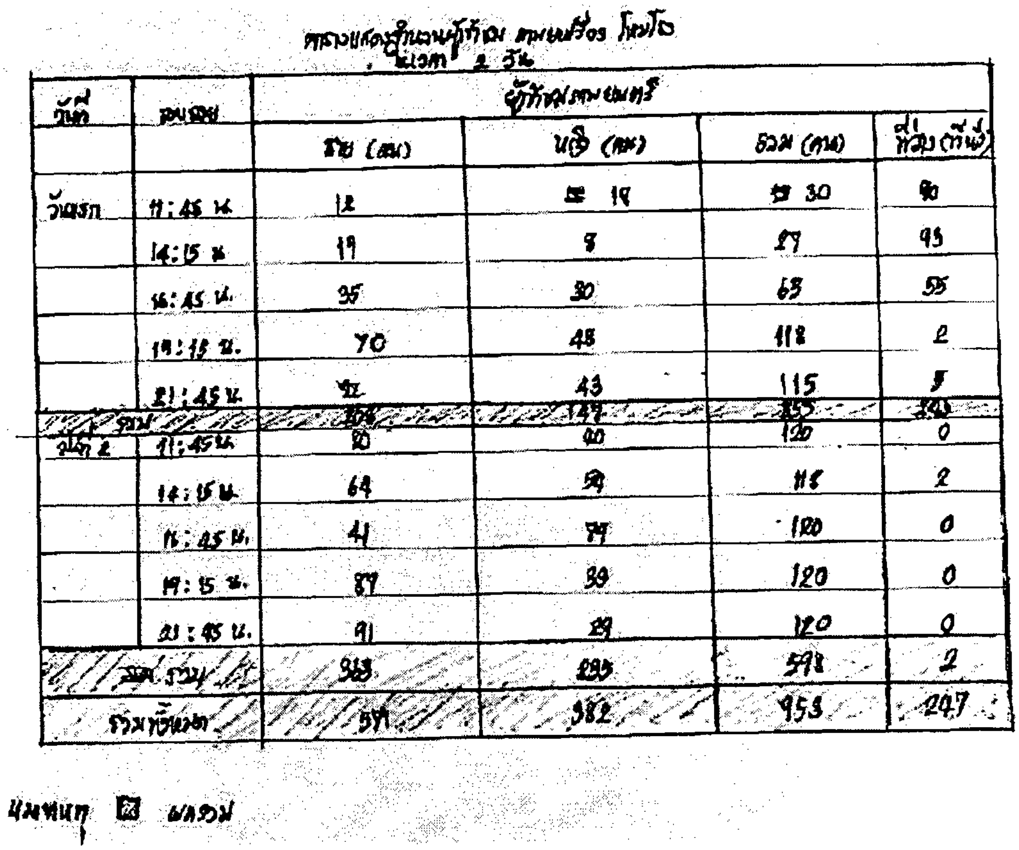
FIGURE 47 ROJ'S RESPONSE ON CINEMA PROBLEM

More over, it could be seen that Pathra (Mathayomsuksa three) presented the same data as Nisa and Roj, but she combined the column of time in each day together because it was the same session (Figure 48).

รอบฉายหนัง	วันที่ 1			วันที่ 2		
	จำนวนคน			จำนวนคน		
	ช.	ญ.	รวม	ช.	ญ.	รวม
11:45 น.	12	18	30	80	40	120
14:15 น.	19	8	27	64	54	118
16:45 น.	35	80	[illegible]15	41	79	120
19:15 น.	70	48	118	87	33	120
21:45 น.	72	43	115	91	29	120

FIGURE 48 PATHRA' S RESPONSE ON CINEMA PROBLEM

S ome students interpreted the question and expanded their thinking providing much more than the question asked. The data in the table was clear and had many dimensions. For example, Chira who was studying in Mathayomsuksa three presented the number of the empty seats in the theater in each session as shown in Figure 49.



ตารางแสดงจำนวนผู้เข้าชมภาพยนตร์เรื่อง โหมโรง ในเวลา 2 วัน

วันที่	รอบฉาย	ผู้เข้าชมภาพยนตร์			
		ชาย (คน)	หญิง (คน)	รวม (คน)	ที่ว่าง (ที่นั่ง)
วันแรก	11:45 น.	12	18	30	90
	14:15 น.	19	8	27	93
	16:45 น.	35	30	65	55
	19:15 น.	70	48	118	2
	21:45 น.	72	43	115	5
รวม		208	147	355	245
วันที่ 2	11:45 น.	80	40	120	0
	14:15 น.	64	54	118	2
	16:45 น.	41	79	120	0
	19:15 น.	87	33	120	0
	21:45 น.	91	29	120	0
ผลรวม		363	235	598	2
รวมทั้งหมด		571	382	953	247

หมายเหตุ ▨ ผลรวม

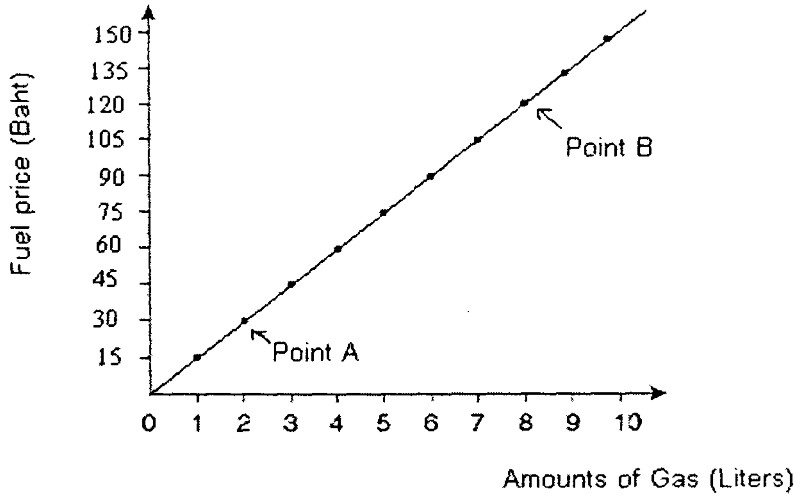
FIGURE 49 CHIRA'S RESPONSE ON CINEMA PROBLEM

In summary, Level 4 students construct the table that presents all the given data correctly and is organized systematically and clearly. Some students expand their thinking in the data and represent the data much more than the question require. The data in the table is clear and has many dimensions.

Interpreting the data in a Graph

Students were given the gasoline problem (Problem 7) with a graph of fueling gas and were asked to read and compare points on a graph and also interpret the graph as shown in Figure 50.

7. *Gasoline graph*: Graph shows fueling gas, which costs 17.50 baht per liter. Answer the following questions.



1) What information that you get from point A?
2) What information that you get from point B?
3) Besides information you get from 1) and 2), what do you know from the given graph?

FIGURE 50 PROBLEM 7 (Gasoline graph)

Four levels of thinking were demonstrated in students' ability to interpret data represented by a graph.

Level 1

Level 1 students could not interpret the points on the graph. Some students did not understand that the given data was called a “Graph”. When asked to explain what the problem required, they read all the questions. They did not say anything about their thinking. Sometimes they guessed the answer.

From the gasoline graph, Rudtai who was studying in Mathayomsuksa two was assigned Level 1 thinking, although in an overall sense she was a Level 2 thinker on representation. She wrote in her paper work that point A indicated the travel distance for which the fuel was able to be supplied. But when asked to explain what she did, she said that she just guessed the answer as her interview dialogue suggests.

Interviewer:	(The interviewer began Problem 7 which the student did from her paper work)
Rutai :	This.....I did not understand at all.
Interviewer:	Try again, this is a graph that shows price of gasoline. I want to know what information that you get from the point A?
Rutai :	I don’t know, I just guess.
Interviewer:	Pardon......
Rutai :	I said that the distance while filling the gasoline. [*SIC*]

Suthi who was studying in Mathayomsuksa two did not write any answer in his paper work for this question. When he was interviewed, he read all the words of the question but did not explain his understanding in own words. Moreover, when asked about the given graph, he said that he did not know and guessed the answer as the interview dialogues shows.

Interviewer:	What does the problem ask you?
Suthi:	(the student reads through the question)
Interviewer:	What do you call this? (point to the graph).
Suthi:	I don’t know.
Interviewer:	What does the first question require?
Suthi:	What does the point A inform?

(the student reads through the question)

Interviewer:	What does the point A tell you? (item 1)
Suthi:	I don't know.
Interviewer:	What does the point B tell you about? (item 2)
Suthi:	I don't know also.

In summary, students at this level did not understand the problem. They did not capture the meaning of the given graph. When asked to explain what the problem required, they just read all the words of the problem and didn't respond. Sometimes, they guessed the answer.

Level 2

Students who gave Level 2 responses showed that they started to interpret the data on the graph by reading the points on one axis only. Students at this level understood that the problem involved a "graph" which displayed the fuel price that you have to pay. Samples of students' responses are presented below.

Sasi who was studying in Mathayomsuksa three read the point on the graph by looking at the data on the vertical axis which was the amount of money. She was not interested in the data on the horizontal axis as the dialogue indicates:

Interviewer:	What does the problem ask you?
Sasi:	Tells me to answer the questions.
Interviewer:	Well, answer from what......
Sasi:	From the graph, which shows the fuel price which we must pay.
Interviewer:	From this graph you see, what does the point A tell you? (item 1)
Sasi:	Amount of money.
Interviewer:	Amount of money, how much?
Sasi:	The amount of money is 35 Baht.
Interviewer:	And what about the point B? (item 2)
Sasi:	It also tells about the amount of money.

Interviewer: Tell the amount of money.

Sasi: Yes, the amount of money is 122.50 Baht.

In summary, students at this level interpreted the points on the graph by reading the data points on only one axis.

Level 3

The students providing Level 3 responses demonstrated their ability to interpret the points on the graph with respect to both axes, but they could not compare the points. Some students tried to compare data from the graph but they could not.

Surat who was studying in Mathayomsuksa three could tell the points on the graph correctly, but when he was asked about the relationship on the graph, he could not answer as the following student response illustrates.

Interviewer: What is this line? (the horizontal axis)

Surat: The amount of fuel.

Interviewer: What does this line in vertical axis show?

Surat: The amount of money.

Interviewer: The line in the middle, what does it display? (point to the graph)

Surat: The line tells the quantity of fuel and money.

Interviewer: Point A, what does it show you? (item 1)

Surat: Fuel price and amount per liter.

Interviewer: Tell the fuel price and amount of liters, How much?

Surat: 2 liters.

Interviewer: How much does its cost?

Surat: 35 baht.

Dialogue of another interview question which required Surat to explain the gasoline graph:

Interviewer: What did you know from the graph, except item 1 and 2? (item 3)

Surat: Money to be paid.

Interviewer: What do you mean?

Surat: Money to be paid.

Interviewer: Can you give me an example of what you mean?

Surat: I don't know.

Interviewer: Anything else you know from this graph?

Surat: No.

Orchid (Mathayomsuksa two) was assigned Level 3 thinking for this problem, although in an overall sense he was a Level 2 thinker on representation. Orchid could also read the data from the gasoline graph. He tried to compare two points on the graph but was not clear as the following interview shows.

Interviewer: What did you know from the graph?

Orchid: I compared the amount of money: 35 baht is able to buy 2 liters but 140 baht is able to buy 8 liters.

Interviewer: What is the difference?

Orchid: It will be increased following the amount of money. If we add more money, we can add more liters of fuel.

In summary, it can be inferred that students at this level can understand and also interpret the given graph which focuses on both axes, but they do not compare the points on the graph.

Level 4

Level 4 students could interpret the points on the graph and compare data with respect to both axes. Moreover they could explain the relationships between the data on the graph. Samples of students' responses are presented as follows.

Pathra who was studying in Mathayomsuksa three described the given graph as the relationship of the data and constructed a general formula to describe the relationship.

Interviewer: What did you know from the given graph?

Pathra: If 3 liters of fuel cost 52.50 baht. Then 4 liters of fuel cost 70 baht. Then 5 liters of fuel cost 87.50 baht. Then 6 liters cost 105 baht, then 7 liters of fuel is 122.50 baht, then 8 liters is 140 baht, then 9 liters cost 157.50 baht, then 10 liters cost 175 baht, If fuel is N liters, I get the formula to find the cost as 17.50N.

Chira who was studying in Mathayomsuksa three described the relationship from the given graph differently from Pathra. She looked at the relationship in the graph as a direct variation as the following interview shows.

Interviewer: What did you know from the given graph?

Chira: The fuel price is varied directly to the amount of fuel in liter because I noticed that when we added more fuel, for example, one liter 17.50 baht, 2 liters will cost more... 35 baht, 3 liters will cost more... 52.50 baht. So, it's 17.50 baht added to get the price of fuel that increasing. The increasing is consistent.

Interviewer: O.k.

Chira: Well, when we knew the amount of fuel in liter, we were able to calculate the price. For example, we buy 5 liters, we could calculate that 5 liters cost 87.50 baht, so if we know the amount of money, we also know the amount of fuel in liters.

Roj (Mathayomsuksa three) was also classified at Level 4 thinking. He could compare and interpret the points on the graph and was also able to describe the graph in the context of other social situations involving rates.

Interviewer: When we buy something in the unit of dozen, such as books or snacks, its cost will be cheaper than buying by pieces. How about this case of fuel?

Roj: It depends on the different kind of shop.

Interviewer: For this graph, is it possible to decrease the price?

Roj: No, because it is already set.

In summary, students at this level can interpret and compare the points on the graph with respect to both axes. Moreover, they can explain the given graph correctly.

Students' Thinking on Variable

Variables involve understanding the role of variable. This indicator is concerned with students' ability to understand the role of variable as generalized number in algebraic expressions, equations and inequalities.

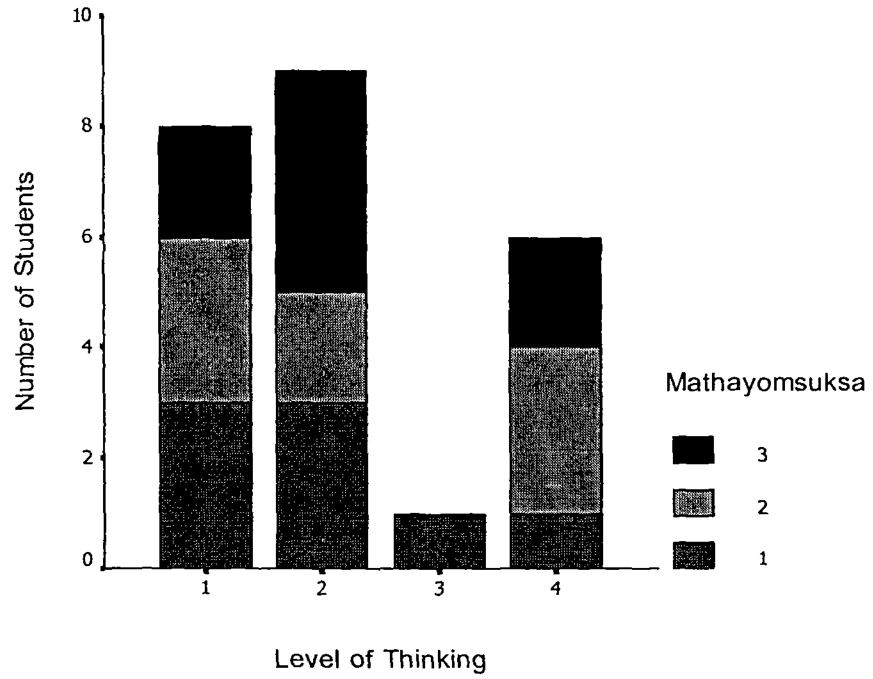


FIGURE 51 STUDENTS' LEVEL OF THINKING ON VARIABLE

Figure 51 shows students' level of thinking on variable. Nine students demonstrated Level 2 thinking, 4 students from Mathayomsuksa three, 2 students from Mathayomsuksa two and 3 students from Mathayomsuksa one. Eight students performed at Level 1 in their thinking, 2 students from Mathayomsuksa three, 3 students from Mathayomsuksa two and 1 student from Mathayomsuksa one. Six students were Level 4 thinkers, 2 students from Mathayomsuksa three, 3 students from Mathayomsuksa two and 3 students from Mathayomsuksa one. Only one student demonstrated Level 3 thinking who was Mathayomsuksa one. The result indicated that students in Mathayomsuksa one illustrated all 4 levels of thinking, while Mathayomsuksa two and three students spread over

Level 1, 2, and 4, none was in Level 1. It is remarkable that students' thinking on variable was not much different between grade levels.

In this study, students were confronted with one problem involving variable which consisted of 7 questions (see Figure 52).

8. Answer the question below and explain how you get the answer.

8.1) Which is larger, $n+n$ or $n+6$?

8.2) What can you say about b If $b+d=18$ and $b \langle d$?

8.3) $x(\frac{1}{x})=1$, find the value of x

8.4) $a=a\times 1$, find the value of a

8.5) What can say about y if $y=\frac{1}{2}x$ and x is any number?

8.6) $2x-3=\frac{4(2x-3)}{4}$, find the value of x

8.7) $(-a)\times b=(-b)\times a$ or not? Why?

FIGURE 52 PROBLEM 8 (Variable).

This problem was intended to explore students' algebraic thinking at all levels. The following descriptions and illustrations reflect students' thinking about variable at each level.

Level 1

Students at this level were unable to understand the question. Some students failed to engage the problem. Veera exemplified the students at this level. He was studying in Mathayomsuksa two. In this problem, he could not understand the question which required the English alphabet in the role of variable or unknown. This prevents him to engage in working on the question as the following interview indicates.

Interviewer: What happened in your work? Why didn't you work on it?
Veera: ......I cannot read the English alphabet.... I don't know.
Interviewer: Do you know what the question asks?
Veera: ....I don't know.

When they were asked about other items, they would answer that "I don't know" and sometimes they avoided the answer as Orchid did. He was studying in Mathayomsuksa two.

Interviewer : What's this? (Point to question 8.3 that is $x(\frac{1}{x})=1$, what is x)
Orchid : ....I don't know.
Interviewer : You don't know what this is.
Orchid : ....Yes.
Interviewer : And what's this? (Point to question 8.3; what can you say about y if $y=\frac{1}{2}x$ and x is any number?)
Orchid : ....I don't know either.
Interviewer : This? (Point to y)
Orchid : ....I don't know.

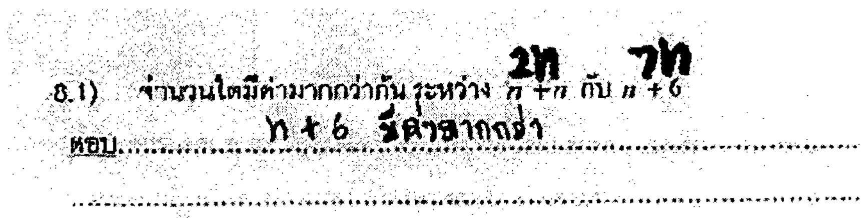
Some students answered the question based on irrelevant data such as the interview dialogue of Thavee. He was studying in Mathayomsuksa three.

Interviewer : Which is larger, $n+n$ or $n+6$?
Thavee : ....I don't know.
Interviewer : what do you mean?
Thavee : May be...... $n+6$ because n has no value but 6 has value.

Surat (Mathayomsuksa three) was assigned Level 1 thinking for question 8.1, although in an overall sense he was a Level 2 thinker on variable. He thought of the variable in a similar way to other students. He guessed the answer for this question which asks, "which is larger, $n+n$ or $n+6$?"

Interviewer :	Which is larger, $n+n$ or $n+6$? What 's your answer?
Surat :	Seven.
Interviewer :	Why did you answer 7?
Surat :	I estimated it.

Sara, a Mathayomsuksa three student, was assigned Level 1 thinking for question 8.1, although in an overall sense she was a Level 2 thinker on variable. She has a misconception in adding variables. She calculated the result of $n+n$ to be $2n$, but for $n+6$ she answered $7n$. Then, she said $n+n \langle n+6$ (see Figure 53).



8.1) จำนวนใดมีค่ามากกว่ากัน ระหว่าง $n+n$ กับ $n+6$ 2n 7n

ตอบ n + 6 มีค่ามากกว่า

FIGURE 53 SARA'S RESPONSE IN QUESTION 8.1

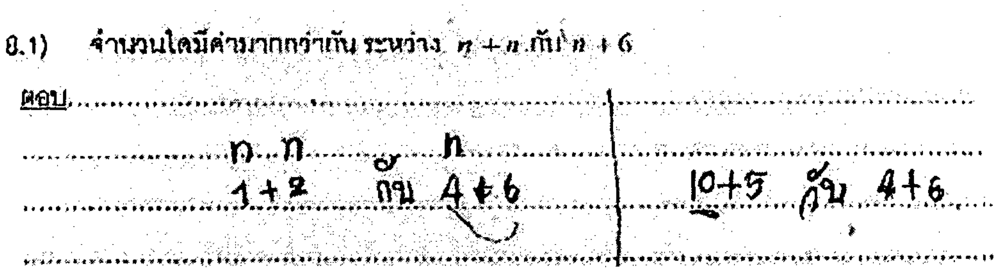
Some students did not know the meaning of the variable and sometimes they represented the same variable with different values as is the interview dialogue of Thep. He was studying in Mathayomsuksa one and was assigned Level 1 thinking for question 8.1, although in an overall sense he was a Level 2 thinker on variable.

Interviewer :	I want to know which is larger, $n+n$ or $n+6$? Your answer is........
Thep :	$n+6$.

Interviewer : Why do you think that it is $n+6$?

Thep : The value of $n+n$..... I don't know, but the value of $n+6$.... I know because I know that 6 is a positive integer.

Some students had a misconception about the meaning of variable. They represented the same variable with 2 values. For example, Wadee, a Mathayomsuksa one student, did not understand the meaning of variable. She thought that the given variable (n) in question 8.1 "which is larger, $n+n$ or $n+6$?" was not the same. For $n+n$, she substituted 1 and 2, respectively and represented the value of n from $n+6$ by 4. She represented each value differently as shown in Figure 54.



8.1) จำนวนใดมีค่ามากกว่ากัน ระหว่าง $n+n$ กับ $n+6$

ตอบ

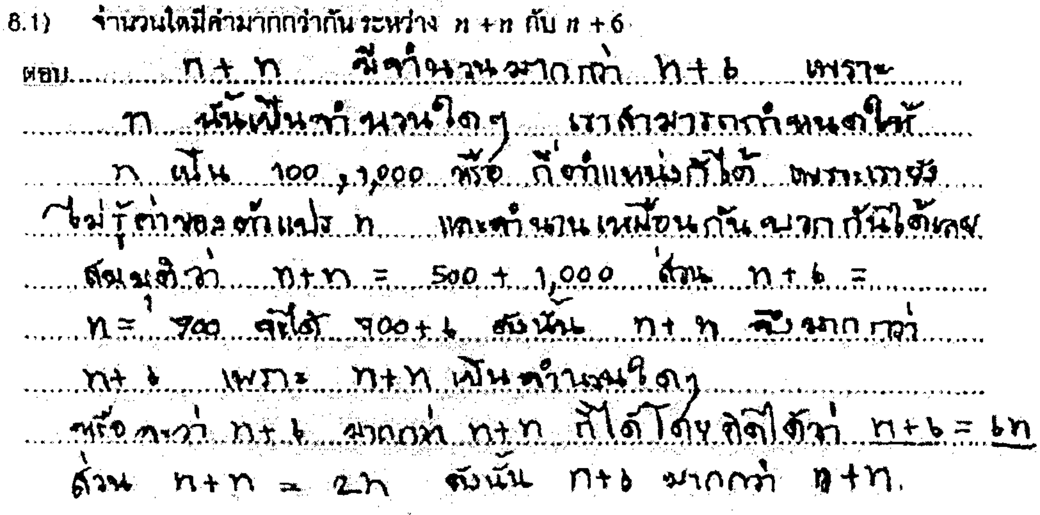
n n n

1+2 กับ 4+6

10+5 กับ 4+6

FIGURE 54 WADEE'S RESPONSE IN QUESTION 8.1

Panee (Mathayomsuksa three) used different numbers to represent each value of n and also added the specific numbers with the variable, although she thought of n as a generalized number in question 8.1 (see Figure 55).


8.1) จำนวนใดมีค่ามากกว่ากันระหว่าง $n+n$ กับ $n+6$

ตอบ n + n มีจำนวนมากกว่า n + 6 เพราะ n นั้นเป็นจำนวนใดๆ เราสามารถกำหนดให้ n เป็น 100, 1,000 หรือ ก็ตำแหน่งก็ได้ เพราะเราจึงไม่รู้ค่าของตัวแปร n และจำนวนเหมือนกัน มากกันได้เลย สมมุติว่า n+n = 500 + 1,000 ส่วน n + 6 = n = 700 จะได้ 700+6 ดังนั้น n+n จึงมากกว่า n+6 เพราะ n+n เป็นจำนวนใดๆ หรือจะว่า n+6 มากกว่า n+n ก็ได้โดยสิ่งที่ได้ว่า n+6 = 6n ส่วน n+n = 2n ดังนั้น n+6 มากกว่า n+n.

FIGURE 55 PANEE' S RESPONSE IN QUESTION 8.1

In summary, Level 1 students cannot understand the question or fail to engage the questions. Some students find the answer based on irrelevant data. Sometimes they guess the answer. Also, some students do not understand the meaning of the variable and sometime they substitute the same variable with a different value. Students at this level have misconceptions about operations in equations, expressions and inequalities.

Level 2

Students at Level 2 used only one specific number to make a general conclusion, when they responded to questions with variables. This following students' interview dialogues illustrate their thinking with respect to variables.

Saijai (Mathayomsuksa two) substituted 10 for the variable n in question 8.1 that asked students to choose between which expression was larger $n+n$ or $n+6$ as is shown in Figure 56.

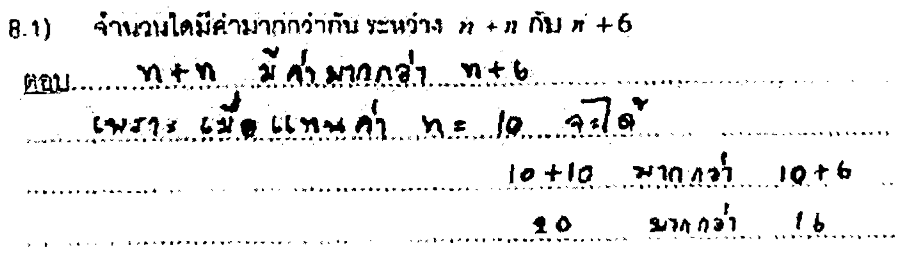


FIGURE 56 SAIJAI'S RESPONSE IN QUESTION 8.1

S asi who was studying in Mathayomsuksa three also used only one number, 10, to find out the result of the equation in question 8.2 as shown in Figure 57.

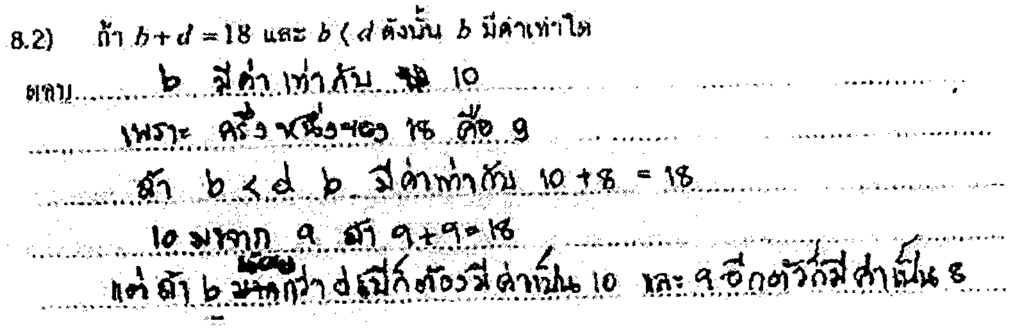


FIGURE 57 SASI'S RESPONSE IN QUESTION 8.1

Similar to Sasi's solution, Thep who was studying in Mathayomsuksa one represented the variable in question 8.2 with only one number and then made the conclusion as in his interview dialogue.

Interviewer :	For this equation (What can you say about b, if $b+d=18$ and $b \langle d$?), your answer is b equals 8. How did you find it?
Thep :	I knew the value of d, that is 10 and then b must be 8.
Interviewer :	If a plus b equals 10 and b is any number, how can you find a?
Thep :	10 divided by 2 equals five.

Interviewer : For what?

Thep : For finding a, so that a equals 5.

Interviewer : Why do you choose 2 to divide 10?

Thep : Because 2 is a half of 10.

Sara was studying in Mathayomsuksa three. When she solved the equation $x(\frac{1}{x})=1$ to find the value of x, she used only one number to represent the value of x as in the interview dialogue.

Interviewer : Find out the value of x in $x(\frac{1}{x})=1$, your answer is........

Sara : x equals 1.

Interviewer : How do you know?

Sara : I try to represent the variable of x with 1 which is $\frac{1}{1}=1$. Therefore, x equals 1.

Wut (Mathayomsuksa one) was assigned Level 2 thinking for the question 8.1, although in an overall sense he was a Level 1 thinker on variable. He substituted 6 for the variable n in this question. His answer was $n+n=n+6$ as in the interview dialogue.

Interviewer : Which is larger between $n+n$ or $n+6$? Your answer is.....

Wut : Equal.

Interviewer : Why do you answer like that?

Wut : Because this is 6. (n equals 6).

Interviewer : You substitute n with 6.

Wut : Yes, 6 plus 6 equals 12. Therefore, both are equal.

In summary, Level 2 students deal with variables using only one specific number. When they solve a question with variables, they substitute the given variable with one specific number and then make a general conclusion.

Level 3

Only one student responded consistently at Level 3. This student used several values of the variable to make conclusions. There were a number of other students who exhibited Level 3 thinking on a few questions. However, these latter students preferred to use one number only to represent the variable. Hence, they often missed the key idea in an expression on equation involving a variable and were assigned to Level 2 thinkers.

Saijai (Mathayomsuksa three) was assigned Level 3 thinking for the question 8.2, although in an overall sense she was a Level 2 thinker on variable. She answered this question using 1, 2, 3, 4, 5, 6, 7, and 8 to examine the equation $b+d$, $b<d$ (see Figure 58). Her answer was not complete.

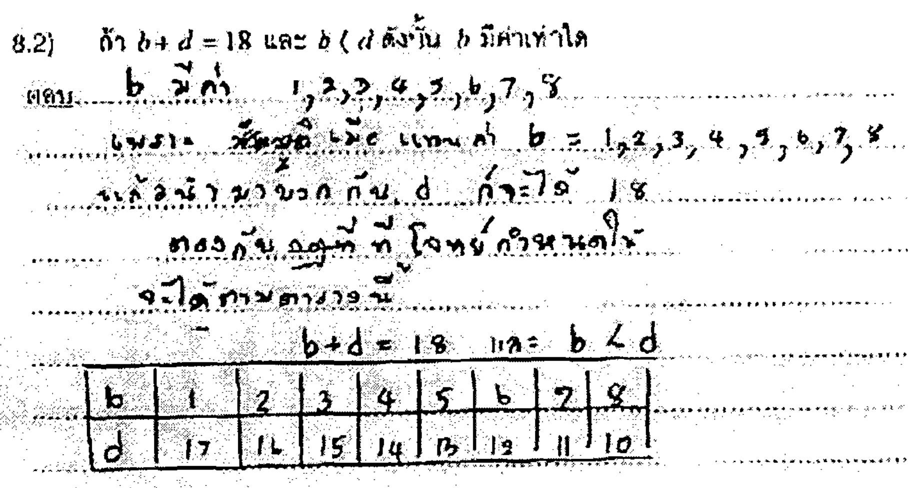
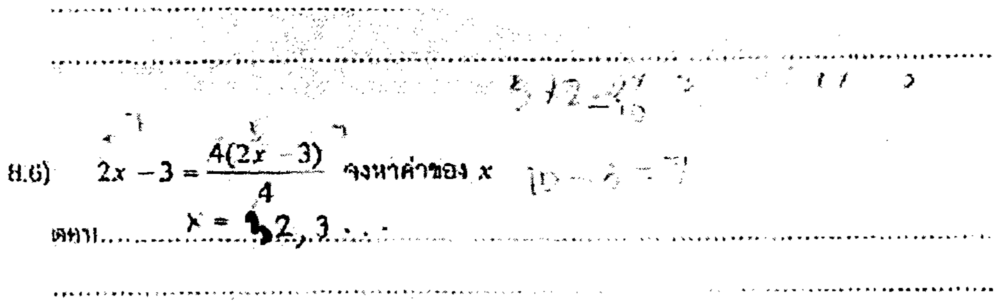


FIGURE 58 SAIJAI'S RESPONSE IN QUESTION 8.2

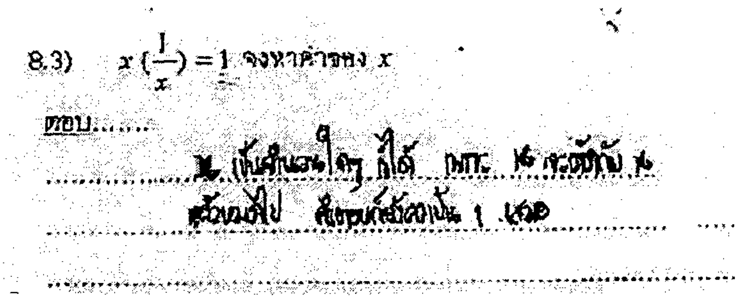
For the question 8.6, Riya (Mathayomsuksa two) was assigned Level 3 thinking, although in an overall sense she was a Level 4 thinker on variable. She was asked to find the value of x when $2x-3=4\left(\frac{2x-3}{4}\right)$. Her answer was 1, 2, 3, 4,...... which was not complete as Figure 59 shows.


8.6) $2x - 3 = \frac{4(2x - 3)}{4}$ จงหาค่าของ x

ตอบ $x = 1, 2, 3 \ldots$

FIGURE 59 RIYA'S RESPONSE IN QUESTION 8.6

Students at this level did not recognize the condition on the given variable, that is $x \neq 0$. For example, Chira who was studying in Mathayomsuksa three was assigned Level 3 thinking for the question 8.3, although in an overall sense she was a Level 4 thinker on variable. She stated that x could be any number in the equation $x(\frac{1}{x}) = 1$, but she did not recognize that x could not be 0 (see Figure 60).


8.3) $x(\frac{1}{x}) = 1$ จงหาค่าของ x

ตอบ

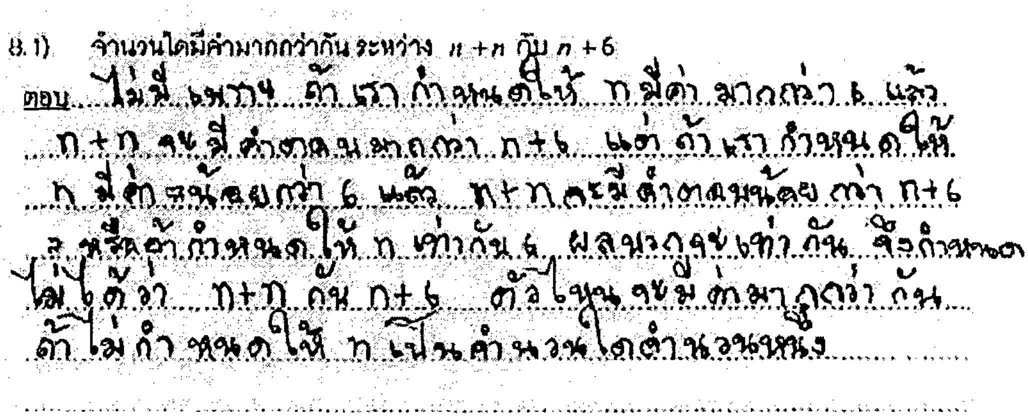
FIGURE 60 CHIRA'S RESPONSE IN QUESTION 8.3

In summary, students exhibiting Level 3 thinking with respect to variable use several specific numbers to make a general conclusion, they do not fully understand the role of a variable. Most students focus only on the counting numbers when they solve the questions. Moreover, students at this level do not recognize the condition of variable that was not explicitly stated.

Level 4

Students at Level 4 thinking understood the role of variable as generalized numbers. They substituted values for the given variable and used the variable to stand for any number in a set, such as the set of whole number or the set of rational number. They also gave a well reason for general conclusion. Illustrations of their thinking are presented below.

Roj who was studying in Mathayomsuksa two perceived the role of variable as generalized number and gave certain conditions for a variable such as when $n<6$, $n=6$ and $n>6$ which was a well reasoned conclusion as the interview dialogue below (see Figure 61).


8.1) จำนวนใดมีค่ามากกว่ากันระหว่าง $n+n$ กับ $n+6$

ตอบ ไม่มี เพราะ ถ้าเรากำหนดให้ n มีค่ามากกว่า 6 แล้ว n+n จะมีค่ามากกว่า n+6 แต่ถ้าเรากำหนดให้ n มีค่าน้อยกว่า 6 แล้ว n+n จะมีค่าน้อยกว่า n+6 หรือถ้ากำหนดให้ n เท่ากับ 6 ผลลัพธ์จะเท่ากัน จึงกำหนดไม่ได้ว่า n+n กับ n+6 ตัวไหนจะมีค่ามากกว่ากัน ถ้าไม่กำหนดให้ n เป็นค่าหนึ่งค่าใดค่าหนึ่ง

FIGURE 61 ROJ' S RESPONSE IN QUESTION 8.1

Interviewer :	The question asked which is larger, $n+n$ or $n+6$? What do you answer?
Roj :	It depends on the value of n. If n is more than 6, then the value of $n+n$ is more than the value of $n+6$. If n equals 6, then both are equal. If n is less than 6, the value of $n+n$ is less than the value of $n+6$.

Chuti (Mathayomsuksa two) displayed her understanding of variable as a generalized number and also recognized the condition of variable in such a way as to solve $x(\frac{1}{x})=1$ (see Figure 62). Her answer was x can be any number except 0.

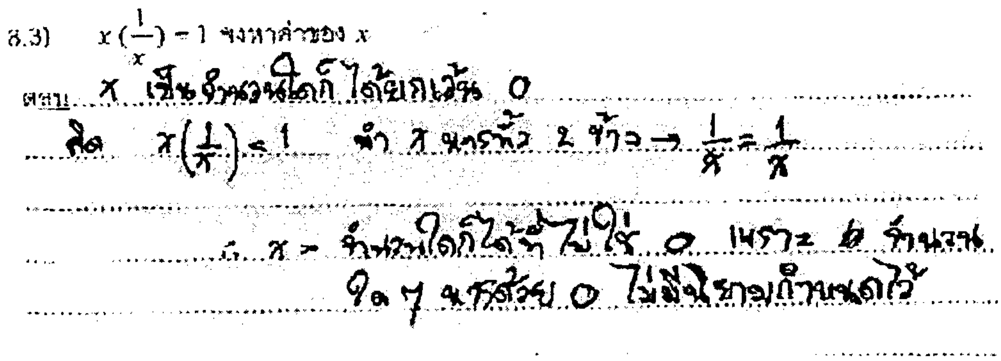


FIGURE 62 CHUTI'S RESPONSE IN QUESTION 8.3

When asked to find the value of a, when $a\times 1$ equals a in question 8.4, Chira (Mathayomsuksa three) answered that a could be any number because one multiplied by any number would get the same value.

Nisa who was studying in Mathayomsuksa, her reason for why $(-a)\times b=(-b)\times a$ in Figure 63 one indicated that she understood that a and b could represent any number.

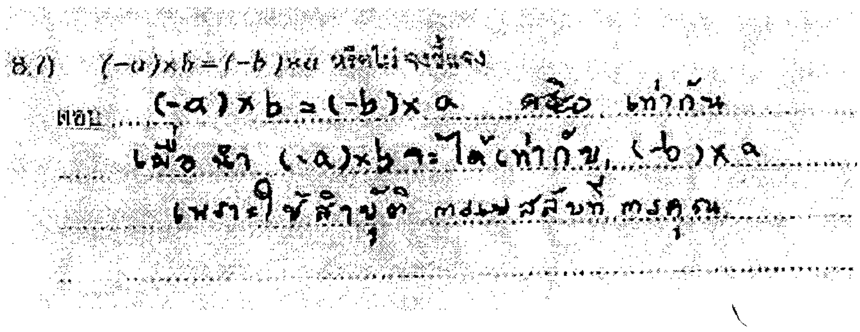


FIGURE 63 NISA'S RESPONSE IN QUESTION 8.7

In summary, Level 4 students' thinking on variable enables them to see and use a variable as a generalized number. When they solve the questions with a variable, they substitute values for the given variable by perceiving every number and give a well reasoned general conclusion. They also recognize all conditions on the variable; even these were not explicitly stated.

Part III: Algebraic Thinking Level Summaries

After analyzing all 24 students' papers and interview responses, the descriptors in the algebraic thinking framework were refined. Examples of each level in algebraic thinking across the three indicators were presented in the second part of this chapter. However, in order to generate a final picture of each of the levels of students' algebraic thinking, the researcher chose a student who had the same level of thinking across three indicators. Such a student was representative of his/her level and exemplified algebraic thinking at that level. Summaries of the four levels of thinking are based on the students who were consistent across all indicators in the earlier analysis.

Level 1 Summary

Students exhibiting Level 1 algebraic thinking were unable to understand the questions or were confused about them. They avoided answering the question or answered by guessing based on irrelevant data. Sometimes they observed the given information in the problem, but they made no logical connections between the given information. When they were asked about the given problems, they restated the questions and also were unable to describe their understanding. Sometimes they made no attempt to give an answer.

Suthi who was studying in Mathayomsuksa two was selected to be the representative of Level 1 algebraic thinking. For the *pattern* problems, Suthi knew that the given data was a pattern, but did not know how to use the given pattern to answer the question. In Problem 1 (Arrow pattern) Suthi knew that there were 10 given arrows. When asked to find the next arrow, Suthi could not answer the question. He just said "I don't know" and it seemed that Suthi did not know how to analyze the pattern and made no attempt to find the answer. However, when confronted with semi-concrete problems, Suthi attempted to solve them but he failed to engage them. For example, in Problem 2 (Table pattern) he was asked to find the number of chairs when 54 tables were put next to each other. He understood that the given data was a pattern, but he did not see the given pattern as an extended pattern and so could not analyze it. Although he considered the set of 3 tables with 8 chairs, he used only one term of the pattern to help him in drawing and counting the number of chairs. His method was not correct and was inconsistent for finding the value of other terms. For instance, when asked to find another number of chairs with 5 tables, he just used other terms of the pattern which was suitable for the question. Suthi combined 2 tables for 6 chairs and 3 tables for 8 chairs to find the number of chairs with 5 tables that is 6 + 8 = 14 chairs but his answer was not correct and he did not find the value of the next term. Sometimes, Suthi used every given term of the pattern which was possible for finding the answer but did not make logical connections. With the rest of the pattern problems, Suthi avoided answering the questions. He did not write any word on his paper. When asked to explain his understanding of the problem, he just read all the words in the problem.

For the *representation* problems, Suthi could not understand the problem and avoided to answer the question. When asked to represent the time and distance of Mr. Big's journey in the form of a graph, Suthi did not construct the graph. He did not write anything on the paper. Suthi restated the problem when asked to solve the question and although he was asked what question he needed to answer, he just said "I don't know". Also, Suthi's response to Problem 6 (Cinema problem) about constructing a table to represent the people in the cinema was the same as his response to Problem 5 (Mr. Big's journey). When Suthi was given the line graph of fueling gas in Problem 7 (Gasoline graph) and asked to read the points on the graph, he did not know that the given graph was called a "graph".

In investigating the role of *variable*, questions were used in the form of an equation, an expression and also an inequality. Suthi failed to engage each question of the problem. He also said "I don't know" to almost every question. Sometimes he read part of a question but did not answer correctly such as $n+n$, he would read $x+x$ and also did not know the meaning of variable. In another example, Suthi hold a misconception in using operations with numbers and variables. He did not know that in the equation $2x-3=4\left(\frac{2x-3}{4}\right)$, $2x$ means 2 multiplied by x. Also he did not know the meaning of parenthesis in this equation.

Level 2 Summary

Level 2 students showed their thinking that went beyond Level 1. They could engage in the problem and knew what the question required, but did not know how to answer it. Students at the second level could give the correct answer, but not consistently. They also used only one relevant aspect of the information in the problem to find the answer.

Nat who was studying in Mathayomsuksa one was chosen as the example of Level 2 algebraic thinkers. With the *pattern* problems, Nat could find the value of the next term of a given pattern, but when finding the value of a higher term her response was irrelevant. For example, with Problem 4 (Number sequence), Nat could recognize that the values of T increased by 3 when she was presented with four terms of a number sequence. Although Nat recognized only the relationship between the values of the given terms, she did not use the relationship to find the value of a higher term. Therefore, she could find the value of the fifth term of a given pattern (the next term) that is the value of T equals 16 when S equals 5. But for the higher terms, Nat gave an irrelevant answer. She multiplied 70 by 3 which was 210 to find the value of T when the value of S equals 70. For the first pattern problem (Arrow pattern), Nat could find the next arrow of the 10 given arrows and also recognized that there were the repeating arrows in some term as part of the conditions of the arrow pattern. When asked to find the arrow at the 63rd term, a higher term, she counted each arrow until the 10th arrow and then skipped to count the 11th arrow as the 2nd arrow. It seems that she recognized that the values of some term are the same. She found the value of a specific term of the pattern by counting every previous term from the pictures of the 10 given arrows. She did this systematically up to the 63rd pictures of arrows. Nat also used concrete objects, 10 arrow pictures, to help her find the value of a specific term of the pattern. Although Level 2 students tried to generalize from only one given term or only one aspect of the given pattern which the generalization was not correct, Nat did not use in this strategy.

In assessing algebraic thinking on *representation*, Nat could understand the question and the condition of the problem. When asked to represent the given data, Nat did not know exactly how to construct a graph or table. For example, Nat understood that she was asked to construct a graph to represent the time and distance of Mr. Big's journey in Problem 5 (Mr. Big's journey), but what she constructed did not show the relationship between time and distance. Instead, she drew diagram in the form of a wave and called it "a graph" which was not what she was asked to do. With respect to Problem 6 (Cinema problem), Nat constructed a table to present the data of the people who went to see the movie but it was specific to the number of men and women. She did not present the total number of the audience in each session, that is she represented only some of the data accurately. Also, her table was difficult to read because it was not well organized.

In order to assess Nat's understanding of the role of *variable*, she was asked to solve the equations, expressions or inequalities. She gave only one specific number for the value of variable. Although the variable was satisfied by many numbers, for example in the equation $x(\frac{1}{x})=1$, Nat just said that the value of x was 1. Moreover, Nat used only one number, 10, to find out the value of b in question 8.2 (What can you say about b $b+d=18$ and $b \langle d$?). With an expression, Nat indicated that the value of n in the question 8.1 (which is larger, $n+n$ or $n+6$?) could not be identified since she had not been studied.

Level 3 Summary

Students exhibiting Level 3 algebraic thinking tended to use two or more aspects of the data to find the answer, but they failed to link those aspects to each other. However, students at Level 3 could express many more ideas than those students at Level 2. However, they did not cover all the ideas and also did not focus on relationships of the given data. Sometimes, it looked like they were able to be Level 4 thinkers but their thinking fell short of generalization.

Latda who was studying in Mathayomsuksa one was selected as the sample of a Level 3 algebraic thinker. When confronted with the *pattern* problems, Latda recognized only the relationship between the values of the given terms and used this relationship to find the value of a higher term. Although she did not relate the value of a term to its term number in the given pattern, she used the relationship between the values of the given terms to generate a systematic method to find the value of any specific term of the given pattern without finding every previous term. For example, with the first pattern problem (Arrow pattern), Latda recognized that there were three cycles in the given pattern and that each cycle had 3 arrows. In order to find the nature of the 63rd arrow, Latda drew 2 more arrow signs to the given pattern to create the unit of 12 arrows. In this process, Latda multiplied 5 by 12 arrows to find the 60th arrow and then counted the first picture arrow as the 61st arrow and the second picture arrow as the 62nd arrow and also the third picture arrow as the 63^{rd} arrow. Sometimes Latda tried to generalize the given pattern from one value in the sequence to the next value without relating the value of a term to the term number, but her generalization was not correct. Latda was assigned Level 2 thinking for Problem 4 (Number sequence), although in an overall sense she was a Level 3 thinker on pattern. Latda generalized that there was a difference between the numbers of T in the given pattern which was the feature of Level 3 thinkers. She noted that the values of T increased by 3, she focused only on the values of T without relating to the values of S. Thus, she gave the answer of T when S equals 5 as 16 because she added 3 to the value of T when S equals 4. When asked to find the value of T when S is 70, she did not use what she noticed. Latda used only the 5^{th} term to find the answer which is not correct.

For *representation*, Latda gave Level 2 and Level 4 responses in representing data with a graph and a table respectively. Although students at this level showed their ability to represent some data correctly and clearly in the form of a graph or table, they left out some data or conditions and did not recognize all conditions of the problem when constructing a graph, i.g. Latda just drew a map to represent the distance and time of Mr. Big's journey in Problem 5 (Mr. Big's journey). She failed to construct a graph, although she understood that she had to represent the distance and time of Mr. Big's journey in the form of a graph. On the other hand, in Problem 6 (Cinema problem) Latda could represent all the given data in the form of a table. She mentioned that she was familiar with the table problem from her class. However, Latda responded to the problem of interpreting the given graph using Level 3 thinking. She was able to interpret points on the graph of fueling gas in Problem 7 (Gasoline problem), but did not compare it to the other points. She just stated that the cost of gas was 17.50 baht per liter which was already stated in the problem.

In the part on *variable*, Latda did not recognize any conditions on the variable that were not explicitly stated such as finding the value of x in the equation $x(\frac{1}{x})=1$. She answered 1 and 0 when 0 was not acceptable for this equation. Latda also did not fully understand the role of a variable as a generalized number. She substituted for the variable with some specific number such as in question 8.2. She found that the value of b was 1, 2, 3, 4, 5, 6, 7, and 8 when $b+d=18$ and $b \langle d$. It showed that she did not fully understand the role of a variable as a general number. Although she used several specific numbers to make a general conclusion, her answer was not complete.

Level 4 Summary

Students who were at Level 4 of algebraic thinking could see relationships between the given data and were able to demonstrate an integrated understanding of the relationships among various aspects of the data. They also used all aspects of the data when confronted with problems.

Pathra who was studying in Mathayomsuksa three was the exemplar of students who were at Level 4 across all three indicators. She also demonstrated Level 4 thinking on all problems of pattern. When confronted with *pattern* problems, Pathra recognized the relationship between the value of the term and the term number and generalized the relationship for the pattern in both verbal and symbolic forms. For example, Pathra recognized the given pattern in Problem 4 (Number sequence); that is, there was a relationship between S and T that applied in a general form. The value of S was started by 1 and increased by 1; meanwhile the value of T was started by 4 and increased by 3. Pathra constructed the formula $3m+1$ to find the value of T when S equals m. With other pattern problems, Pathra used the same method to generalize a symbolic form for the given pattern; thus her symbolic form focused on the relationship between the value of a term and the term number in the given pattern.

With respect to *representation*, Pathra presented correctly and systematically all data in the form of a graph and a table that recognized all the conditions and relationships in the given data. For example, Pathra draw a valid graph to represent the relationship between the time and distance of Mr. Big's journey in Problem 5. Her graph showed that she recognized the difference between walking and running distance for the same length of time. Pathra said that the distance for walking was less than the distance for running when using equal time. When she was confronted with the line graph of fueling gas in Problem 7 (Gasoline graph), Pathra could compare the points on the graph and explain the relationships of the data in the given graph. Pathra also constructed a formula to find the cost of N liters of gas. She said, "if I fill the car with N liters of gas, the cost will be 17.50N baht."

For the role of *variable*, Pathra understood the role of a variable as a generalized number and gave a well reasoned for general conclusion. For example, in the question 8.4 that she was asked to find the value of a when $a = a \times 1$, she mentioned that a could be any number except 0 because these numbers when multiplied by 1 will equal themselves. Pathra gave the similar answer with other questions such as in the equation

$2x-3=4\left(\frac{2x-3}{4}\right)$. She said that x was any number because when the equation was solved, she found that x equals x; that is, x could be any number. Pathra also recognized all conditions on the variable even they were not explicitly stated. For example, in question 8.1 (which is larger between $n+n$ and $n+6$?). Pathra said it depended on the value of n. If n is less than 6, then the value of $n+n$ is less than the value of $n+6$. If n equals 6, then both are equal. If n is more than 6, then the value of $n+n$ is more than the value of $n+6$. Also for the equation $x(\frac{1}{x})=1$, she said that x should be every number except 0.

CHAPTER 5

CONCLUSION AND DISCUSSION

Summary of the Study

In this study, the researcher formulated and validated a framework to describe and predict lower secondary school students' algebraic thinking. The formulation process was begun with the study of previous research on students' algebraic thinking and the use of the SOLO model of Biggs and Collis (1991) to describe cognitive processes. The initial framework described students' algebraic thinking on three indicators of algebra: pattern, representation, and variable. For each of these indicators, four levels of algebraic thinking were identified according to the four SOLO cognitive levels (Biggs; & Collis. 1991): Level 1 (prestructural), Level 2 (unistructural), Level 3 (multistructural), and Level 4 (relational). Hence, this framework consisted of four levels of thinking across three indicators.

An algebraic test was developed comprising a number of items for each indicator of the framework: 4 pattern tasks, 3 representation tasks and a variable task incorporating 7 questions. This test was administered to 48 lower secondary school students (Mathayomsuksa one, two and three). These students were purposefully chosen with eight students from each of three grades in two schools. Two students at each grade were sampled from the high achievement group, four from the average achievement group and two from the low achievement group. These achievement groups were based on their previous school assessment.

In the process of validating the framework, 24 of the 48 students were chosen for interviewing in order to refine the descriptors in the algebraic thinking framework. The selection of these 24 students mirrored proportionately the sample of 48 with respect to school, grade level and achievement level. They were purposefully selected because their paper works were interesting and clear. Also, their teachers felt they would be able to communicate effectively in an interview situation.

All of these 24 students were interviewed by the researcher to gain further insight into their reasoning. After that the interviews were transcribed and coded with the students' paper works. Three coders independently coded all 24 student interviews and responses from the paper work. The results were used to refine and validate the algebraic thinking framework.

Conclusion and Discussion

The analysis of students' responses was used to validate and illuminate the algebraic thinking framework. In particular, the initial framework was refined and evaluated for stability and interpreted in terms of other literature in the field.

With respect to refinement of the framework, a number of changes were made. For the indicator, *pattern*, five descriptors (P12, P13, P14, P15, and P16) were added and only one descriptor (P17) was modified to Level 1 in the refined framework. These changes were done in order to provide a more complete picture of the difficulties Level 1 students had dealing with pattern problems.

In Level 2 of the pattern indicator, two descriptors (P21, P23) were added and only one descriptor (P22) was modified. These changes were made to accommodate the fact that Level 2 students could find specific terms of a pattern but only by determining all previous terms of the pattern.

In Level 3 of the pattern indicator, one descriptor (P32) was added and only one descriptor (P31) was modified. These changes were made to accommodate the fact that Level 3 students could find a higher term without determining all previous terms of the pattern.

In Level 4 of the pattern indicator, one descriptor (P41) was modified. These changes were inserted to reveal the fact that Level 4 students could generalize the relationship of the pattern.

With the indicator of *representation*, two descriptors (R12 and R13) were added in Level 1. The main changes were made to provide a better picture of how students could not understand the representation tasks.

In Level 2, three descriptors (R22, R24, and R25) were added and three descriptors (R21, R23, and R26) were modified. These changes were made to accommodate the fact that Level 2 students could understand the task but did not know how to construct a graph or table to represent the data. They also could interpret and compare data from a graph with respect to one axis only.

In Level 3, only one descriptor (R32) was added and two descriptors (R31 and R33) were modified. These changes were made to reveal the fact that Level 3 students could represent correctly and clearly for some data in the form of a graph and a table but left out some data or conditions.

In Level 4, one descriptor (R41) was added and two descriptors (R42 and R43) were modified. These changes were made to provide a clear picture of Level 4 students. Students at this level represented all the given data in the form of a graph and a table correctly and systematically, they also recognized all conditions and relationships associated with the data.

For the indicator of *variable*, four descriptors were added in the refined framework. Two descriptors (V13 and V14) were added in Level 1 to accommodate the fact that students at this level did not know the meaning of variable and also they held misconceptions about operations in equations, expressions and inequalities. Only one descriptor (V32) was added in Level 3 to reveal the fact that students at this level did not recognize any conditions on the variable that was not explicitly stated. One descriptor (V42) was also added in Level 4 to provide a clear picture that students at this level recognized all conditions on the variable even these were not explicitly stated. These additions captured thinking of the target students that had not been anticipated in the initial framework.

There is relatively little research on lower secondary school students' algebraic thinking, especially on pattern (Bishop. 2000: 107-126; Orton; & Orton. 1999: 104-120; Pegg; & Redden. 1990: 386-391). It made the task of formulating the descriptors more difficult. In many cases the refinements incorporated aspects of algebraic thinking that were not anticipated from reviewing the literature.

The profile of students' levels of algebraic thinking showed strong stability across the three indicators. Forty-one point six seven percents of the sample students demonstrated consistent thinking levels across the three constructs and 58.33% of students exhibited consistent thinking level on two indicators. The stability of this framework compared favorably with the stability evident in a number of other studies on cognitive frameworks (Jones; et al. 1996: 310-336; Tarr; & Jones. 1997: 39-59; Mooney. 2002: 23-63). This consistency confirms that the framework provided a cohesive picture of lower secondary school students' algebraic thinking.

The distribution of the algebraic thinking levels over the three lower secondary grades produced some interesting results. There was little difference between the distribution levels among the three grades. In the indicator of *pattern*, at Level 4, the percentages of students in Mathayomsuksas one, two and three were 20%, 60% and 20%, respectively. For Level 3, the percentages of students in Mathayomsuksas one, two and three were 25%, 25% and 50%, respectively. For Level 2, the percentages of students in Mathayomsuksas one, two and three were 50%, 17.7% and 33.3%, respectively. For Level 1, the percentages of students in Mathayomsuksas one, two and three were 0%, 67% and 33%, respectively.

With respect to the indicator of *representation*, at Level 4, the percentages of students in Mathayomsuksas one, two and three were 17.7%, 50% and 33.3%, respectively. For Level 3, the percentages of students in Mathayomsuksas one, two and three were 25%, 25% and 50%, respectively. For Level 2, the percentages of students in Mathayomsuksas one, two and three were 50%, 25% and 25%, respectively. For Level 1, the percentages of students in Mathayomsuksas one, two and three were 50%, 50% and 0%, respectively.

With the indicator of *variable*, at Level 4, the percentages of students in Mathayomsuksas one, two and three were 16.7%, 50% and 33.3%, respectively. For Level 3, the percentages of students in Mathayomsuksas one, two and three were 100%, 0% and 0%, respectively. For Level 2, the percentages of students in Mathayomsuksas one, two and three were 33.3%, 22.2% and 44.5%, respectively. For Level 1, the percentages of students in Mathayomsuksas one, two and three were 37.5%, 37.5% and 25%, respectively.

It is remarkable that students' algebraic thinking across four levels were not much different between grade levels. For example, some students in Mathayomsuksa one (Nisa), in Mathayomsaksa two (Chuti and Riya) responded at Level 4 of algebraic thinking with certain problems similar to Mathayomsaksa three students. This tends to support curriculum development worldwide that algebraic thinking can be started early (NCTM. 1989/2000).

It should be noted that Mathayomsuksa two students had level of thinking higher than Mathayomsuksa three students since they studied the new mathematics curriculum while Mathayomsuksa three students studied the old mathematics curriculum.

In this study, the researcher formulated the initial framework based on the first four levels of the SOLO model (Biggs; & Collis. 1991). The validation process of this study confirmed these four levels of algebraic thinking in that the refined framework coincided with the prestructural, unistructural, multistructural and relational levels of the model of Biggs and Collis.

Students at Level 1 do not understand algebraic tasks. They solve algebraic tasks based on irrelevant data or sometimes they make no attempt to capture or find the answer. Thinking at this level is aligned with Biggs and Collis' (1991) prestructural level, in the sense that the task is engaged but the learner is distracted or misled by an irrelevant aspect.

Students exhibiting Level 2 algebraic thinking are beginning to engage in algebraic thinking. They recognize the relationship between consecutive terms but do not relate the value of a term to its term number. They use the relationship between the given terms to find the value of a specific term by drawing, counting or computing every previous term of the given pattern. Students at this level represent the data using one condition of the data and interpret the graph with only one axis. They also substitute only one number for a variable in making a conclusion. Students at this level reflect their thinking that coincides with Biggs and Collis' (1991) unistructural level; that is, the task is engaged in a relevant domain and one aspect is used or examined.

Students at Level 3 generate a systematic method for finding the value of a specific term of a pattern without finding every previous term. They recognize only the relationship between the values of consecutive given terms. They do not relate the value of the term to its term number. Students at this level represent some data correctly. They represent some data correctly and clearly in the form of a graph or table but leave out other data or conditions. They interpret the graph with respect to both axes but cannot compare the data on the graph. Students at this level also substitute more than one number for the variable before making a conclusion. They exhaust all possibilities but do not cover all the numbers. This ability to use more than one aspect of algebraic thinking is consistent with the multistructural level of Biggs and Collis (1991); that is, multiple ideas are considered and used, but do not integrate them.

Students exhibiting Level 4 algebraic thinking generalize the pattern verbally and symbolically by recognizing the relationship between the value of a term and its term number. They represent data by recognizing all conditions and relationships in the data. Students at this level can also explain relationships within a graph. They understand variable as a generalized number and recognize all conditions applying to the variable. Students at this level demonstrate thinking parallel to the relational level of Biggs and Collis (1991) that is students integrate the parts with each other, so that the whole has a coherent structure and meaning.

As already noted, the results of the validation confirm that the algebraic thinking framework in this study has four levels of thinking. This appears to be consistent with the number of levels in several other cognitive frameworks: number sense (Jones; et al. 1996: 310-336), probability (Jones; et al. 1997: 101-125), statistical thinking (Jones; et al. 2000: 269-307; Mooney. 2002: 23-63) and stochastic thinking (Shaughnessy. 1992: 465-494). By way of contrast, a number of other cognitive frameworks describe students' thinking according to three levels (Campbell; Watson; & Collis. 1992: 279-298; Callingham. 1994; Pegg; & Davey. 1998: 109-135; Watson; et al. 1995: 247-275). Interestingly, the van Hiele model (van Hiele. 1985: 243-252) describes geometrical thinking in five levels.

In this study, a framework for characterizing lower secondary school students' algebraic thinking was formulated and validated. It is claimed that the validated framework represents a coherent picture of students' algebraic thinking that provides insight into how algebraic learning of lower secondary school students develops. While no claim is made that the framework is generalizable to all students, it provides teachers with knowledge of students' algebraic thinking that is applicable well beyond the classroom in which the study was conducted. As a result, this landscape of lower secondary school students' algebraic thinking can be used by curriculum developers and teachers to improve instruction and assessment (Cobb. 2000: 307-334; Fennema; & Franke. 1992: 147-164; Simon. 1995: 114-145).

Applications and Suggestions

Applications for Curriculum and Instruction

The framework, generated in this study, characterizes lower secondary students' algebraic thinking and as a consequence it can be used in the design, implementation and assessment of classroom instruction.

For the *design* of instruction, the algebraic thinking framework identifies key algebraic indicators of learning algebra in the lower secondary school. It also provides curriculum developers and teachers with a cohesive picture of students' algebraic thinking across four levels. In particular, curriculum developers can use this framework in the design of a curriculum strand on algebra. Such information will provide teachers with a background on each student's thinking and will facilitate for making of decisions on instruction. Similar to other cognitive models, this framework also be used by the teachers to design learning goals, learning activities, problem tasks, and also a conjectured learning process. This is, it can be used as a basis for Cognitively Guided Instruction (CGI) (Carpenter; et al. 1989: 499-532), in the development of hypothetical learning trajectories (Simon. 1995: 114-145) and in the design of teaching experiments (Cobb. 2000: 307-334; Jones; et al. 2001: 109-144).

For the *implementation* of instruction, the algebraic thinking framework can provide teachers with the means for analyzing and characterizing students' responses in the classroom. More specifically, the teacher can use the framework to analyze students' responses to tasks like the ones used in this study during instruction. Similar to the design stage, teachers can use the descriptors of each level in the framework to construct suitable questions during instruction for different grades that can be used to assist students' growth in algebraic thinking.

With respect to the *assessment* of instruction, teachers can use tasks and questions based on the framework to assess students' prior knowledge in algebraic thinking. This background will help the teachers to know their students' levels of algebraic thinking and this in turn will help the teacher to generate instruction that will foster students' algebraic thinking to higher levels. Also this framework can be used to evaluate the effectiveness of classroom instruction by monitoring and assessing students' understanding during and after instruction.

Suggestions for Further Research

Although the refined framework provides a useful tool for characterizing lower secondary school students' algebraic thinking across three indicators, further research on students' algebraic thinking could try out with a larger sample to validate the framework on a more comprehensive scale. Such a study would enhance the current descriptors in the framework by possibly displaying thinking that was not covered in this study. In particular, the modification and addition of some tasks to the assessment protocol would help further researchers capture some descriptors that may not have been revealed in this study because of constraints in the observation and assessment of target students' thinking.

This study initially hypothesized and classified four levels of lower secondary schools' algebraic thinking. Further research might look to formulate and validate a fifth level of algebraic thinking since the results of this study indicated that some students' responses seemed to extend beyond the bounds of the Level 4 descriptors.

Further research also might look to add other indicators of algebraic thinking because the indicators in this study are only part of algebraic thinking. For example, it could include other key indicators like functions, transformations and algebraic modeling which would be helpful in characterizing students' algebraic thinking.

The lower secondary school students' algebraic thinking framework can be used in further research involving instruction in both lower secondary schools and elementary schools, especially Prathomsuksa six or Grade 6. In particular, the framework can be used to inform instruction to facilitate students' algebraic thinking. Such instructional research in algebraic thinking might be parallel the research of Jones and his colleagues (Jones; et al. 2001: 109-144) who conducted teaching experiments based on their statistical thinking framework. Further researchers can use the algebraic thinking framework to develop activities for the classroom including activities that use technology such as graphic-calculators, spreadsheets and graphing programs.

Finally, further researchers can use the design and methodology of the current study to generate an algebraic thinking framework at the elementary school level. In addition, Thai researchers should consider the development of cognitive frameworks in other mathematical domains such as number sense (Jones; et al. 1996: 310-336), probability (Jones; et al. 1997: 102-125) and statistics thinking (Jones; et al. 2000: 269-307; Mooney. 2002: 23-63). It will provide a benchmark for teachers and curriculum developers to develop instruction programs in the critical area of algebraic thinking.

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APPENDIX A

NAMES OF THE EXPERTS AND CODERS IN THIS STUDY

Names of the Experts

There are three experts to recommend the algebraic test and an initial algebraic thinking framework in this study as follow:

1. Assoc. Prof. Prompan Udomsin
   Department of Secondary Education, Faculty of Education,
   Chulalongkorn University.

2. Assoc. Prof. Dr. Preecha Nowyenphon
   School of Educational Study, Sukhothai Thammathirat Open University

3. Arjan Proong Intamat
   Samutsakhonburana School, Samutsakhon Province

Names of the Coders

1. Mr. Vishnu Napaphun
   Mathematics Department, Faculty of Science,
   Thaksin University
2. Mr. Rungson Thongsugnok
   Mathematics Department, Faculty of Science,
   Srinakharinwirot University

APPENDIX B

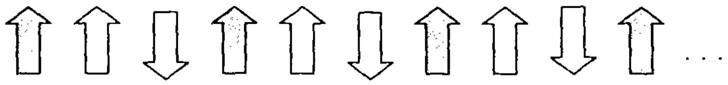
ALGEBRAIC TEST

Algebraic test

Name .... Class ....

School..

1. *Arrow pattern*: From pattern below:



Find (1) What is the next arrow? How do you get it?

(2) What is the 63rd arrow? How do you get it?

(1) Answer

....

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Process of Thinking

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(2) Answer -

....

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Process of Thinking

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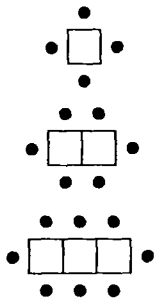
2. *Table pattern*: In the conference room, the tables and chairs are arranged systematically as follows;



1 table with 4 chairs

2 tables with 6 chairs

3 tables with 8 chairs

so on

If we want to arrange tables and chairs according to above pattern by using 54 tables put next to each other to be a long table. How many chairs do we need? Explain how you think along with reason?

Answer

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Process of Thinking

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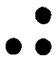
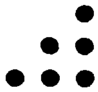
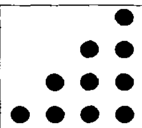
3. *Dot pattern*: From the table below, find the total number of dots in the 5^{th} , 21^{st} and n^{th} position and explain how you get the answer? (n is any positive integer)

Position	1	2	3	4	5	...	21	. . .	n
Picture						...		...	
# of dots	1	3	6	10	$\boxed{}$		$\boxed{}$		$\boxed{}$
					(1)		(2)		(3)

<u>Process of thinking</u>

4. *Number sequence*: From the table below, fill your answer in □ and also explain your thinking along with reason.

S	1	2	3	4	5	...	70	...	m
T	4	7	10	13	□	...	□	...	□
					(1)		(2)		(3)

When m is any positive integer.

<u>Process of thinking</u>

5. *Mr. Big's journey*: Mr. Big has appointment with his friends at school. He lefl home for school by walking. After 10 minutes he realizes that he is going to be late for the appointment which will be in the next 10 minutes. Draw graph showing time and distance of Mr. Big's journey and also explain your thinking along with reasons.

Reason

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6. *Cinema problem*: The data shows the numbers of audience in the Theater over 2 days. This theater is composed of 120 seats and shows 5 sessions of the movie in each day.

The first day

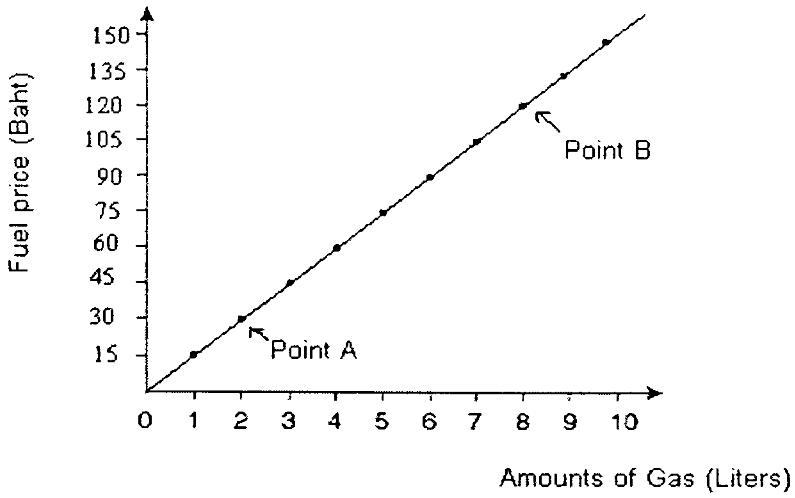
11: 45	12 men and 18 women
14: 15	27 audience and 19 men
16: 45	35 men and 30 women
19: 15	118 audience and 48 women
21: 45	72 men and 43 women

The second day

11: 45	tickets were sold out and 80 men
14: 15	64 men and 54 women
16: 45	tickets were sold out and 79 women
19: 15	87 men and 33 women
21: 45	91 men and 29 women

Represent the data above in the form of table and explain what do you learn from your table.

7. *Gasoline graph*: Graph shows fueling gas, which costs 17.50 baht per liter. Answer the following questions.



1) What information do you get from the point A?

Answer ....

....

....

2) What information do you get from the point B?

Answer ....

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....

3) Besides information you get from 1) and 2), what do you know from the given graph?

Answer

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8. Answer the question below and explain how you get the answer.

8.1) Which is larger, $n+n$ or $n+6$

Answer..

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8.2) What can you say about b If $b+d=18$ and $b \langle d$?

Answer ....

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8.3) $x(\frac{1}{x})=1$, find the value of x

Answer ....

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....

8.4) $a = a \times 1$, find the value of a

Answer ....

8.5) What can say about y if $y = \frac{1}{2}x$ and x is any number?

Answer ....

8.6) $2x - 3 = \frac{4(2x - 3)}{4}$, find the value of x

Answer ....

8.7) $(-a) \times b = (-b) \times a$ or not? Why?

Answer ....

CURRICULUM VITAE

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