

USING INSTRUCTION TO TRACE THINKING AND BELIEFS
IN PROBABILITY OF PRIMARY STUDENTS

A DISSERTATION

BY

SINCHAI JANSEM

Presented in Partial Fulfillment of the Requirements for the
Doctor of Education in Mathematics Education
at Srinakharinwirot University

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USING INSTRUCTION TO TRACE THINKING AND BELIEFS
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AN ABSTRACT
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The purposes of this study were to trace the emergence and evolution of features of primary students' growth in probabilistic thinking, examine the beliefs that these students exhibit as they reason in probability situations, and identify the evolution of sociomathematical norms and classroom mathematical practices that the students develop collectively as they tackle probability problems. A whole class teaching experiment was used as the research method.

Students were from two classes of primary school, one class from each level of Pratomsuksa 3 and 5 respectively. In each class, 6 students were purposefully sampled in order to create a case study. Before the teaching experiment started, these students were interviewed to measure the performance scores and levels of probabilistic thinking (pre-assessment). After instruction, the target students were assessed again two times, one week after the last session (post-assessment) and four weeks after the post-assessment (retention-assessment).

The analyses of students' performance scores revealed that there was a significant difference between pre- and post-assessment scores of all target students ($p < .003$). On the other hand there was no significant difference between post- and retention performance scores ($p = .592$). For Pratomsuksa 3, there was a significant difference between pre- and post-assessment scores ($p < .029$) but no significant difference between post- and retention-assessment scores ($p = .279$). Similarly, for Pratomsuksa 5, there was a significant difference between pre- and post-assessment scores ($p < .029$) but no significant difference between post- and retention-assessment scores ($p = .917$).

Analyses of students' probabilistic thinking development revealed the existence of these patterns: (a) a *stronger part-part schema* that enabled students to reason with consistency when listing the complete set of outcomes of a probability situation and when

ordering probabilities; however, this schema did not enable students to compare probability across sample spaces; (b) an *extreme part-part schema* that was effective in enabling students to identify, order, and sometimes compare probabilities across sample spaces; (c) a *part-whole schema* that students developed after the emergence of *invented informal language*, for describing probability measures.

The students' beliefs about probability that were captured during the assessments and instruction were concerned with representativeness outcome approach, positive and negative recency effect and belief about a non-contiguous probability generator. However, these students' beliefs decreased after instruction.

Sociomathematical norms and their corresponding classroom mathematical practices emerged during the teaching experiment. The emergence of the sociomathematical norm "What counts as a valid way to describe and predict probability" was linked to the classroom mathematical practices "Using invented informal language (e.g., 3 in 8 or 3 from 8)" and "Connecting sample space composition and probability prediction." A second sociomathematical norm "What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations" generated the classroom mathematical practice "Listing all outcomes systematically."

การใช้การสอนเพื่อสืบค้นความคิดและความเชื่อเรื่องความน่าจะเป็น
ของนักเรียนระดับประถมศึกษา

บทคัดย่อ
ของ
สินชัย จันทร์โสม

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่ง
ของการศึกษาตามหลักสูตรปริญญาการศึกษาดุริยบัณฑิต
สาขาวิชาคณิตศาสตร์ศึกษา

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สินชัย จันทน์เสมอ. (2548). *การใช้การสอนเพื่อสืบค้นความคิดและความเชื่อเรื่องความน่าจะเป็นของนักเรียนระดับประถมศึกษา*. ปรินญานิพนธ์, กศ.ด. (คณิตศาสตร์ศึกษา). กรุงเทพฯ: บัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ. คณะกรรมการควบคุม: ผู้ช่วยศาสตราจารย์ ดร. สุพจน์ ไชยสังข์, รองศาสตราจารย์ ดร. ปิยวดี วงษ์ใหญ่, ผู้ช่วยศาสตราจารย์ ดร. อองอาจ นัยพัฒน์

จุดประสงค์ของการศึกษานี้เพื่อสืบค้นลักษณะการพัฒนาคิดเชิงความน่าจะเป็นของนักเรียนระดับประถมศึกษา เพื่อตรวจสอบความเชื่อที่นักเรียนแสดงออกในขณะที่ทำให้เหตุผลในสถานการณ์ความน่าจะเป็น และเพื่อระบุบรรทัดฐานทางสังคมที่เกี่ยวกับการทำกิจกรรมเชิงคณิตศาสตร์ที่เกิดขึ้นในชั้นเรียน และระบุการฝึกปฏิบัติเชิงคณิตศาสตร์ในห้องเรียนที่นักเรียนร่วมกันพัฒนาขึ้น ในขณะที่ยกปัญหาเรื่องความน่าจะเป็น วิธีการวิจัยเป็นการทดลองสอนแบบการสอนทั้งชั้นเรียน

นักเรียนที่เข้าร่วมในการศึกษานี้เป็นนักเรียนชั้นประถมศึกษาปีที่ 3 และ 5 ชั้นละหนึ่งห้องเรียน ในแต่ละชั้นเรียน ผู้วิจัยสุ่มตัวอย่างนักเรียน 6 คน เพื่อเป็นนักเรียนเป้าหมายในการศึกษารายกรณี และสัมภาษณ์นักเรียนในกลุ่มนี้ เพื่อวัดคะแนนสมรรถนะ และระดับของความคิดเชิงความน่าจะเป็น 3 ครั้ง คือ การประเมินผลก่อนการเรียน การประเมินผลหลังการเรียน และการประเมินผลเพื่อวัดความคงทน

จากการวิเคราะห์ข้อมูลพบว่า คะแนนสมรรถนะของนักเรียนเป้าหมายทั้งหมด ระหว่างการประเมินผลก่อนการเรียนและประเมินผลหลังการเรียนแตกต่างกันอย่างมีนัยสำคัญ ($p < .003$) แต่คะแนนการประเมินผลหลังการเรียนและการประเมินผลเพื่อวัดความคงทนไม่มีความแตกต่างกัน ($p = .592$) เมื่อวิเคราะห์เฉพาะนักเรียนเป้าหมายในชั้นประถมศึกษาปีที่ 3 พบว่าคะแนนสมรรถนะ ระหว่างการประเมินผลก่อนการเรียนและประเมินผลหลังการเรียนแตกต่างกันอย่างมีนัยสำคัญ ($p < .029$) แต่คะแนนการประเมินผลหลังการเรียนและการประเมินผลเพื่อวัดความคงทนไม่มีความแตกต่างกัน ($p = .279$) ในทำนองเดียวกัน เมื่อวิเคราะห์เฉพาะนักเรียนเป้าหมายในชั้นประถมศึกษาปีที่ 5 พบว่าคะแนนสมรรถนะ ระหว่างการประเมินผลก่อนการเรียนและประเมินผลหลังการเรียนแตกต่างกันอย่างมีนัยสำคัญ ($p < .029$) แต่คะแนนการประเมินผลหลังการเรียนและการประเมินผลเพื่อวัดความคงทนไม่มีความแตกต่างกัน ($p = .917$)

ในการวิเคราะห์เพื่อสืบค้นพัฒนาการด้านความคิดเชิงความน่าจะเป็นของนักเรียนพบว่ามีรูปแบบความคิดของนักเรียน 3 รูปแบบดังนี้ (1) ความคิดที่ใช้ส่วนย่อยต่อส่วนย่อยแบบเข้ม ซึ่งเป็นความคิดที่ให้นักเรียนมีความสม่ำเสมอในการให้เหตุผลด้านการเขียนเซตของผลทั้งหมดในเหตุการณ์ความน่าจะเป็นได้อย่างสมบูรณ์ และมีความสม่ำเสมอในการให้เหตุผลด้านการเรียงลำดับความ

น่าจะเป็น อย่างไรก็ตาม ความคิดนี้ไม่สามารถทำให้นักเรียนเปรียบเทียบความน่าจะเป็นข้ามปริภูมิตัวอย่างได้ (2) ความคิดที่ใช้ส่วนย่อยต่อส่วนย่อยแบบสุดขีด ซึ่งเป็นความคิดทำให้นักเรียนสามารถระบุและเรียงอันดับความน่าจะเป็น และบางครั้งสามารถเปรียบเทียบความน่าจะเป็นข้ามปริภูมิตัวอย่างได้ และ (3) ความคิดที่ใช้ส่วนย่อยต่อจำนวนรวม ซึ่งเป็นความคิดที่นักเรียนพัฒนาขึ้น หลังจากที่นักเรียนสร้างภาษาอย่างไม่เป็นทางการขึ้น ในขณะที่อธิบายความน่าจะเป็น

ในระหว่างการประเมินผล และระหว่างการสอน ผลการวิจัยพบว่า นักเรียนมีความเชื่อเรื่องความน่าจะเป็นในด้าน รูปแบบความเป็นตัวแทน การมุ่งเน้นที่ผล ผลกระทบทางบวกและทางลบจากเหตุการณ์ที่เพิ่งเกิดขึ้น และการไม่ต่อเนื่องของเหตุการณ์แบบเดียวกันในสถานการณ์ความน่าจะเป็น สถานการณ์หนึ่ง อย่างไรก็ตามความเชื่อเหล่านี้ของนักเรียนลดลงหลังจากการทดลองสอน

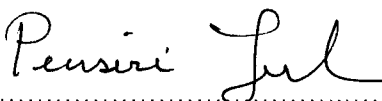
ในระหว่างการสอน ผลการวิจัยพบว่ามี การเกิดขึ้นของบรรทัดฐานทางสังคมที่เกี่ยวกับการทำกิจกรรมเชิงคณิตศาสตร์ และการฝึกปฏิบัติเชิงคณิตศาสตร์ในห้องเรียน ในการเกิดขึ้นของบรรทัดฐานทางสังคมที่เกี่ยวกับการทำกิจกรรมเชิงคณิตศาสตร์แบบ "วิธีการใดที่จะนับว่าเป็นวิธีการที่ดีในการอธิบายและทำนายความน่าจะเป็น" นำไปสู่การเกิดขึ้นของการฝึกปฏิบัติเชิงคณิตศาสตร์ในห้องเรียนแบบ "การใช้ภาษาอย่างไม่เป็นทางการที่สร้างขึ้น (ตัวอย่างเช่น 3 ใน 8 หรือ 3 จาก 8)" และ "การเชื่อมโยงระหว่างความน่าจะเป็น และปริภูมิตัวอย่าง" ส่วนบรรทัดฐานทางสังคมที่เกี่ยวกับการทำกิจกรรมเชิงคณิตศาสตร์แบบ "วิธีการใดที่จะนับว่าเป็นวิธีการที่ดีในการเขียนผลทั้งหมดที่เกิดขึ้นได้ในปริภูมิตัวอย่าง ที่เป็นสถานการณ์แบบ 2 และ 3 มิติ" นำไปสู่การฝึกปฏิบัติเชิงคณิตศาสตร์ในห้องเรียนแบบ "การเขียนเซตที่สมบูรณ์ของผลทั้งหมดอย่างมีระบบ"

The dissertation titled
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by

Mr. Sinchai Jansem

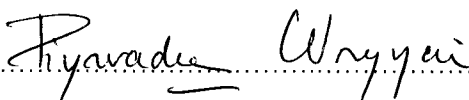
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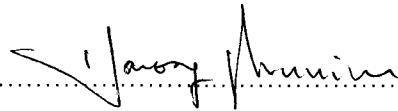
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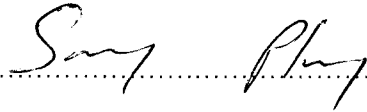
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CHAPTER 1

INTRODUCTION

The problem and its background

Today Thailand is in a period of education reform. During this reform, there were topics discussed among Thai educators about what and how should be taught in classrooms. Consequently, the curriculum was changed under this reform. In the mathematics curriculum, there are new topics that were added in the new curriculum. Probability topics, for example, were added in the Prathomsuksa 4 - 6 curriculum, whereas in the prior primary school mathematics curriculum, there was no probability topic. In this period of education reform, one of the key issues that confront mathematics educators or teachers is what teaching might be like if it were built on an epistemological view of knowledge development (Simon. 1995 : 115). An approach to addressing this problem is using research based knowledge of students' thinking to inform instruction (Carpenter & Fennema. 1989 : 501). Therefore, it is necessary for Thai educators to develop research about teaching and learning. The knowledge from those studies may provide suitable teaching strategies for Thai education while using the new curriculum.

In the area of research on teaching and learning probability, there are a number of studies involving clinical interviews of individual students confronting probability tasks, and they are categorized in two ways. The first category is the studies that are concerned with observing students' thinking over time and mapping out stages of cognitive development in probabilistic thinking (Fischbein. 1975; Piaget & Inhelder. 1975; Falk & Levin. 1980; Fischbein & Gazit. 1984; Fischbein & Snarch. 1997; Jones; et al. 1997). The second category is the studies that are concerned with examining students' probabilistic conceptions and misconceptions (Green. 1983; Kahneman & Tversky. 1972; Tversky & Kahneman. 1983; Konold. 1988; Fischbein, Nello & Marino. 1991; Konold; et al. 1993). The study of Piaget & Inhelder (1975) is accepted as the seminal study of research on probabilistic thinking (Polaki. 2000 : 27). They interviewed young children to mark a turning point in what was then known about the development of probabilistic thinking. They posited the existence of three stages through which children progress in their development of

probabilistic thinking: (a) inability to distinguish between certain and possible events, and the absence of the concept of uncertainty, (b) recognition of the distinction between certain and possible events, and unsystematic approaches to listing outcomes, and (c) development of facility for combinatorial analyses, and an understanding of probability as a limit of relative frequency. Another well-known study is that of Fischbein (1975). He characterized the development of children's intuitions across a range of probability constructs including chance, relative frequency, and combinatorial operations.

Although the above paragraph reveals a lot of studies of students' probabilistic thinking, there are very few studies (Tarr. 1997; Jones; et al. 1999; Polaki. 2000) that have attempted to use research-based knowledge of students' thinking to inform instruction in probability. Jones; et al., using an instructional program based on a framework that described students' probabilistic thinking, examined North America third-grade students' growth in probabilistic thinking across four constructs: sample space, probability of an event, probability comparisons and conditional probability. Using a similar approach, Tarr traced the growth in probabilistic thinking of North American students (Grades 7-8) across two constructs; namely, conditional probability and independence. Working with different grades, Polaki designed, implemented, and evaluated two versions of a teaching experiment to trace elementary students' growth in probabilistic thinking in Lesotho, Southern Africa. These studies show evidence that students made significant gains in their probabilistic thinking and they also revealed cognitive mechanisms by which students built and understood probability.

There are very few studies of students' probabilistic thinking that have been conducted in a non-English speaking culture (Polaki. 2000). Especially in Thailand, there is no study in this area. Similar to children in any culture, Thai students have different cultural norms of thinking and interacting among themselves. The study of Amir & Williams (1999 : 409-413) provides the evidence that culture influences children's probabilistic thinking. Therefore there should be studies on students' thinking and interactions while they are doing probability in a Thai context, especially in primary school where this topic is just beginning to be introduced. The findings of the study could help Thai primary school teachers facing the problems of teaching this new content. More especially, it could inform teaching

methodologies for improving students' probabilistic thinking. Such methodologies will be appropriate for the purposes of the new curriculum.

Purposes of the study

For the reasons described in the prior section, there is a need for research that traces students' thinking in probability as they are engaged in an instructional program in a Thai context. This study addressed that need by carrying out a teaching experiment. More specifically, this study was designed for the following purposes:

1. Trace the emergence and evolution of features of primary students' growth in probabilistic thinking;
2. Examine the beliefs that students exhibit as they reason in probability situations.
3. Identify and describe the evolution of sociomathematical norms and classroom mathematical practices that the students develop collectively as they tackle probability problems.

Scope of the study

Population. The population for this study was the Prathomsuksa 3 and 5 students of Wat Bunyarasri School in Chonburi, Thailand.

Subjects. The subjects of this study were the target students. In each class of a Prathomsuksa 3 class and a Prathomsuksa 5 class, there were 6 target students. These students were purposefully sampled in order to create a case study.

Content. The content that was incorporated in the instructional program consisted of 4 concepts of probability constructs: 1) Sample Space; 2) Probability of an Event; 3) Probability Comparisons; and 4) Conditional Probability.

Definition of terms

Probabilistic thinking. In this study the term probabilistic thinking was used to describe children's thinking in response to any probability situation (Jones; et al., 1999). Students may not always use probabilistic thinking when responding to a probability

situation. In fact neither deterministic thinking nor subjective reasoning is strictly probabilistic thinking even though each is used by students in this study.

Teaching experiment. Cobb (1999 : 6-13) described *teaching experiment* as a conceptual tool for documenting changes in students' thinking in mathematics domain. Elements of a teaching experiment methodology include a sequence of teaching episodes. The people who are involved in the teaching episode are a teacher (or researcher/teacher), a witness, and one or more students. During the instructional session, the teacher poses mathematical tasks for the students. As the student is attempting the task, posed by the teacher, the witness and researcher may pose probing questions to facilitate or reveal student understanding. Each session is carefully recorded, through some combination of videotape, audio-tape, teacher notes, witness notes, and student artifacts. Between sessions, the teacher and witness conduct retrospective conceptual analysis of prior sessions in planning for subsequent episodes. The teaching experiment that was used in this study focused on the development of students' thinking in probability.

Sample space. A sample space is the complete set of outcomes for a random experiment (Hogg & Tannis. 1997: 15-28). A probability situation (or situation involving uncertainty) is an activity or random experiment in which multiple outcomes are possible. Thus tossing two coins simultaneously is a probabilistic situation since there are four possible outcomes: HH, TH, HT, and TT. Students are regarded as understanding sample space if they are able to list a complete set of outcomes for both a 1-dimensional experiment (e.g., tossing a coin) and a 2-dimensional experiment (tossing two coins). The ability to construct, recognize and list the elements of a sample space for a random experiment constitutes a starting point to understanding probability of an event and other probability concepts.

Probability of an event. According to Hawkins & Kapadia (1984 : 350-351), there are four conceptions of probability of an event, namely (a) subjectivist theory, (b) classical theory, (c) relative frequency theory and (d) formal probability theory. In this study, the theoretical probability of an event was understood as classical theory. This theory holds that the probability of an event is the ratio of the number of favorable outcomes to the total number of outcomes, assuming that all outcomes are equally possible.

Probability comparisons. According to Jones; et al. (1997 : 105), students' understanding of probability comparisons is measured by their ability to determine and justify: (a) which of two or more probability situations is more likely to generate the target event in a random draw; or (b) whether two probability situations offer the same chance for the target event. In this study, a probability comparison was the concept of comparing or ordering the probabilities of target events.

Conditional probability. This study used a definition of "conditional probability" which recognizes that in some random experiments there is interest only in those outcomes that are elements of a subset B of the sample space S (Hogg & Tanis. 1997 : 245). An example of conditional probability is a random experiment involving without-replacement situations. In this study, students' understanding of conditional probability was demonstrated by their ability to recognize and adjust the probability of an event when it had been changed by the occurrence of another event.

Sociomathematical norm. A sociomathematical norm is the classroom participation structures. It includes making sense of explanations and questioning alternative viewpoints where disagreements exist. These aspects of the participation structure are typical of the mathematical discourse in the classroom (Cobb. 1999 : 6-7). They include, for example, what counts as (a) a mathematical solution, (b) a sophisticated mathematical solution, and (c) an acceptable mathematical solution. These sociomathematical norms are often re-negotiated in the course of a teaching experiment.

Classroom mathematical practices. Classroom mathematical practice is the taken-as-shared way of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas (Cobb. 1999 : 9). Mathematical practices include individual students' mathematical interpretations and solution strategies that become generally understood, accepted and used by the entire classroom community.

Beliefs about probability. Students' conceptions of probability often involve the use of intuitions, misconceptions, and judgmental biases that are grounded in beliefs (Konold. 1993 : 393-395; Tversky & Kahneman. 1974 : 1124-1131). This study considered

students' intuitions, misconceptions and judgmental biases in probability situations as students' beliefs about probability.

Theoretical considerations

To investigate elementary students' growth in probabilistic thinking and to trace students' individual and collective thinking, this study used a whole class teaching experiment (Cobb. 1999; Steffe & Thompson. 2000). A teaching experiment is a conceptual tool for understanding and documenting the development of students' thinking over an extended period (Cobb & Steffe. 1983 : 9-11; Steffe & Thompson. 2000 : 280-282). This study took the form of what Cobb; et al. (1991 : 11) called a "whole class experiment". A whole class teaching experiment was driven by the theoretical assumption that is referred to as the "emergent perspective" (Cobb. 1999 : 13). According to this perspective, knowledge acquisition is not solely based on individual constructions. The relationship between individual constructions and social processes is an important effect for knowledge construction. Thus one of the purposes of a whole class teaching experiment is to analyze classroom interactions. This analysis focuses on students' thinking as individuals and as members of a classroom community.

The teaching experiment in this study is based on the work of Cobb and his colleagues and incorporates the general developmental research cycle (Cobb; et al. 1991 : 123; Cobb. 1999 : 10). This cycle comprises two phases: instructional development, and classroom-based analyses. Instructional development entails designing a sequence of learning activities based on specific instructional theories. Classroom based analysis involves analysis of students' perceptions and interaction. During the process, these phases cycle back on each other.

Figure 1 (next page) graphically depicts the theoretical framework of this study. The figure is used to show the relationship between theories and theoretical indicators in each phase of the teaching experiment that relates to this study. These theories will be described in each phase of the teaching experiment.

Instructional Development Phase. Figure 1 shows the instructional development phase. In this study, this phase is driven by cognitively guided instructional theory

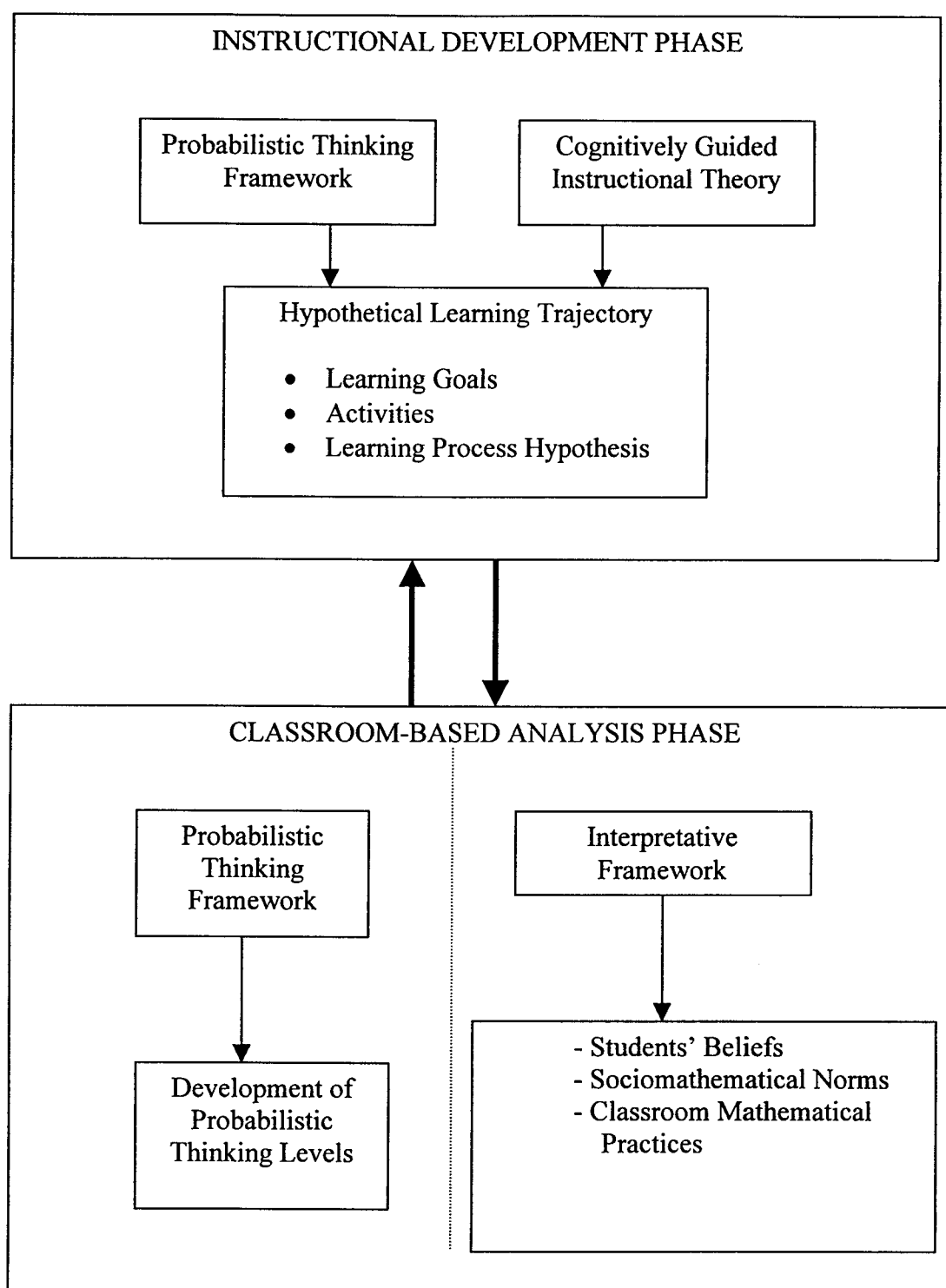


Figure 1 Theoretical Framework

(Carpenter & Fennema. 1989 : 504-506) and a probabilistic-thinking framework (Jones; et. al. 1997 : 11). That is, the probabilistic thinking framework provides the research base to inform instruction using the cognitively guided model. More specifically the probability-thinking framework was used to design and refine a hypothetical learning trajectory (Simon. 1995 : 135-136) for the teaching experiment. In the following paragraphs these theories are described in detail.

Cognitively Guided Instructional theory. The cognitively guided model of instruction is based on the premise that research-based knowledge of students mathematical thinking should be used to inform instruction (Carpenter & Fennema. 1989 : 504-506; Fennema; et. al. 1993 : 302-305). Fennema; et. al. maintain that a teacher with knowledge of research on students' mathematical thinking can use this information combined with individual student problem-solving strategies to select problems that help develop students' mathematical thinking. Teachers who use cognitively guided instructional theory monitor student thinking so as to make sense of student learning in the context of their own knowledge. Much of the early use of research-based knowledge of students' thinking to inform instruction was focused on arithmetic (Carpenter; et al. 1989). Other domains include geometry (Fuys; et al. 1988), fraction (Mack. 1995), ratio and proportion (Lamon. 1993) and probability (Jones; et al. 1999).

Probabilistic Thinking Framework. The Probabilistic Thinking Framework (Framework) constructed by Jones; et al. (1997) is a framework (Figure 2) that describes students' probabilistic thinking. This framework enables young children's probabilistic thinking to be described across four levels for each of four constructs: sample space, probability of an event, probability comparisons and conditional probability. Level 1 is associated with subjective thinking, Level 2 is seen to be transitional between subjective and naive quantitative thinking, Level 3 involves the use of informal quantitative thinking and Level 4 incorporates numerical reasoning.

In the Level 1, students' intuitive thinking often involves myths and mental imaging and is consistent with reasoning that is subjective or focused on irrelevant aspects. In contrast, when faced with probability situations, students at Level 2 demonstrate a readiness to recognize the significance of quantitative measures. They appear to be in a

CONSTRUCT	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Quantitative	Level 4 Numerical
SAMPLE SPACE	lists a complete set of outcome for a one-stage sample space	lists a complete set of outcome for a one-stage sample space, and sometimes list a complete set of outcomes for a two-stage experiment using limited and unsystematic strategies	consistently lists the outcomes of a two-stage experiment using a partially generative strategy	adopts and applies a generative strategy with enables a complete listing of the outcome for two- and three-stage case
PROBABILITY OF AN EVENT	- predicts most/least likely event based on subjective judgments - recognizes certain and impossible events	predicts most/least likely event based on quantitative judgments but may revert to subjective judgments	- predicts most/least likely event based on quantitative judgments including situations involving non-contiguous outcomes - uses numbers informally to compare probabilities - distinguishes "certain," "impossible," and "possible" events, and justifies choice quantitatively	- predicts most/least likely events for single-stage experiments - assigns a numerical probability to an event (it may be a real probability or a form of odds)
PROBABILITY COMPARISONS	- compares the probability of an event in two different sample spaces, usually based on various subjective or numeric judgments - cannot distinguish "fair" probability situations from "unfair" ones.	- make a probability comparisons based on quantitative judgments (may not quantify correctly and may have limitations where non-contiguous events are involved) - begin to distinguish "fair" probability situations from "unfair" ones	- makes probability comparisons based on quantitative judgments - justify with valid quantitative reasoning, but may have limitations where non-contiguous events are involved - distinguishes "fair" and "unfair" probability generators based on valid numerical reasoning	- assigns a numerical probability measure and compares - incorporates non-contiguous and contiguous outcomes in determining probabilities - assigns equal numerical probabilities to equally likely events
CONDITIONAL PROBABILITY	- following one trial or a one-stage experiment, does not give a complete list of outcomes events though a complete list was given prior to the first trial - recognizes when certain and impossible events arise in non-replacement situation	recognizes that the probability of some events changes in a non-replacement situation, however recognition is incomplete and is usually restricted only to events that have previously occurred	- can determine changing probability measures in a non-replacement situation - recognizes that the probability of all events change in a non-replacement situation	- assigns equal numerical probabilities in replacement and non-replacement situation - distinguishes dependent and independent events

Figure 2 Probabilistic Thinking Framework

From: Jones; et al. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics*.

period of transition that is characterized by students' naive and often inflexible attempts to quantify probability. However, they sometimes revert to subjective judgments. Students exhibiting Level 3 thinking typically use quantitative reasoning when dealing with probability tasks. They use more generative strategies in listing outcomes of 2-dimensional situations and have the ability to coordinate and quantify thinking about sample space and probability. Students exhibiting Level 4 thinking make precise connections between sample-space outcomes and their probability, and they are able to use valid numerical measures to describe the probability of events and conditional events.

There are four constructs in the Probabilistic Thinking Framework. This study also focused on all constructs in tracing students' thinking in probability. Students were regarded as understanding sample space if they could provide complete sets of outcomes for 1-dimensional, 2-dimensional and 3-dimensional probability situations. They were identified as understanding probability of an event and probability comparisons if they could predict and justify the most or least likely event. Students' understanding of conditional probability was measured by their ability to recognize when the probability of an event was and was not changed by the occurrence of another event (Jones; et al. 1997 : 104-106).

Figure 1 shows that the Framework is incorporated in two phases of the teaching experiment. The Framework descriptions were used to (a) select or design appropriate probability tasks and learning activities, (b) interpret and support students' probabilistic conception during the teaching experiment, and (c) evaluate the extent to which the students had made gains in their probabilistic thinking. Activities (a) and (b) were concerned with the developmental instruction phase, whereas activity (c) was concerned with the classroom-based analysis phase.

Hypothetical Learning Trajectory. A hypothetical learning trajectory (Simon, 1995) consists of a learning goal, a plan of instructional lessons and activities, and an hypothesis of the learning process for the class. The overall learning goal for this teaching experiment was the development of an understanding of probability with respect to sample space, probability of an event, probability comparisons and conditional probability. The plan of instructional lessons and activities was designed to provide activities and tasks that the teacher used for teaching. The hypothesized learning process was the anticipated way that

the teacher expected students' learning and understanding to evolve when the learning activities took place in the classroom. This process, as formulated by the teacher-researcher, determined the exact nature of activities and problems as well as modifications to the learning process or activities. During the teaching experiment the hypothetical learning trajectory was continuously revised and reformulated as determined by the classroom-based analysis. In this study, both Cognitively Guided Instruction and the Framework informed the hypothetical learning trajectory.

Classroom-based analysis phase. In this study, one purpose was to evaluate classroom instruction. More specifically in this phase the researcher considered students' individual thinking, students' collective thinking, students' beliefs, sociomathematical norms, and classroom-mathematical practices. Figure 1 depicts how the interpretative framework constructed by Cobb (2000 : 308) is used to examine students' individual thinking, students' collective thinking and students' beliefs in probability. In other words, this interpretative framework is used to examine growth in students' probabilistic thinking from both the psychological and the social perspective. The interpretative framework is further discussed in the next paragraph.

Interpretative Framework. Cobb (2000 : 308) developed the interpretative framework to capture different aspects of students' thinking. In this study, the framework was used to analyze classroom activities. There are two dimensions in this framework (Figure 3). The first dimension is the social perspective, and the second dimension is the psychological perspective. The social perspective consisted of three components: classroom social norm, sociomathematical norms, and classroom mathematical practices. The psychological dimension consists of mathematical beliefs, representations and conceptions.

The psychological dimension of Cobb's (2000) interpretative framework is designed to capture aspects that constitute *individual mathematical thinking*. They include beliefs about the nature of a mathematical activity, the role of the individual and others in that activity, *mathematical beliefs* or values, and mathematical conceptions.

<i>Social Perspective</i>	<i>Psychological Perspective</i>
Classroom social norm	Beliefs about our own role, others' roles, and the general nature of mathematical activity
Sociomathematical norm	Specifically mathematical beliefs and values
Classroom mathematical practice	Mathematical conceptions and activity

Figure 3 Interpretative Framework.

From: Cobb, P.(2000). Conducting teaching experiments in collaboration with teachers. in *Handbook of Research Design in Mathematics and Science Education*. A. E. Kelly & R. A. Lesh, editors, p. 308.

Students' beliefs about probability. Beliefs held by students can influence how they reason about a probability situation. In other words, reasoning and beliefs together determine how a student approaches and thinks about problems. There are a number of studies that show that beliefs play a major role in students' reasoning about probability (Fischbein & Gazit. 1984; Fischbein & Schnarch. 1997; Garfield & Ahlgren. 1988; Piaget & Inhelder. 1975; Shaughnessy. 1992). That is, students' conceptions of probability often involve the use of judgment heuristics, such as representativeness, availability, and the outcome approach, that are grounded in beliefs (Konold. 1991(a) : 145; Konold; et al. 1993 : 339-404). Moreover, students' beliefs in such judgment heuristics indicate that they may not have developed a cognitive framework in which to understand the mathematical relevance of a probability situation (Shaughnessy. 1992 : 154-157). In order to cast as wide a net as possible in relation to students' beliefs, Schoenfeld's (1985 : 123) characterization was used in this study. Schoenfeld viewed beliefs as a lens through which an individual sees and approaches mathematics. Students' conceptions of probability often involve the use of intuitions, misconceptions, and judgmental biases that are grounded in beliefs (Konold. 1991(a) : 145; Konold; et al. 1993 : 154-157). In this study, one purpose was to examine the beliefs that students exhibited as they reasoned about a probability situation. Thus,

students' intuitions, misconceptions and judgmental biases, were considered as students' beliefs.

Significance of the study

In view of the void in the literature regarding students' growth in probabilistic thinking in non-English speaking contexts, this study added knowledge and concerns in a new cultural context.. Especially in Thailand and Asia, there was no study in this area and such research was becoming increasingly important with probability being introduced into the elementary curriculum.

Moreover, the findings derived from theoretical and practical implications could provide guidelines for teaching and learning probabilistic thinking in elementary schools. Tasks designed before and during the research period also provided the means of developing Thai students' probabilistic thinking. Mathematics teachers can use the finding and the tasks as a source of reference in designing and carrying out probability lessons for elementary school students.

Finally, the results of this study provided information for the mathematics curriculum developers for judging probability content and sequence in elementary mathematics.

CHAPTER 2

REVIEW OF RELATED LITERATURE

This chapter provides a general overview of research on children's thinking in probability and a discussion of theories that are concerned with this study. There are four parts to this chapter: research on students' probabilistic thinking, related theories on probability instruction, research on students' beliefs about probability, and a review of the method of qualitative analysis.

The first part consists of three sections. The first section discusses the early research on development of probabilistic thinking of Piaget & Inhelder (1975) and Fischbein (1975). The second section examines research about the probabilistic thinking of individual students. It begins with research about students' probability conceptions and misconceptions and then focuses on studies about the probabilistic thinking of individuals during instruction. Finally, the third section discusses teaching experiments that deal with probabilistic thinking.

The second part describes the details of the methodology that is associated with the teaching experiment, learning trajectories, cognitively guided instruction, the interpretative framework, the Probabilistic Thinking Framework.

The third part describes the details of students' beliefs about probability and provides the research that relates students' beliefs about probability. Finally, the fourth part provides the data analysis to be used in the process of qualitative analysis.

Research on probabilistic thinking

Early research on probabilistic thinking. A seminal study of probabilistic thinking is the study of Piaget and Inhelder (1975). They studied children's thinking in probability by using interviews and experiments. They found that it was possible to identify three stages of probabilistic thinking. Students in the first stage (aged about 4 to 7 years) did not differentiate between the possible and the necessary; that is, between random and deterministic situations. Supported by the later study of Falk (1983 : 714-716) who studied children aged 5 to 7. She asked the children to determine which urn should be selected to

get the desired color of balls. Children selected their favorite color, or simply chose a color and indicated that the color they had selected would definitely occur; they did not describe likelihood or chance. Way (1996 : 419-526) found supporting evidence when children were asked to place colored bears into a box to match the structure of a biased spinner. Of the 32 students of ages 5-12 reported on, fewer than 20% had responded correctly. No children under 9 years of age were correct. This may be an indication of how difficult it is for children to make connections and comparisons among different probability generators.

In Piaget's second stage (aged about 6 or 7 to 10 or 12 years), students indicated that probabilistic experiments would be reversible, much as subtraction reverses addition. For example, a child might predict that tipping a tray with marbles would rearrange the marbles, but repeating the process would return the marbles to their original location. The child may have demonstrated a global concept of random mixture, but did not really comprehend what the process of randomization meant. Supporting, in part, Piaget's description of children at the second stage, Green (1988 : 287-291) studied 1600 primary school students. He found that children had a sound conceptual awareness of randomness. Nevertheless, a child in this stage might find it difficult to estimate or compare probability. Falk (1983 : 714-716) found that children in this age range were able to predict more successfully than were younger children which urn would be more likely to yield a desired color ball. However, there were still limitations: the older students were not always correct in their responses and they used informal reasoning to justify their predictions.

In Piaget's third stage (aged about 11 or 12 though adulthood), students might apply deduction and logic to probabilistic situations. They recognized that if a board with marbles was to be tripped and returned, it would take a large number of trials before the marbles might return to their original locations and that this would be a very rare occurrence. It might well be that many students would not achieve this stage without instruction.

Although Fischbein (1975 : 5-20) reached conclusions that were consistent with the theory developed by Piaget and Inhelder (1975) with regard to stages of development of probabilistic thinking. However, there is a number of ways in which Fischbein differs from Piaget in this description of children's development of probabilistic ideas. Whereas Piaget

tended to focus on the development of theoretical aspects of probability, Fischbein investigated the intuitive foundations of probabilistic thinking.

More importantly, Fischbein (1975 : 21-31) characterized the development of children's intuitions across a range of probability constructs including chance, relative frequency, and combinatorial operations.

According to Fischbein, the development of primary probability intuitions amongst young children is marked by: (a) subjectivism which is underscored by the tendency to impose arbitrariness where objective randomness exists, (b) passive induction which involves the tendency to reach decisions on the basis of the immediately preceding facts, (c) the belief that random events can be controlled, and (d) inability to draw a distinction between random and certain events. Contrary to Piaget and Inhelder's (1975) finding, Fischbein observed that (a) an instructional program was able to correct some children's probabilistic intuition, and (b) the absence of the so called proportionality schema amongst these children did not appear to hinder their development of probability concepts.

This section has presented some of the research on the most significant probabilistic thinking of students. These studies have been the basis of the later research on probabilistic thinking including the study of Jones; et al. (1997) which developed the Probabilistic Thinking Framework. As mentioned in Chapter I, the Framework will provide the research-based knowledge for informing the instruction in the instructional development phase of the teaching experiment. Understanding of these original studies will provide a background to understanding the Framework.

Research on probabilistic thinking of individual students. The studies of students' probabilistic thinking that are discussed in this section are about students' probabilistic conceptions and misconceptions. Those studies show aspects of students' thinking in probability individually and provide background knowledge for doing research in the area of probabilistic thinking.

The studies of Kahneman & Tversky (1972 : 440-442), Tversky & Kahneman (1982 : 122-123), Konold (1989 : 81-84) and Konold; et al. (1993 : 402-404) suggest the existence of two major thinking tendencies that often become evident when people make probabilistic judgments: (a) normative reasoning, and (b) judgmental heuristics. According to

Shaughnessy (1992 : 791), normative reasoning involves a belief that the outcomes of a probability experiment are essentially random in nature, and do not depend on some pre-conceived ideas such as patterns of previously observed outcomes. In contrast, judgment heuristics, for example representativeness, involve the tendency to estimate the likelihood of an event according to some criteria based on the extent to which the event resembles the parent population. Variants of this tendency include availability and gambler's fallacy. For example, a student who uses availability might argue that a coin toss is likely to produce heads "because I often get a head when I toss the coin." Unlike a student who uses availability as a basis for making probabilistic judgments, a student who exhibits the gambler's fallacy might expect tails to come after a short run of heads.

Green (1983), Fischbein; et al. (1991) and Tarr (1997) examined middle school students' conceptions of probability. Green tested a group of over 3,000 British students of ages 11-16 on probability tasks. He found that the majority of them tended to confuse "certainty" with "highly probable," and appeared to believe that all outcomes of a random experiment were equally likely. Similarly, Tarr who studied North American fifth graders found that students had a tendency to use the "50-50" phrase in describing (a) any probability situation for which all events were equally likely, and (b) probability situations whose sample space contained two elements that were not equally-likely.

Another misconception has been referred to as the equi-probable bias (Lecoutre. 1992 : 557-568; Konold. 1989(b) : 203-209). It is similar to what Tarr (1997 : 208) described as one misuse of the "50-50 phrase". It is often marked by a tendency to describe all events as equi-probable. Lecoutre found the result of this misconception from what is called the "chance model". According to this "chance model" all events are described as having an equal chance of occurring, irrespective of sample space composition, because their occurrence depends on chance. Lecoutre described a number of instructional conditions that were undertaken in order to address the equi-probable bias. These included the provision and use of frequency information as a basis for determining the most likely event.

Watson; et al. (1997 : 68-69) found that when Australian students in grades 3, 6 and 9 were challenged to make a probabilistic prediction, some of them used phrases such as "it might land on anything" or " no one knows what will happen." The researchers

concluded that these students were acknowledging the existence of a chance aspect in the problem situation even though they stopped short of quantifying it. Konold (1989 : 81-84) made similar observations amongst college students. However, he claimed that the students were using the "outcome approach" in that they tended to interpret a probability situation as requiring the prediction of a single outcome rather than a sequence of outcomes.

I turn now to research that focuses on the probabilistic thinking of individual students during clinical and individual interviews and also on studies that examine students' probabilistic thinking during instruction. The first group of these studies is the research that developed the frameworks to describe students' probabilistic thinking. The framework developed by Jones; et al. (1997) evolved from a review of the literature and observations of young children's thinking. Jones et al. validated this framework by interviewing eight third-grade students across four constructs: sample space, probability of an event, probability comparisons, and conditional probability. This framework suggests that the development of probabilistic thinking proceeds via four distinct levels: subjective, transitional, informal qualitative and numerical. More importantly, these levels are consistent with those that constitute the general development model described by Biggs and Collis (1991 : 66-72).

Using the framework developed by Jones; et al. (1997) as a starting point, Polaki, Lefoka & Jones (2000) developed a framework that described Basotho elementary and middle school students' probabilistic thinking across five constructs: sample space, probability of an event, probability comparison, conditional probability, and independence. Polaki et al. modified a number of original descriptors to reflect different aspects of Basotho students' probabilistic thinking, and were therefore added to the original framework. The framework is situated in Case's theory of cognitive development rather than that of Biggs and Collis (Polaki. 2000 : 19-21).

The second group of studies that is discussed in the following paragraphs concerns probabilistic thinking of individual students and students' probabilistic thinking during instruction.

Hovarth and Lehrer (1998) investigated instruction involving the models of chance and uncertainty with children in second grade. They found that all students made a clear

distinction between activities with certainty like walking from one place to another and activities with uncertainty like rolling dice, but the children still had a "lucky number." Students talked about controlling the outcome of an experiment by rolling the dice in a certain manner. Most students understood that single events had an uncertain outcome but came to believe, through a class activity involving the use of bar graphs, that there were patterns. In this activity, the class collected the data from all students' trials of rolling one or more dice, and recorded the results. For instance, a student might have recorded 5 when using one dice or 1 and 3 when using two six-sided dice. Displaying the data in a bar graph was helpful to students' understanding. Nevertheless, students at this age still accounted for residuals between their predicted and actual results by resorting to conceptions of lucky numbers and other subjective explanations.

English (1993 : 155-273) reported that young children were able to use efficient strategies for listing the outcomes of 2- and 3-dimensional experiments. In particular, these children used six solution strategies ranging in degree of sophistication from trial-and-error to a systematic "odometer strategy." The use of the odometer strategy involves holding one item unchanged while systematically matching it with other items. For instance, in listing the outcomes of an experiment involving a six-sided die and a coin, a student who uses this strategy might first match each of the six possible outcomes of die with a head, and then repeat the process using a tail.

Acredolo; et al. (1989 : 933-945) found that young children were capable of using part-whole relationships in their reasoning even though they often chose not to do so. More specifically, Acredolo; et al. identified three strategies used by children in making probabilistic predictions when doing probability-comparison tasks: (a) a numerator strategy in which they focus on the part that corresponds to the event, (b) an "incomplete denominator" strategy in which students look at the complement of an event, and (c) an integrating strategy in which they recognize the reciprocal relationship between parts and the whole.

Teaching experiment involving probabilistic thinking. Teaching experiments have become an important investigative tool into students' individual and collective thinking. In particular, teaching experiments have been used to illustrate how students reason

statistically and probabilistically in an instructional environment. The teaching experiment also provides the opportunity for teacher-researcher questioning of student reasoning in order to plan the hypothetical learning trajectories. Tarr (1997), Jones et al. (1999) and Polaki (2000) used the teaching experiment as the methodology to trace students' probabilistic thinking in their studies. These studies are described in the following paragraphs.

Jones; et al. (1999) studied the thinking of 37 North American 3rd-grade students in relation to an instructional program in probability. A research-based framework (Jones et al., 1997 : 11) that included a description of students' probabilistic thinking at four levels informed the instructional program. The framework was used to (a) assess students' probabilistic thinking three times during the instructional program: prior to instruction, immediately after instruction, and three months after instruction, (b) design a sequence of suitable learning activities for students, (c) implement the sequence of learning activities, and (d) evaluate the effectiveness of the instructional program. Both an early- and a delayed-instruction group participated in the program. Qualitative evidence from 4 target students revealed that overcoming a misconception in sample space, applying both part-part and part-whole reasoning, and using invented language to describe probabilities were key patterns in producing growth in probabilistic thinking. Moreover, 51% of the students exhibited the latter 2 learning patterns by the end of instruction, and both groups displayed statistically significant growth in probabilistic thinking following the intervention.

Polaki (2000) used two versions of a teaching experiment to compare elementary students' growth in probabilistic thinking in Lesotho. Polaki had 4 theoretical perspectives for conducting this investigation. The first perspective was the teaching experiment which is a tool for finding out the development of students' thinking over a limited period. Also, according to a constructivist teaching experiment, students' knowledge forms by interacting with the social process. The second perspective is based on instructional theory. It guided the researcher to (1) inform the instrument design phase of each version of the teaching experiment; (2) design and continually modify instructional activities on the basis of emerging students' response; and (3) choose appropriate learning activities. The third perspective is the probabilistic thinking framework (Polaki; et al. 2000 : 11) consisting of 5

constructs: sample space, probability of an event, probability comparisons, conditional sample space, and independence. However, the researcher studied only sample space and probability of an event. In addition, there are 4 levels of each construct: (1) subjective, (2) transitional, (3) informal quantitative, and (4) numerical. The last perspective is the interpretative framework. It was used as a guideline for analyzing individual and collective classroom activities associated with both the social and psychological perspectives.

Polaki collected information by designing and implementing 2 versions of the instructional activities in the presence of a research witness. The participants were 6 students from Grade 4 and the other 6 from Grade 5 of an elementary school in a semi-urban area of Lesotho. At each grade level, 2 students had achievement levels at the upper quartile, middle quartile, and lower quartile respectively. The researcher selected the students of the opposite sex from each level to prevent bias. The 12 students were randomly assigned to group A and B with equal numbers from each grade and achievement level. Group A focused on analyses of sample space composition and small sample data while group B examined large-sample experimental data prior to analysis of sample space. Each group met twice a week for 6 weeks. Each meeting/instruction lasted 45 minutes. Prior to the instruction on probability, the researcher visited the two classes and observed them doing mathematics. Data was collected from 6 sources: (1) students' responses to the Assessment Protocol, (2) researcher evaluation of each student's thinking, (3) witness's narratives on students' thinking, (4) students' journal entries, (5) audio and video recordings of students' interaction, and (6) individual and collective students' responses to probability tasks. The researcher also used triangulation from video and students' responses to the protocol before and after instruction.

Polaki and another researcher used a double-coding procedure to determine the thinking level at each assessment period. Non-parametric statistics were used to evaluate growth in levels of probability thinking. The results of the analysis showed that both versions of the teaching experiment had a profound impact on students' thinking. The researcher analyzed the qualitative information collected via the witness's narratives, students' journal entries, audio and video recordings, and individual and collective responses to probability tasks. The study claims the existence of 3 students' thinking patterns—(1) sample space

misconception, (2) a weak and often unstable part-part schema; and (3) a stronger and more stable part-part schema. The main findings were that 2 students sampled from the lower quartile had the sample space misconception and had weak part-part schema to describe 1-dimension situations in probability. With regard to 2-dimensional situations, both students had difficulty in listing complete sets of outcomes and made no connection between probability measures and sample space composition. Two students of the middle quartile could list complete sets of outcomes but struggled in ordering probability. Only one student of this quartile had matured to Level 4 and was able to coordinate part-part and part-whole in ordering probability in both 1-and 2-dimensional situations. Two students of the upper quartile could integrate strategy into logical and meaningful structure. They could make and justify predictions although they experienced difficulty in ordering probabilities in 2-dimensional situations and those using part-whole schema. Based on just witness's narratives on students' thinking, the subjects used minimal student-to-student communication. They were reluctant to sustain discussion. Therefore the researcher and witness had to play the role of mentors in small group settings. Some students showed odometer strategies when listing all possible ways of sample space such as using abbreviations and multiplication rule. Further, the study generated mathematical practices and sociomathematical norms. The mathematical practices that emerged from the teaching experiment included the use of invented mathematical language, a systematic method for generating two-dimensional outcomes, and the use of the composition of the sample space to reason about probability problems. The mathematical practices were considered instrumental in building students' conceptual understanding of probability.

The last study, which used a teaching experiment to trace students' probabilistic thinking and is described, is the study of Tarr (1997). This study traced the growth in probabilistic thinking of North American students (Grades 7-8) across two constructs; namely, conditional probability and independence. The instructional program of his study was developed on the basis of the framework of students' probabilistic thinking namely: The Extended Framework (Jones, Thornton & Tarr. 1999 : 38). Tarr used this framework to generate a series of problem solving tasks that provided a focus for instruction. He reported that the students had a tendency to use the "50-50" phrase in describing (a) any probability

situation for which all events were equally likely, and (b) probability situations whose sample space contained two elements that were not equally-likely. However Tarr noted that whole class discussions to which the students were exposed during the instructional program tended to help them overcome their misuse of the "50-50" phrase.

Related theories

Teaching experiment. Teaching experiments as an instructional and interpretive methodology are relatively new in mathematics education research. In a teaching experiment, the researcher and teacher work closely, often the same person takes on both roles. A teaching experiment may involve a teacher working with an entire class (e.g. Cobb. 1999) or with an individual or group of students (e.g. Simon.1995). In this methodology, each instructional session is planned based upon the analysis of prior sessions, content knowledge, and knowledge of children's thinking. Teaching experiments provide an opportunity for the researcher to gain insight into children's thinking and to note the impact of instructional strategies on learning.

History of teaching experiments. Teaching experiments came to be used in mathematics education research in the U.S. in about 1970 (Steffe & Thompson. 2000 : 269). The two most important reasons for the development of teaching experiments have been: (a) to adapt existing theoretical models and create new models to account for progress that students make as a result of interactive mathematics communication; and (b) to bridge the chasm between practices of research and teaching. Before this time, research on teaching was considered similar to agricultural research, in which the effect of change in treatment on students' performance was measured and tested statistically. This scientific research methodology tried to measure the interactions between students and between students and teachers through frequency counts rather than measures of quality. More qualitative aspects of the students' and teacher's involvement in the learning process were ignored. Classical experiment design inhibited efforts to investigate students' sense-making constraints. Conceptual analyses of mathematics understanding and mathematics performances were too often missing. In the early research, psychometrics dealt with significant differences and student performance was accounted for as the actual score

being made up of a "true score" plus "error." This approach assumed that the "error" aspect was in some sense random, counter to the instincts of mathematics educators. Current mathematics educational research reads errors as indicators of children's misconceptions in their mathematical thinking.

The teaching experiment has been accepted by the research community for several reasons. First, versions of the teaching experiments had already been used successfully by researchers in the Academy of Pedagogical Science in the then U.S.S.R. This methodology seemed intuitively correct to mathematics educators in that children's learning was traced in an instructional environment. Moreover, mathematics education was entering its post-modern period and there were epistemological shifts in the way that mathematics knowledge was understood.

Description of a teaching experiment. Steffe and Thompson (2000 : 267) described what they considered to be the underlying principles and essential elements of a teaching experiment.

A primary purpose for using teaching experiment methodology is for the researchers to experience, firsthand, students' mathematical learning and reasoning. Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students construct or even for suspecting that these concepts and operations may be distinctly different from those of the researcher.

The goals of the researcher in a teaching experiment are to construct models of students' mathematics, and to look for rational grounds for what the students say and do, through conceptual analysis. The teaching experiment methodology is based upon the researcher understanding the progress of students over time, and assisting in the organization of instructional activities and interpretations. It is a living methodology that was initially designed for exploration and explanation of students' mathematical activity. Steffe and Thompson (2000: 278) stated, "The teaching experiment's coherence resides in what

the teacher-researcher can say about bringing forth, sustaining, and modifying students' mathematical schemes."

Elements of a teaching experiment methodology include a sequence of teaching episodes. Three or more people are involved in the teaching episode: the teacher, a witness, and one or more students. During the instructional session, the teacher poses mathematical tasks for the students. As the student is attempting the task, the teacher poses and the witness may pose probing questions to facilitate or reveal student understanding. If more than one student is involved, students may interact in this process. Each session is carefully recorded, through some combination of videotape, audio-tape, teacher notes, witness notes, and student artifacts. Between sessions, the teacher and witness conduct retrospective conceptual analysis of prior sessions in planning for subsequent episodes. A teaching experiment is exploratory teaching, an opportunity for the researcher to learn the students' mathematical thinking firsthand. Throughout this process, the teacher/researcher must attempt to put his/her own concepts and operations to the side and to focus on exploring students' ways and means of operating. The researcher's mathematics concepts may be used for orienting, but are not necessarily what the students should learn. The researcher is trying to learn about what the students can and cannot do.

In describing constraints of the teaching experiment, Steffe and Thompson distinguished between *students' mathematics*, what goes on in the students' minds, and *the mathematics of the students*, the researcher's interpretation of students' mathematics. They suggested that the constraints on research are such that there may always be a difference between the two and that the research must attempt to accept the student's mathematical reality as being distinct from his/her own.

In a teaching experiment, the term teaching is used in a very restrictive sense. Piaget's terms *assimilation* and *accommodation* are used in describing students' processes during a teaching experiment. Accommodation refers to the student's process of adapting or changing his/her own reasoning and understanding of concepts based upon the experiences of the instructional session. Assimilation refers to the process of adopting, or taking as a whole with little change, concepts from the activities experienced during the instructional session. Because a researcher in a teaching experiment often engages in

responsive and intuitive teaching interactions with students, learning how to interact with students is central to this methodology. Sometimes the researcher is unaware of why he/she responds in a given way. During this process, the researcher formulates an image of the students' mental operation and an itinerary of what the students might learn and how they might learn it. This "itinerary" is detailed further below in the discussion of the "learning trajectory" Simon (1995 : 135). The researcher continuously modifies the structure of a students' learning trajectory as it is developed, to fit the mathematical activity of the students. No single observation is taken as an indication of learning or development; a collection of observations that form a pattern allow the researcher to form an hypothesis about students' learning. The hypothesis is then tested in subsequent episodes.

The witness assists the researcher as an additional observer and interpreter. The witness is especially helpful in providing a second opinion when encountering unanticipated and apparently novel ways of operating by the students, along with unexpected mistakes and an inability to operate. The witness, as a person outside the interaction, may interpret students' actions differently and catch important elements of the students' actions that could be missed by the teacher/researcher. Teaching is predicated on understanding of human beings as self-organizing and self-regulating. A witness provides another perspective to assist in understanding the students' reasoning and in planning for the next teaching episode.

Examples of teaching experiments. Tzur(1999) used a teaching experiment with 2 fourth graders to study the co-emergence of teaching and children's construction of an improper fraction concept. The teaching experiment provided an opportunity for Tzur's instructional strategies and the children's fractional reasoning to develop simultaneously through their interactions. Tzur had paired these students because of a common established concept. He conducted 29 videotaped teaching sessions, each about 30 minutes long, one to two times per week. During these sessions, Tzur posed tasks (and the children also posed tasks) for them to solve using a microworld simulation for equi-partitioning and partitive schemes. This microworld allowed the students to investigate fractions through an iterative process. The researcher performed an intertwining analysis of the development of students' understanding and the teacher's adaptation of learning tasks

as well as the teacher-learner interactions that took place during the constructivist teaching experiment. This teaching experiment was an example that is comparable in approach (but not in content or duration) to my teaching experiment.

Confrey (1998) examined how students built mathematical structure to learn basic skills as interconnected ideas within a network of mutually supporting argumentation. She chose the teaching experiment methodology for this research because it allowed her to investigate children's thinking in tool-based investigations grounded in modeling activities; that is, she could learn as she observed and interacted with children who used manipulatives in solving problems. The methodology supported articulation of students' mathematical voices and allowed for careful elaboration, refinement, and differentiation of concepts as intellectual tools through intra- and inter-activity sequencing; that is, sequencing of tasks was analyzed both within one activity and between several different activities. Reflection and practice helped students to stabilize their knowledge and encouraged teachers to examine their own perspectives. Confrey's results indicated a decreasing gap between high and low achievers, showing the viability of a modeling approach with heterogeneous groups through the use of the teaching experiment methodology. Confrey's work provides another example of a successful teaching experiment in which students learned mathematics and the researcher gained insight into children's mathematical thinking.

Cobb (1999) recently completed a classroom teaching experiment focused on statistics. His research goal was to clarify how to analyze students' mathematical reasoning as acts of participation in the mathematical practices established by the classroom community. He found an interconnectedness between instructional development guided by domain-specific instructional design theory and classroom-based analyses guided by an interpretive framework. Analyses of sessions were used to develop successive reorganizations of tasks based upon students' reasoning. Cobb's overarching goal in this teaching experiment was to develop materials that could be distributed for use in other classrooms. This is an example of a teaching experiment used for the development of curriculum.

Hypothetical learning trajectory. Simon (1995 : 132-143) described the pedagogy of the constructivist teaching experiment as it evolved from work on the Construction of Elementary Mathematics (CEM) Project. This project was a 3-year study on the mathematical and pedagogical development of pre-service elementary school teachers. Although his research involved classroom experiences of undergraduate students, his learning trajectory illustrations apply to any teaching experiment. Simon considered teaching to be an active, constructivist process that included both advanced planning and spontaneous decision-making.

Simon (1995 : 135) used the term *hypothetical learning trajectory* to refer to the teacher's (or researcher's) prediction of the path for student's learning. Simon's hypothetical learning trajectory model has been described as consisting of learning goals, learning activities, and a conjectured learning path. The term hypothetical is used because the actual path may differ, since the nature of the child's learning cannot be known in advance. A student's hypothetical learning trajectory is theorized by the teacher/researcher based upon knowledge of student's mathematical thinking, actions, and understandings, and is linked to the researcher/teacher's knowledge of agreed-upon mathematical concepts and strategies. The researcher must use all he/she knows about the child's mathematical knowledge and experiences to create meaningful mathematical tasks that link the student's informal thinking with accepted mathematical reasoning.

The goal of the researcher/teacher is to create a task that will enable the child to build understanding of key mathematical concepts from existing experience and understanding. This learning trajectory may be conjectured before the first session. After each session, the researcher revisits the initially hypothesized learning trajectory and, in the light of the latest session and revisited analysis of prior sessions, the researcher adjusts the hypothesized learning trajectory for each student. Development of each theorized learning trajectory is based upon all the researcher's knowledge of the individual student's thinking and reasoning, analysis of all prior instructional sessions, content knowledge, and pedagogical content knowledge. Development of a learning trajectory is a complex, iterative process.

Cognitively Guided Instructional theory. The instruction used in this study was informed by cognitively guided instructional theory (CGI) (Carpenter & Fennema. 1988). CGI is a theory that integrates what is known about students' cognitive processes with pedagogy. In other words, a basic tenet of CGI is that the teacher makes instructional decisions based on what is known about student's thinking. In their study, Carpenter; et al. (1989 : 504-506) implemented CGI by first informing the teachers on current research on student reasoning about whole number. Teachers were then encouraged to use this information and their own assessment of student knowledge and ability to inform their instruction. Carpenter; et al. found that teachers who put into practice a CGI strategy were more likely to have focused on developing student knowledge through a problem-solving approach than were the control group. Additionally, the CGI teachers tended to promote student use of multiple problem-solving strategies. CGI emphasizes the process of student solutions rather than the solutions themselves in order to provide the teacher with an opportunity to assess student thinking and at the same time encourage a variety of problem-solving strategies so students may connect these strategies to their own knowledge and thus build on it.

The four steps of CGI classroom. Carpenter; et al. (1999 : 60-85) described the four important steps in the process of teaching in a CGI classroom—the teacher poses the problem; the students solve the problem; the students report their solutions and strategies; the teacher and the students discuss the solutions and the strategies. However, there is no optimal way to organize a CGI class. Whatever organization enables a teacher to get the students to solve problems and to listen to the students' problem-solving strategies is the optimal organization for the teacher. The teacher can arrange the students into either a small group or a large group.

The first step of activity in a CGI classroom is asking students to solve the problem. Teachers can select the problems for their purposes. The teacher also selects problem types and number sizes depending on the teacher's intent. For example, if students have trouble with some problems, teachers give them the same type of problem again. For another example, if teachers want students to improve their reporting skills, they select relatively simple problems. Sometimes teachers select problems to encourage students to

move to mature strategies, while other teachers select problems to give children rich experiences in solving a variety of problems.

The contexts of problems posed in a CGI classroom should be interesting, engaging and understandable for children. The contexts should relate to something else the children are doing currently, such as a book they are reading, a social studies unit they are pursuing, or a field trip they are going to take.

After posing the problem, the second step is for the teacher to let students spend a period of time trying to understand the problem and to help them until the teacher is sure that students understand it all. The important thing during students' solving of problems in a CGI classroom is that students have to be comfortable and use materials and tools if they need to. Teachers should provide manipulative materials and tools frequently in a CGI classroom.

After the teacher has selected a problem, posed it to the students, and waited a suitable period of time, the teacher asks the children individually to report how they solved the problem to the class or a small group (the third step). The teacher should ask several students to report their solution strategies. During students' reports of their strategies, the teacher asks questions to ensure that students' reports represent their thinking. For example, the teacher might ask the following questions: Why did you start with...? How did you solve that problem? Tell me how you got the answer? Can you tell me what are you thinking?

The last step of teaching in a CGI classroom is discussion of students' solutions and strategies. After a number of students have reported, the rest of the students discuss different solutions and strategies they have learned. The teacher can be the leader of the discussion by asking a question such as: How are two solutions alike or different and has anyone solved it a different way? The teacher should continue the discussion by asking probing questions based on what the students say or do.

The role of a CGI teacher. The important activity in a CGI classroom is to understand students' mathematical thinking and the role of the teacher is asking students to solve problems. The strategies students use to solve problems help the teacher understand how students think. One of the teacher's roles is to provide manipulative materials and the

teacher should prepare them before class. The other important role of a CGI teacher is that he or she has to encourage students to report their thinking and attend carefully to what other students report. For this purpose, the teacher should create a learning environment where all students are able to communicate about their thinking and feel good about themselves in relation to mathematics. The teacher should select activities that will engage all children and enable their mathematical knowledge to grow. Additionally, teachers also learn about their students as they observe, listen to, and question children during the ongoing instructional process. Moreover, teachers must always update their knowledge about how each student thinks.

Interpretative Framework. There are numerous studies that question an exclusive preoccupation with the individual learner in mathematics classroom (e.g. Cobb & Bauersfeld. 1995; Greeno. 1991; Lave. 1988 and Saxe. 1991). The context and culture of a classroom are regarded as important factors in students' mathematical thinking. The study of M. Simon (1995), the researcher who took a strong psychological perspective, acknowledges the influence of social interaction and the relationship between individual psychological processes and the classroom social process. The study of Vygotsky tends to elevate social processes above psychological processes. For example, it is argued in some theories that the quality of students' mathematics thinking is generated by or derived from the organizational features of social activities in which they participated (Cobb. 1993 : 25). From these studies, Cobb (2000 : 311) mentioned that individual students' activity is seen to be located both within the classroom microculture and within broader systems of activity that constitute the sociopolitical setting of reform in mathematics education.

The classroom teaching experiment described by Cobb (1999 : 5-13), was designed to address ways of investigating students' mathematical learning in alternative classroom contexts developed in collaboration with teachers. The focus on students' mathematical development leads to a number of related issues. One of these concerns is the social context of development. The classroom microculture in which students participate influences profoundly the goals they attempt to achieve, their understanding of what counts as an acceptable mathematical explanation, and their general beliefs about what it means to know and do mathematics in school. Consequently, it is essential to

document the microculture established by the classroom community even if the intent is to formulate a psychological model of processes by which students transform their mathematical activity. To capture these processes, Cobb (1995 : 26) focused on what he called the emergent perspective. One of its assumptions is that learning can be characterized as both a process of active individual construction and a process of mathematical enculturation.

Cobb (1999 : 13-27) described the emergent perspective into two different functions. On the one hand, the emergent perspective goes beyond exclusively psychological approaches by viewing students' mathematical activity as being necessarily socially situated. Therefore, the products of students' mathematical development—increasingly sophisticated ways of reasoning—are seen to be related to their participation in particular communities of practice such as those constituted by the teacher and the students in the classroom. On the other hand, the emergent perspective questions the subordination of psychological processes to social processes and attributes a central role to analyses of individual students' mathematical activity. The basic relationship posited between students' constructive activities and the social processes in which they participate in the classroom is one of reflexivity in which neither is given preeminence over the other. In this view, students are considered to contribute to evolving classroom mathematical practice as they reorganize their individual mathematical activities. Conversely, the ways in which they make these organizations are constrained by their participation in the evolving classroom practices. Finally, Cobb concluded that a basic assumption of the emergent perspective is that neither individual students' activities nor classroom mathematical practices can be accounted for adequately except in relation to the other; and theoretically, the emergent perspective emphasizes the importance of analyzing students' mathematical activity as it is situated in the social context of the classroom.

For analyzing individual and collective mathematical activity in the classroom, Cobb (2000) developed an interpretative framework (see Figure 3). The framework has two columns, social perspective and psychological perspective that refer to the two reflexively related perspectives. The psychological perspective comprises the emergent viewpoints

that correlate social perspective. The details of this correlation will be explained in the following paragraphs.

Classroom social norms. Although Cobb; et al. (1991 : 5-8) claimed that classroom social norms pertain to classroom participation structure and include making sense of explanations and questioning alternative viewpoints where disagreements exist, Cobb (2000 :309-311) focused on only the norm of classroom discussion which he called the re-negotiation process. Example of norms include explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations where a conflict in interpretations or solutions has become apparent. From the emergent perspective, in a classroom where disagreements exist, the role of the teacher is in initiating and guiding the re-negotiation process. The students have to play their part in contributing to the establishment of social norms. In making these contributions, students reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity. As a consequence, these beliefs are taken to be the psychological correlates of the classroom social norm (see Figure 3).

Sociomathematical norms. The sociomathematical norms are mathematics-specific social norms. Only the renegotiations of mathematical activities are focused to account as sociomathematical norms. Examples of such sociomathematical norms include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation and justification. The negotiation process of different mathematical solutions, for example, emerges because students do not know what would constitute a mathematical difference until the teacher and other students accept some of their contributions but not others.

When students establish sociomathematical norms, students are constructing specifically mathematical beliefs and values that enable them to act as increasingly autonomous members of a classroom mathematical community (Yackel & Cobb. 1996 : 461-463). These beliefs and values are psychological constructs and constitute a mathematical disposition. As shown in Figure 3, they are taken to be the psychological correlates of the sociomathematical norms.

Classroom mathematical practices. Cobb (1999 : 9) defined a *mathematical practice* as " the taken-as-shared way of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas." Mathematical practices include individual students' mathematical interpretations and solution strategies that become generally understood, accepted and used by the entire classroom community. In a classroom, for example, a student might find it useful to rewrite $59 + 8$ as $60 + 7$, which easily simplifies to 67 (Polaki. 2000 : 69). As dictated by sociomathematical norms, such a student is often obliged to provide a reasonable explanation of how and why the strategy works. In the course of discussion, the strategy becomes taken-as-shared by the classroom community. Thus the strategy is translated into the status of a classroom mathematical practice because it no longer needs detailed explanation.

Although classroom mathematical practices are established in the classroom community, they can be seen to constitute the immediate local situation of the students' mathematical development. In identifying sequences of such practices, the research analysis documents the evolving social situations in which students participate and learn. Individual students' mathematical conceptions and activities are taken as psychological correlates of these practices, and the relationship between them is considered to be reflexive.

Examples of studies involving sociomathematical norms and classroom mathematical practices. The studies of Polaki (2000) and Zimmermann (2002) investigated probabilistic thinking and also examined the emergence of sociomathematical norms and mathematical practices. Polaki focused on mathematical practices that emerged during a teaching experiment in a probability classroom. The mathematical practices that emerged from the teaching experiment of his study were (1) invented informal language, (2) odometer strategy, (3) basing predictions on sample space composition, and (4) describing probabilities quantitatively. These mathematical practices were considered instrumental in building students' conceptual understanding of probability. The detail of these classroom mathematical practices described by the researcher will be summarized in the next paragraphs.

One example of invented informal language was the phrase that students used to describe probability quantitatively. In the first session, one student used the phrase "5 out of 8" to answer the probability problem for which the answer was $\frac{5}{8}$. By the second session more than half the students had begun to use "... out of ..." for describing probabilities. Polaki noted that the invented informal language had become taken-as-shared because most students liked to use this phrase for answering the problems.

The odometer strategy is a generative strategy for listing a set of outcomes for 2-dimensional and higher experiments. In the study of Polaki, at the beginning of the teaching experiment all students used non-generative strategies to list the set of possible outcomes. Subsequently, there were two students who could use the odometer strategy after they were asked to list the set of outcomes systematically. This strategy had become taken-as-shared by the tenth instructional session in that all students had adopted it as their own.

The classroom mathematical practice "basing predictions on sample space composition" is the classroom mathematical practice where students make predictions of probability problems by considering sample space composition. Polaki found that some students could make connections between sample space composition and probability predictions. Nevertheless, this mathematical idea became taken-as-shared only in 1-dimensional situations.

Describing probabilities quantitatively is when students reason and order probabilities by using numbers. Polaki (2000 : 99-100) showed that there were some students who used the phrase "... (number) ... out of ... (number) ..." or a fraction to order or describe probability situations. However, describing probabilities qualitatively was never sufficiently understood in a collective sense to become taken-as-shared.

The study of Zimmermann (2002) examined the role instruction played in changing students' individual and collective reasoning and beliefs about probability simulation. It also examined the impact technology had in these changes. A teaching experiment was the research methodology of this study. As students reasoned collectively during the teaching experiment, a number of sociomathematical norms emerged about how students justified valid simulation components. These sociomathematical norms were: (1) what counts as a valid simulation, (2) what counts as a valid probability generator, (3) what counts as a valid

trial, and (4) what counts as a sufficient number of trials. For each sociomathematical norm, there were corresponding classroom mathematical practices. Figure 4 in the next page shows the summary of the sociomathematical norms and classroom mathematical practices that emerged in this study.

<i>Sociomathematical Norms</i>	<i>Classroom Mathematical Practices</i>
What counts as a valid simulation	● Must use a valid probability generator ● Need to ensure a trial is valid ● Must conduct at least 30 trials per group
What counts as a valid probability generator	● Must have 1-1 correspondence to probabilities given in problems ● Will accept equivalent probability generators ● Will use the calculator as the mechanism for generating a trial
What counts as a valid trial	● Number of outcomes in a trial must correspond to target question
What counts as a sufficient number of trials	● Will combine results of 10 trials per person in a group ● Number of trials depends on whether we are dealing with an "ideal" or a "practical" situation

Figure 4 Summary of Sociomathematical norms and Classroom Mathematical Practices Of Zimmermann. From: Zimmermann, G. (2002). *Students' reasoning about probability simulations during instruction*. Unpublished doctoral dissertation, Illinois State University. Normal. Photocopies. p 149.

Probabilistic Thinking Framework. On the basis of the cognitive framework (Figure 2), Jones; et al. (1997 : 102) claimed that probabilistic thinking is multifaceted and develops slowly over time. To capture the manifold nature of probabilistic thinking and its interconnections, in this framework they incorporated four key constructs: sample space, probability of an event, probability comparisons, and conditional probability, as they related to one-stage (e.g., tossing a coin) or two-stage (e.g., tossing two coins) probability situations. In this study a probability situation, or a situation involving uncertainty, referred to an activity or random experiment for which multiple outcomes are possible; that is, the actual outcome cannot be predetermined exactly. However, the set of possible outcomes for the probability situation can be determined as can the long-term distribution of these outcomes (Hogg & Tanis, 1997 : 22).

The framework was initially constructed from a 2-year study of students' probabilistic thinking and from a synthesis of previous probability research in sample space, e.g., Borovcnik & Bentz. (1991); English. (1993), in probability of an event, e.g., Acredolo et al. (1989); Fischbein et al. (1991), in probability comparisons, e.g., Falk. (1987); Fischbein; et al. (1991), and in conditional probability, e.g., Borovcnik & Bentz. (1991).

Through validation of the initial framework (Jones; et al. 1997 : 107), the researcher confirmed the existence of four levels of probabilistic thinking across each of the four constructs. Level 1 is associated with subjective thinking; Level 2 is transitional between subjective and naive quantitative thinking; Level 3 involves the use of informal quantitative thinking; and Level 4 incorporates numerical reasoning. These levels of probabilistic thinking appear to be consistent with a neo-Piagetian general developmental model developed by Biggs & Collis (1991 : 55-57)

Incorporated in this developmental model are five modes of functioning (sensorimotor--from birth, ikonic--from around 18 months, concrete-symbolic--from around 6 years, formal--from around 14 years, and postformal--from around 20 years) and four cognitive levels (prestructural, unistructural, multistructural, and relational) that recycle during each mode and represent shifts in the complexity of students' reasoning. According to Biggs & Collis (1991), each of the five modes of functioning emerges and develops in a way that incorporates the continuing development of earlier modes. The modes that are

most applicable to the students in their research are the ikonic and the concrete-symbolic, although such students will also use some sensorimotor functioning.

Jones; et al. claim that the four levels of probabilistic thinking described in their framework (subjective, transitional, quantitative, and numerical) correspond to the four levels of cognitive thinking identified by Biggs & Collis (1991 : 65-68). Students exhibiting Level 1 thinking are narrowly and consistently bound to subjective reasoning, that is, reasoning like "Red, because it is my favorite color." As such, these students appear to reflect what Biggs & Collis describe as the prestructural thinking level of the learning cycle: the learner is engaged in the task but is then distracted or misled by an irrelevant aspect. According to Biggs & Collis, although the prestructural level is associated with the concrete-symbolic mode, prestructural responses really belong in the ikonic mode. In this latter mode students' intuitive thinking often involves myths and mental imaging and as such is consistent with reasoning that is subjective or focused on irrelevant aspects.

In contrast to students at Level 1, students exhibiting Level 2 thinking, when faced with probability situations, demonstrate a readiness to recognize the significance of quantitative measures. Consequently, Level 2 appears to be a period of transition that is characterized by students' naive and often inflexible attempts to quantify probabilities. Their thinking is more indicative of what Biggs & Collis (1991 : 72-74) term the unistructural level in the sense that the task engages the students in a relevant way, but generally only one aspect is pursued. Biggs and Collis claimed that these students are beginning to function in the concrete-symbolic mode and, as such, have the potential to represent their thinking using mathematical language and symbols. However, there is potential for conflict when such students try to reconcile ikonic intuitions and concrete-symbolic reasoning (Watson, Collis & Moritz. 1997 : 70-72), and this conflict appears to manifest itself through reversions to subjective judgments.

Students exhibiting Level 3 thinking typically use quantitative reasoning when dealing with probability tasks. Thinking at this level is manifested through students' use of more generative strategies in listing the outcomes of two-stage experiments and through the ability to coordinate and quantify thinking about sample space and probability. Students assessed at Level 3 appear to exhibit characteristics of both the multistructural and

relational levels within the concrete-symbolic mode (Biggs & Collis. 1991: 72-74) in the sense that they recognize more than one relevant feature of a task and try to integrate features such as changing sample spaces and probabilities in without-replacement situations. Students exhibiting Level 4 thinking make precise connections between sample-space outcomes and their probabilities, and they are able to use valid numerical measures to describe the probabilities of events and conditional events. Their observations indicate that their thinking is consistent with the characteristics of the relational level within the concrete-symbolic mode. That is, they now integrate relevant aspects of the task into a meaningful structure and generate abstractions in probability as long as these abstractions are related to their "experienced world". Notwithstanding the capabilities of students who exhibit Level 4 thinking, they may be limited by the scope of their mathematical experiences, which generally do not include fractions and combinatorics.

In formulating and refining the framework (Figure 2), Jones; et al. (1997 : 110-114) attempted to achieve stability for a student's thinking across the four constructs; that is, when a student's probabilistic thinking was at Level 2 in sample space, it was also expected to be at Level 2 for probability of an event, probability comparisons, and conditional probability. Stability was important because they wanted each level in the framework to represent a consistent picture of probabilistic thinking across all four constructs. From a practical perspective this feature was intended to make the framework more "user-friendly" for teachers when monitoring a student's probabilistic thinking. Although, in the validation study (Jones; et al. 1997 : 120) they demonstrated that the framework provided a coherent picture of probabilistic thinking at each of the four levels of the framework, there was "static in the system" in that each student's thinking levels were not always uniform. They concluded that this static resulted from students' unpredictable tendencies to regress to subjective judgments and from task-related factors associated with the assessment of sample space and conditional probability.

Students' beliefs about probability

A number of studies show that beliefs play a major role in students' reasoning about probability (Fischbein & Gazit. 1984 : 15-16; Fischbein & Schnarch. 1997 : 96-105;

Garfield & Ahlgren. 1988 : 51-52; Piaget & Inhelder. 1975 :105-107; Shaughnessy. 1992 : 478). That is, students' conceptions of probability often involve the use of intuitions, misconceptions and judgmental biases that are grounded in beliefs (Konold. 1991 : 145). Moreover, students' beliefs in such judgment heuristics indicate that they may not have developed a cognitive framework in which to understand the mathematical relevance of a probability situation (Shaughnessy. 1992 : 480). The following paragraphs describe the students' misconceptions and common heuristics along with what is currently known about how people reason in probabilistic situations using each cognitive process.

Representativeness. People who apply the representativeness heuristic are making judgments according to how well an outcome represents some general characteristic or a parent population (Kahneman & Tversky. 1972 : 444-445). Given a fair coin, for example, a person using representativeness would choose an outcome HTHHTT to be more likely to occur than an outcome HHHHHT since the first example illustrates a more even distribution of head and tails. This response is compelling for students using representativeness even though the two outcomes both have probabilities of $1/64$.

Positive and negative recency effect. The misconception of positive recency effect or the causal thinking misconception is the deterministic perspective of a probabilistic situation in which a student indicates that a second trial would be a repeat of the first, with the same result caused by the same action (Borovcnik & Bentz 1991 : 269-270). For example, if a student were to place a dice with the number 6 on top and roll it, getting a 2, that student might expect the same result each time the dice is rolled, with the 6 on top.

The misconception of a negative recency effect takes place when a student expects the result on one trial to be different from the result on a prior trial (Greeno. 1991 : 322-323). For example, if a coin was tossed and heads had come up for one trial, the student might predict tails for the next trial, even though a head and a tail are equally likely. Green studied middle and high school students. He found that 28% of children aged 11 and 12% of those aged 16 selected an incorrect response based upon the negative recency effect misconception. Also, Shaughnessy (1983 : 790-791) reported this same misconception among college age students who assumed that one trial would automatically yield a different result than the prior.

Verbal understanding and ignoring base data misconception. There are two kinds of misconception that relate to a student's understanding about tasks, verbal understanding and ignoring base data (Benson. 2000 : 24-25). Borovcnik and Bentz (1991 : 202-204) discussed the verbal understanding misconception in which the student is influenced by the statement of the task and, as a result, may or may not select the correct response, depending upon what parts of the initial verbal statement the student focuses on. For example, a student is asked to place a two-colored disk red-side-up in hand and to flip the disk. One student may focus on the verbal statement that the disk was held red-side-up and consequently think that the result should be the same on each toss as it was on the first toss because the disk began with the same side up. Another student might focus on the verbal statement that the disk flipped in the air, assuming that the disk would land green-side-up because it began with red and the flip would always change the result side up. This misconception is particularly difficult to distinguish and it may lead to a correct response.

A similar misconception involves ignoring base data. In this situation, students pay no attention to part of the basic information of the task, thus making the probabilistic situation simpler (Bentz & Bonovcnik. 1984 : 268-269.) For example, a student is asked to model a 2- dimensional situation of randomly selecting one of six shirts and one of two pairs of pants for an outfit. One student may simplify the task to selection of a shirt, a 1- dimensional task. Another student may focus on selecting the pants. In either of these cases, the student has simplified the probabilistic reasoning involved by ignoring part of the basic information. Another student might carry out the selection of shirts and pants independently, then pair them to form the shirt-pants outfits.

Availability. The availability heuristic is the determination of the likelihood of an event based on an easily recalled event process. For example, if asked if letter R is more likely to appear in the first position of a word or in the third position, some people may respond that it is more likely to appear in the first position, even though R is more likely to appear in the third position. These persons respond like that because they were able to recall words beginning with R more easily on more recently than words with R as the third letter.

Conjunction fallacy. Many people, both statistically naïve and trained, will claim that some types of conjunctive events are more likely to happen than the parent event. For example, Tversky and Kahneman (1973 : 212) reported that college subjects considered the event that a person was 55 and had a heart attack to be more likely than the event that a person had a heart attack, regardless of age. One possible explanation for this phenomenon may be that the person was using a representativeness or availability heuristic (Tversky & Kahneman. 1973 : 214-215). In other words, people naturally connect the attributes of age with heart attack and, therefore, consider the conjunctive event more likely. Shaughnessy (1992 : 485) suggested that subjects were in fact confusing conjunction and conditional probability.

Outcome approach. Konold; et al. (1993 : 398) stated that students who reason using the outcome approach hold the belief that they are being asked to predict the outcome of a single event even when they are being asked to find the probability based on the distribution of occurrences. For example, if a student were asked the likelihood of getting a 5 on the roll of one dice, the student might respond that he/she would get a 2, not that a 5 was as likely as any number. Then, if he/she rolled a 3, the student would say, "I was wrong."

The above heuristics provide valuable tools that help understand and explain common misconceptions or intuitions students have in developing and applying probabilistic reasoning. Heuristics can also prove to be beneficial when developing a framework to assess students' understanding of probability. As an example of how heuristics have been used to study probabilistic reasoning, Fischbien & Schnarch (1997 : 103) found that contrary to their research hypothesis, intuitions of students ranging from 10 years old through college age were not necessarily enhanced with age. The effect of availability decreased with age, and yet the conjunction fallacy remained strong through grade 9 and then decreased in older students. By the way of contrast, the misconception manifest in the availability heuristic gained strength over time. Fischbein & Schnarch revealed that a mixture of problem content, cognitive conceptions, and intuitions influenced the complexity of students' reasoning.

Some research that has examined students' beliefs about probability has focused on the effect of instruction. Because instruction will be germane to this study, the following is a review of some of those studies. Fischbein & Gazit (1984) explored the effects of instruction on probability conceptions of students in Grade 5, 6, 7. It was the hypothesis of these researchers that instruction could be used to alter and develop new intuitive attitudes through participatory activities involving probability concepts. Student understanding did appear to increase with age, however, many of the concepts proved to be too difficult for fifth-grade students. It was also found that instruction which involved students in practical activities had the effect of improving some students' intuitive misconceptions.

Using instruction to change students' misconceptions, delMas & Bart (1989 : 46-50) found that students possess schemata that they bring into the class with them, and these schemata may or may not be accurate. These schemata act as a filter when processing information. If the information fits the schema then it may work to reinforce the process model. Otherwise, if the information is counter to existing schemata, the student may very well ignore it.

Fast (1999 : 237) suggested that the use of analogies to teach probability could influence conceptual change. In his study, students completed version A of a probability testing instrument and then immediately proceeded to version B. Version B was designed to use analogies that were meant to elicit correct responses along with appropriate justifications. For example, in version A, students were asked with birth order if either of BGGBG or BBBBB, was more likely to occur. In version B, students were provided the contextual situation of a 4-digit lottery and asked to compare the chances that a ticket with number 2222 would win with the chances that a ticket with number 2332 would win. In the interview stage, students who incorrectly answered the problem on version A but correctly answered the analogous situation in version B were provided the opportunity to correct their work in version A. Fast concluded that the analogy probe worked to build conceptual knowledge, and students were often able to realize and correct their original mistakes. However, most students did so only with probing questions and guidance by the researcher. The results tend to underscore the importance of the role of the teacher in helping students reflect on common misconception.

This study traced the growth of students' thinking in probability, examined their beliefs about probability and identified the sociomathematical norms and classroom mathematical practices that emerged during instruction. Qualitative analysis will be a key method in addressing these three purposes. For this reason and because qualitative analysis is relatively new in Thailand, the literature on qualitative analysis will review in the next section.

A review of the method of qualitative analysis

As mentioned the main purpose of this study is to trace students' probabilistic thinking. The study is based on the premise that students can develop probability concepts over time; it also adopts the view that mathematical learning is an interactive as well as a constructive process (Jones; et al. 1999 : 491-492). Consistent with this position, students' probability knowledge arose from students' attempts to solve problems, to build on and reorganize their informal knowledge, and to resolve conflicting points of view. This perspective is consistent with an inquiry approach, which requires frequent opportunities for students to "discuss, critique, explain, and when necessary, justify their interpretations and solution" (Cobb; et al. 1991 : 6). The procedure of interviewing and observing will be the process of data collection. Consequently, the qualitative method of analysis needs to analyze that data. The qualitative methods that will be discussed in this section consist of a case-study approach, coding, and three part of analysis.

Case-study approach. The case study approach used in this study is the one that Stake (1998 : 437) named the "intrinsic case study". Stake identified three types of case study namely: intrinsic case study; instrumental case study; and collective case study. An intrinsic case study is a case study that enables a researcher to gain a better understanding of this particular case. The purpose of this type of case study is not theory building. Study is undertaken because of an intrinsic interest in, for example, this particular child, clinic, conference, or curriculum. Whereas an instrumental case study is the particular case that is examined mainly to provide insight into a more general situation. The case is of secondary interest, it plays a supportive role, and it facilitates our understanding of something else. The case still is looked at in depth, its contexts scrutinized, its ordinary activities detailed,

but all because this helps the researcher to pursue the external interest. The case may be seen as typical of other cases or not.

With even less intrinsic interest in one particular case, a researcher may jointly study a number of cases in order to investigate a phenomenon, population, or general condition. It is called a collective case study. Essentially, it is an instrumental study extended to several cases. Individual cases in the collection may or may not be known in advance to manifest some common characteristic. They may be similar or dissimilar, redundancy and variety are both important. They are chosen because it is believed that understanding them will lead to a better understanding about a still larger collection of cases.

Stake (1998 : 448) summarized the major conceptual responsibilities of the qualitative case researcher in the following:

1. Bounding the case, conceptualizing the object of study;
2. Selecting phenomena, themes, or issues—that is, the research questions—to emphasize;
3. Seeking patterns of data to develop the issues;
4. Triangulating key observations and bases for interpretation;
5. Selecting alternative interpretations to pursue;
6. Developing assertions or generalizations about the case.

Except for the first of these, the steps are similar to those taken by other qualitative researchers. The more the researcher has an intrinsic interest in the case, the more the focus of study will be on the case's uniqueness, particular context, issues, and story.

Stake (1998 : 448) has also described the weak and strong features of the case study approach. He mentioned that whereas single or a few cases are a poor representation of a population and poor grounds for advancing a grand generalization, a single negative case example can establish limits to a grand generalization. For example, researchers lose confidence in the generalization that a child of separated parents is better off placed with the mother when research find a single instance of resultant injury. Case

studies are of value in refining theory and suggesting complexities for further investigation, as well as helping to establish the limit of generalizability.

The purpose of a case study is not to represent the world, but to represent the case. Criteria for conducting the kind of research that leads to valid generalization need modification to fit the need for effective particularization. The utility of case research to practitioners and policy makers is in its extension of experience. The methods of qualitative case study are largely the method of disciplining personal and particularized experience.

Codes and coding. Coding is the early step in qualitative analysis. To review a set of field notes, transcribed or synthesized, and to dissect them meaningfully, while keeping the relations between the parts intact, is the stuff of analysis. This part of analysis involves how to differentiate and combine the data that is retrieved and reflected on by the researcher. Codes are tags or labels for assigning units of meaning to the descriptive or inferential information compiled during a study. Codes usually are attached to "chunks" of varying size—words, phrases, sentences, or a whole paragraph, connected or unconnected to a specific setting. They can take the form of a straightforward category label or a more complex one.

Codes are used to retrieve and organize the chunks. The organizing part will entail some system for categorizing the various chunks, so the researcher can quickly find, pull out, and cluster the segments relating to a particular research question, hypothesis, construct, or theme. Clustering, and, as we will see, display of dense chunks, then sets the stage for drawing a conclusion.

Types of codes. There are three types of coding (Miles & Heberman. 1994 : 57) namely: descriptive codes, interpretive code, and pattern codes. A descriptive code entails little interpretation. A descriptive code describes the aspects of the phenomena. On the other hand, an interpretative code is a code where the researcher is attributing a class of phenomena to a segment of text. The same segment could be handled more interpretively.

A class of codes that are pattern codes is even more inferential and explanatory. A coded segment of field notes illustrates an emergent leitmotiv or pattern that researchers have discerned in local events and relationships.

Reliability of coding. Miles & Huberman (1994 : 64) suggest the method of check-coding to ensure that codes are fit for the chunk. This method is called "Double Coding". A double coding procedure is a process where two persons code a transcript independently. The reliability of coding will be determined by using the formula:

$$reliability = \frac{number\ of\ agreements}{total\ number\ of\ agreement + disagreement}$$

To increase the reliability of coding, each coder is well advised to check-code the first dozen pages of field notes or transcripts, once right away and again a few day later. The disagreements have to be discussed before the coding process of the next dozen pages. By this method, the agreements of the two coders will increase. Miles & Huberman (1994 : 64) also mention that check-coding two-thirds of the way through the study is beneficial. A coder may drift into an idiosyncratic sense of what the codes—revised or original—mean in light of new, often unshared insights. And the more interpretive codes, up one notch from the literal sense of the text, are more slippery when more than one researcher is scanning for them.

Qualitative analysis. Miles & Hubermann (1994 : 10-12) define qualitative analysis as consisting of three concurrent flows of activity: data reduction, data display, and conclusion drawing and verification. These processes will be described in the following paragraphs.

Data reduction. Data reduction refers to the process of selecting, focusing, simplifying, abstracting, and transforming the data that appear in field notes or transcriptions. This process occurs continuously throughout the life of any qualitatively oriented project. During the process of data reduction, the researcher will make the decision as to which data chunks to code and which to pull out, which patterns best summarize a number of chunks, and which have an evolving story to tell. Data reduction is a form of

analysis that sharpens, sorts, focuses, discards, and organizes data in such a way that conclusions can be drawn and verified.

Data display. A data display refers to the process of organizing and comparing the data after the process of data reduction. Generically, a display is an organized, compressed assembly of information that permits conclusion drawing and action. There are many ways to display data. They include many types of matrices, graphs, charts, and networks. This process can help the analyst to look over all of the sources of data.

Conclusion drawing and verification. Although, from the start of data collection, the qualitative analyst is beginning to decide what things mean—regularities, patterns, explanations, possible configurations, causal flows, and proposition, it still cannot be the final conclusion. The researcher can draw the final conclusion only when data collection is over depending on the data sources. The researcher can draw a conclusion by considering data in the process of data display. Conclusions are also verified as the analyst proceeds. Verification may be as brief as a fleeting second thought crossing the analyst's mind during writing. Moreover, the analyst and colleagues can collaborate to verify a conclusion by developing the "intersubjective consensus."

CHAPTER 3

METHODOLOGY

The key purposes of this study were to trace the emergence and evolution of features of primary students' growth in probabilistic thinking and examine the beliefs that these students exhibit as they reason in probability situations. A whole class teaching experiment (Cobb, 1999 : 6-13) was used as the research method.

Cognitively guided instructional theory (Carpenter & Fennema, 1989 : 504-506; Fennema; et. al. 1993 : 302-305) and the Probabilistic Thinking Framework constructed by Jones; et al. (1997 : 111) were used as the theories that informed the instruction. Moreover, the Framework was used in order to trace cognitive changes in students' probabilistic thinking. The interpretative framework constructed by Cobb (2000 : 308) was considered to examine growth in students' probabilistic thinking from the social perspective.

During instruction sessions, the researcher was both the teacher of the classroom and also the researcher who conducted the whole-class teaching experiment. Students were exposed to problems involving sample space, probability of an event, probability comparisons and conditional probability. The instruction provided opportunities for students to make or critique conjectures to accommodate the social dimension of their thought processes.

Additionally, the researcher worked with two witnesses whose roles were to provide alternative views of what happened during each instructional session and also to collaborate with the researcher in the process of data analysis. This chapter will describe the processes and procedures that were used in this study. It includes specifics about participants, instrumentation, procedure, data collection, and data analysis.

Participants

The participants of this study consisted of students and two witnesses. The students were from two classes at a primary school in Chonburi, one class from each level of Prathomsuksa 3 and 5 respectively. A case study analysis (Miles & Huberman, 1994) was used to provide a more in-depth view of students thinking and beliefs. In each class, 6

students were purposefully sampled to provide the target students for the case study. For purposefully sampling students from each grade level, the researcher attempted to select students that represented the following proficiency categories with mathematics: (a) lower quartile (2 students), (b) middle quartile (2 students), and (c) upper quartile (2 students). The levels of proficiency were based on students' school records of performance in mathematics. The other criterion for selecting the case-study students also included the students' ability to discuss and express ideas. The researcher asked for permission from the students' parents by sending a consent letter. The letters explained the objectives of the study and included a part indicating that the parents were or were not willing to let the students be the participants of this study.

There were two witnesses associated with this study. One is the lecturer in the mathematics department in a university and the other is a primary school teacher. Before beginning the process of the study, the researcher and the witnesses discussed theories and procedures associated with the study so as to make sure that both witnesses understood all of aspects of the research. During the process of this study, a witness was present during the entirety of the whole-class teaching experiment. The witness's role was to act as another pair of eyes in the classroom. The witness recorded what transpired during instructional lessons, as well as collaborated with the researcher to develop and revise the hypothetical learning trajectories for the subsequent instructional sessions. More specifically, the witness focused on the 6-students case-study group of each class and attempted to identify the emergence and evolution of classroom sociomathematical norms, mathematical practices and students' beliefs about probability. The other witness was concerned with the process of data analysis. The witness was included in this process because they could validate the findings of the researcher.

Instrumentation

The interview assessment protocols. In this study the interview assessment protocols captured students' probabilistic thinking. The questions for the assessment were adapted from those used by Polaki (2000) and Jones; et al. (1999). There were four kinds of questions.

The first kind of question was designed to probe into students' understanding of sample space. These questions focused on what outcomes were possible in a one-stage, two-stage and three-stage random experiment. The second kind of question was provided to capture students' understanding about probability of an event. These questions required students to determine the probability of a random event such as spinning a spinner. The third kind of question was about probability comparisons. These questions required students to determine and justify which probability situation afforded a better chance for the target event. The fourth kind of question captured students' understanding about conditional probability. These questions focused on the probability of a nonreplacement situation. The contexts of the problem were chosen to be accessible to Thai students. However, the tasks involving more than one construct could be included within one context (see Appendix A, Pre-Assessment Protocol)

Although the pre-, post- and retention assessments used different interview assessment protocols, they were parallel. All the interview assessment protocols involved interviewing only the target students.

The Instructional program: Hypothetical learning trajectory. Before the teaching experiment started, the researcher developed a hypothetical learning trajectory for subsequent instructional sessions. The hypothetical learning trajectory comprised learning goals, instructional activities, and conjecture learning processes. According to the studies of Jones; et al. (1999) and Polaki (2000), these studies used the Probabilistic Thinking Framework (Jones; et al. 1997 : 11) to inform the planning and implementation of the probability instruction. The activities that were used in the classroom of the teaching experiments of those studies were developed based on the Framework. Therefore the activities of the studies of Jones; et al. (1999) and Polaki (2000) could be adapted for use in this study. In other words, the researcher developed learning goals and then instructional activities by adapting from the studies of Jones; et al and of Polaki. The instruction focused on the development of 4 mathematical ideas: (a) ability to list a complete set of outcomes for 1-, 2- and 3-dimensional probability situations; (b) recognition of the relationship between sample space composition and probability of an event; (c) ability to describe and order

probabilities; and (d) ability to recognize when the probability of an event is and is not changed by the occurrence of another event.

After the researcher had developed learning goals and activities, the researcher and witnesses collaborated in using research-based knowledge of probability to construct hypotheses about the learning processes. In other words, the researcher, teachers, and the witness wrote down anticipated responses and actions students might exhibit as they worked through the instructional activities. The researcher adjusted the activities in each session if the hypothesis about the learning processes showed that students could not access the activities. This process was done when the hypothesis was suitable.

There were 16 sessions in the instructional program. Each session used the tasks that were based on one or more of the key constructs. The class started with a whole-class-opening problem. Next, the students made conjectures about how to solve the problem. After acting out the problem (playing a game) to generate a small set of data, they then reconciled their conjectures with the way game turned out. Next, discussions focused on figuring out a winning strategy by analyzing sample space composition (see Appendix C, Session 1-2).

Procedure

Two weeks before the teaching started the researcher observed in the mathematics classes of Prathomsuksa 3 and 5. Then, 6 students from each class were chosen to be target students. These students from the case-study groups were *interviewed* to measure their performance scores and levels of probabilistic thinking (pre-assessment). In each class, the researcher taught twice a week for eight weeks (45 minutes per period). The witness observed the classes. After the teaching sessions, the target students were assessed again two times, one week after the last session (post-assessment) and four weeks after the post-assessment (retention-assessment).

During the teaching experiment, the researcher and the witness kept adjusting the hypothetical learning trajectory. After the first session the researcher and the witness who is a primary school teacher discussed the conjectured learning processes of the first session that were written before the teaching experiment started. The conclusion of this process

was the information for improving the second session of the hypothetical learning trajectory, especially the activities and the conjectures. This process was done for every session.

Data collection

This study collected data from the following sources: (a) pre-, post-, and retention assessments; (b) videotapes of each classroom event; (c) students' written work, and (d) the witness's field notes and reflections. Students' probabilistic thinking levels and performance in probability were assessed with the target students three times, prior (pre assessment) and twice (post- and retention assessment) after the end of the teaching experiment. The retention assessment was assessed 4 weeks after the post-assessment. Each assessment interview were audio-recorded. For each assessment, the researcher provided the tools for modeling the problems (e.g. manipulative, pencil and paper etc.). These assessments were one source of data that provided a picture of student understanding of probability over time and could capture students' beliefs about probability.

More specifically, each assessment provided data to analyze three ways: 1) students' performance; 2) students' probabilistic thinking levels; and 3) students' beliefs about probability. The detail of the analysis is described in the next section.

Each instructional session was video-recorded. The video focused on students' representations and actions. The videotapes helped to document changes in students thinking and beliefs about probability as well as provided the means to capture the emergence and evolution of classroom sociomathematical norms and mathematical practices.

The students' written work provided the source that enabled the researcher to track and record student thinking and beliefs about probability beyond that of group and classroom discussions.

The field notes of the witness provided data from a different perspective of the researcher. The witness noted as he observed each instructional session. More specifically, he focused on the target students and also was the observer during whole-class discussions.

Data analysis

There were two parts of the data analysis. The first part was concerned with students' responses in the interview assessment protocols. As described in the previous section, the interview assessment protocols were used to assess only the case-study group 3 times, pre-, post- and retention assessment. Each assessment, students' responses were analyzed by three procedures. The first procedure involved double-coding (Miles & Huberman, 1994 : 60) and was used to generate **performance scores**. Each item was scored 2, 1 or 0 according to a three-stage rubric. This rubric is shown by Figure 5. Working independently, the researcher and the witness scored students' responses. Any variations in coding were discussed until consensus was reached. Additionally, the researcher used the Wilcoxon Signed Ranks Test (Siegel & Castellan, Jr. 1988) to compare students' performance scores on assessment protocol of pre- and post-assessment, and also to compare students' performance scores on the assessment protocol of post- and retention assessment.

2 point	Student demonstrates a clear understanding of the concept by providing a correct answer and a complete verbal justification.
1 point	Student demonstrates a limited understanding of the concept by providing either a correct answer with an incomplete verbal justification or providing an incorrect answer but valid verbal justification.
0 point	Student demonstrates little or no understanding of the concept by providing either an incorrect answer with incomplete verbal justification or providing a correct answer with no verbal justification

Figure 5. Assessment rubric

The second procedure also used double coding (Miles & Huberman. 1994 : 60) to establish each student's **probabilistic thinking level** on each of four constructs (sample space, probability of an event, probability comparisons and conditional probability). Working independently, the researcher and the witness (the same as the witness in the first procedure) coded the students' responses on each item of the interview assessment protocols to establish probabilistic thinking levels by using the criteria from the Probabilistic Thinking Framework. Any different codes were discussed until consensus was reached. Moreover, the data gathered during teaching session highlighted factors or conditions that were critical to students' transition from one level to another.

The third procedure was about students' beliefs. Students' responses on the assessment provided the data for analysis of students' beliefs about probability. According to the assessment interview, students could explain the answers orally, so students' misconceptions, intuitions and heuristic biases (students' beliefs) could be captured from their reasons. The transcripts from audio-recorders were coded for the aspects of students' beliefs. Students' beliefs were also captured from participation in the classroom teaching experiment.

The second part of analyzing data was concerned with analysis of the whole-class teaching experiment. This analysis focused on students' individual and collective thinking and beliefs related to probability. In addition classroom sociomathematical norms and classroom practices were identified as they emerged. This analysis consisted of three steps: data reduction; data displays; and conclusion drawing and verification. In the first step, codes from all the data sources generated images and impressions of individual and collective student thinking and beliefs. The analysis began by viewing the daily activities of each class in chronological order on the videotapes. Key episodes were then used to make initial conjectures that were modified based on the analysis of subsequent episodes. The conjectures then became meta-analyzed for patterns and trends. The witness's field notes and students' written work were also coded as were transcriptions of the target student interviews. In the second step, data displays were developed using the codes generated in the first step of the analysis process. These data displays allowed the researcher to compare students' thinking and beliefs across all qualitative data sources. The final step of

the process was generating emergent trends and patterns for probability thinking and beliefs as well as classroom mathematical practices and sociomathematical norms.

Respondent validation that is, the confirmation data by the witness was used to help ensure intersubjectivity between the researcher and the witness. More specifically, the researcher provided the witness with findings related to the whole-class teaching experiment. Using the field notes and videotapes as needed, the witness validated the findings of the researcher. Any discrepancies between the researcher and witness were negotiated until a consensus was reached.

CHAPTER 4

ANALYSIS OF DATA AND RESULT

There were three purposes in this study: (1) to trace the emergence and evolution of features of primary students' growth in probabilistic thinking; (2) to examine the beliefs that students exhibit as they reason in probability situations; and (3) to identify and describe the evolution of sociomathematical norms and classroom mathematical practices that the students develop collectively as they tackle probability problems. According to these purposes, both quantitative and qualitative analyses methods were used. The data collected from the pre-, post- and retention-assessments and the teaching experiment were used to trace the development of probabilistic thinking before, after and during the teaching experiment. These data were used to examine students' beliefs as well as to identify the emergence of sociomathematical norms and classroom mathematical practices.

In this chapter, the analyses of data and the results are described in four parts. The analyses of students' growth in probabilistic thinking are described in two parts—quantitative analyses of students' performance on the assessment and the qualitative analysis of target students during instruction. The third part describes students' beliefs in probability. Finally, the last part discusses about the emergence of sociomathematical norms and classroom mathematical practices.

Analyses of students' performance

To generate performance scores (maximum = 74) for 12 students, 6 in Prathomsuksa 3 and 6 in Prathomsuksa 5, at each of the three assessment points, the researcher and the witness scored students' responses. Agreement was achieved in scoring 1152 of the 1332 questions (86.49 %) in all assessments over the three time periods. Variations were clarified until consensus was reached.

Tables 1 and 2 (page 58) show the performance scores of target students on pre-, post- and retention-assessments. Table 3 (page 59) shows means and standard deviations of the performance scores of these students. The Wilcoxon Signed Ranked Test (WSRT) revealed that the difference between pre- and post-assessment scores of these students

was significantly different ($p < .003$). On the other hand there was no significant difference between post- and retention-assessment ($p = .592$). To compare the difference between scores of case-study group of Prathomsuksa 3, the WSRT revealed that the difference between pre- and post-assessment scores was significantly different ($p < .029$) but there was no significant difference between post- and retention-assessment ($p = .279$). Similarly, to compare the difference between scores of the case-study group of Prathomsuksa 5, the WSRT revealed that the difference between pre- and post-assessment scores was significantly different ($p < .029$) but there was no significant difference between post- and retention-assessment ($p = .917$).

Table 1. The performance score of target students in Prathomsuksa 3

Student name	Performance score		
	Pre-assessment	Post-assessment	Retention-assessment
1. Pirapan	39	66	72
2. Natida	35	63	71
3. Witawat	29	52	53
4. Piyawan	23	53	42
5. Kamol	32	44	41
6. Jiyanee	13	42	40

Table 2. The performance score of target students in Prathomsuksa 5

Student name	Performance score		
	Pre-assessment	Post-assessment	Retention-assessment
1. Tadapon	39	72	72
2. Anotai	43	63	62
3. Wisanu	38	67	65
4. Kanok	41	66	68
5. Kittisak	23	42	48
6. Jirawan	24	36	39

Table 3 Mean and standard deviation of the performance scores of target students

Level of target students	Mean			Standard Deviation		
	Pre-assessment	Post-assessment	Retention-assessment	Pre-assessment	Post-assessment	Retention-assessment
All target students	31.58	55.50	56.08	9.24	12.18	13.61
Prathomsuksa 3	28.50	53.33	53.17	9.33	9.70	14.56
Prathomsuksa 5	34.67	57.67	59	8.82	14.87	12.78

Case-study analyses

To establish the levels of students' probabilistic thinking on each of four constructs (sample space, probability of an event, probability comparisons and conditional probability) of the framework over the three assessment points, the researcher and the witness independently coded the students' thinking levels and reached agreement on 125 of 144 levels, that is, an agreement of 86.06%. Variations were clarified until consensus was reached. Figures 6, 7, 8 and 9 present profiles of thinking levels of each target student at the three assessment points.

Figures 6 and 7 show thinking levels of target students in Prathomsuksa 3. With regard to students' probabilistic thinking development over the four constructs (sample space, probability of an event, probability comparisons and conditional probability), students' profiles show that the target students in Prathomsuksa 3 developed their probabilistic thinking. Pirapan and Natida were in the upper quartile. In pre-assessment point, their probabilistic thinking was at Level 1 and 2 and moved to Level 3 and 4 in post- and retention-assessments. Witawat and Piyawan were in the middle quartile. Their probabilistic thinking moved from Level 2 in pre-assessment to Level 3 in post- and retention-assessments. Jiyani, who was in the lower quartile, developed her thinking from Level 1 in pre-assessment to Level 2 in post- and retention-assessment. Although Kamol was in the lower quartile as Jiyani was, his profile show that he developed probabilistic thinking only in probability of an event. Kamol's thinking about probability of an event

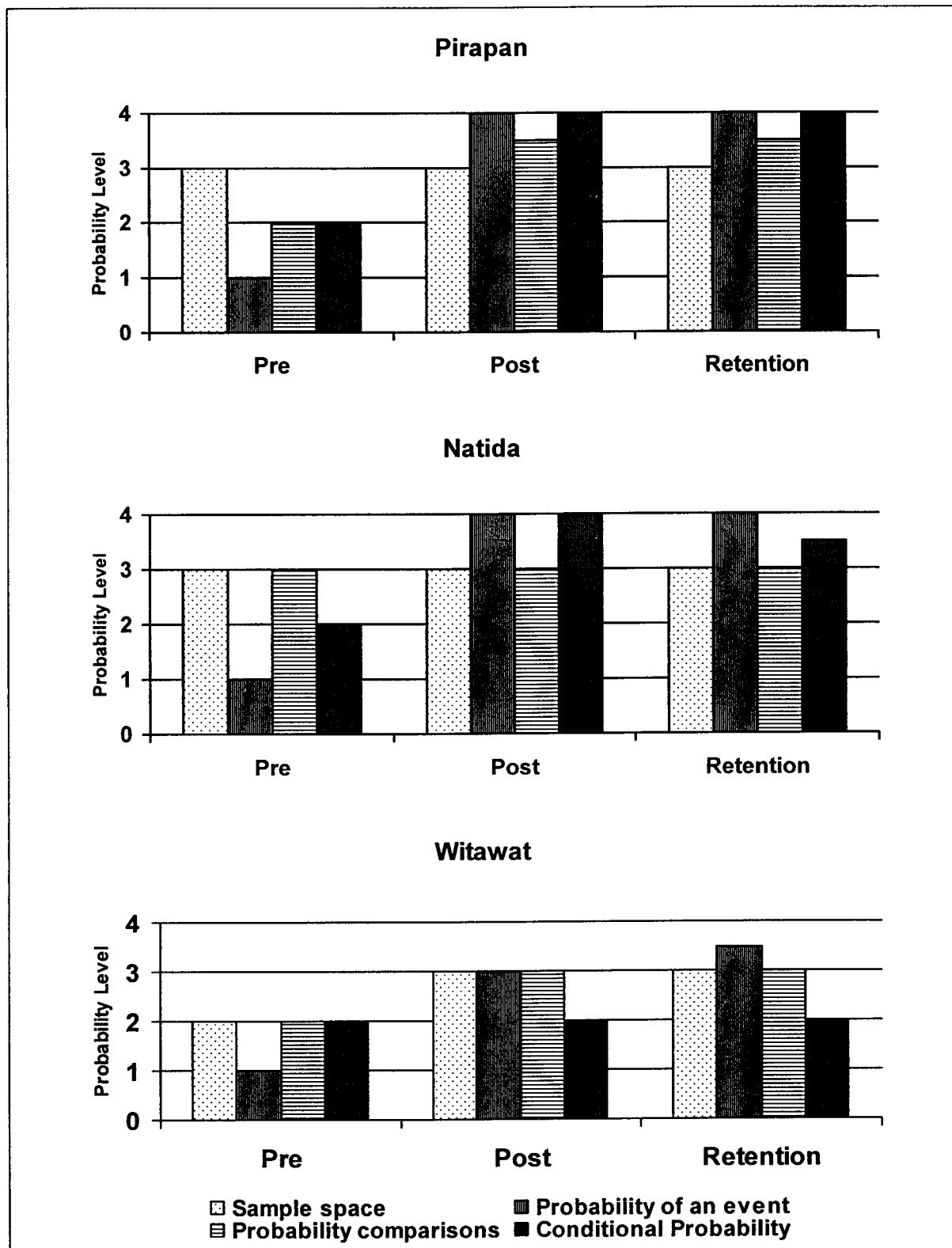


Figure 6. Profiles of probabilistic thinking patterns of target students who were in Prathomsuksa 3

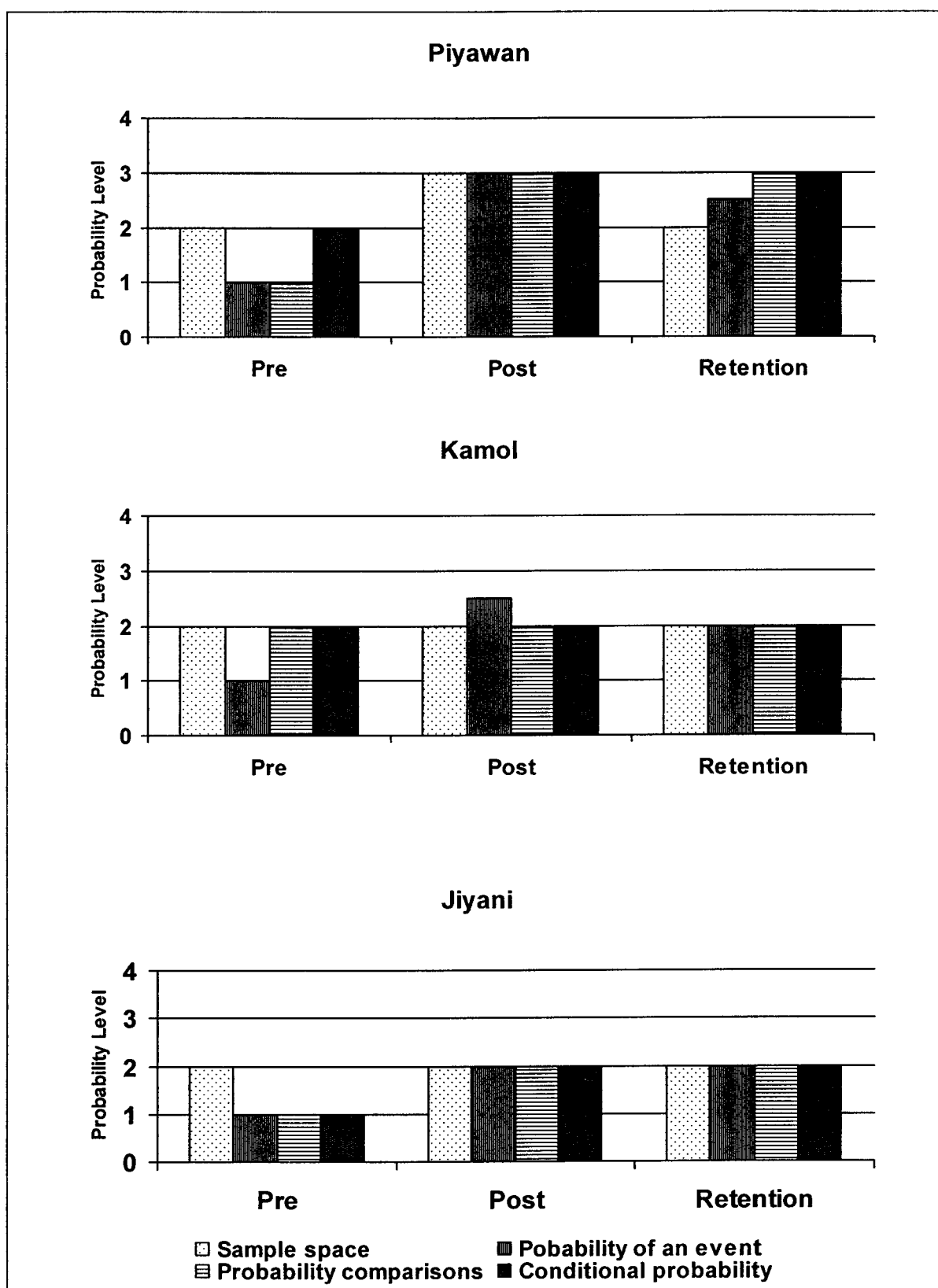


Figure 7. Profiles of probabilistic thinking patterns of target students who were in Prathomsuksa 3 (continued)

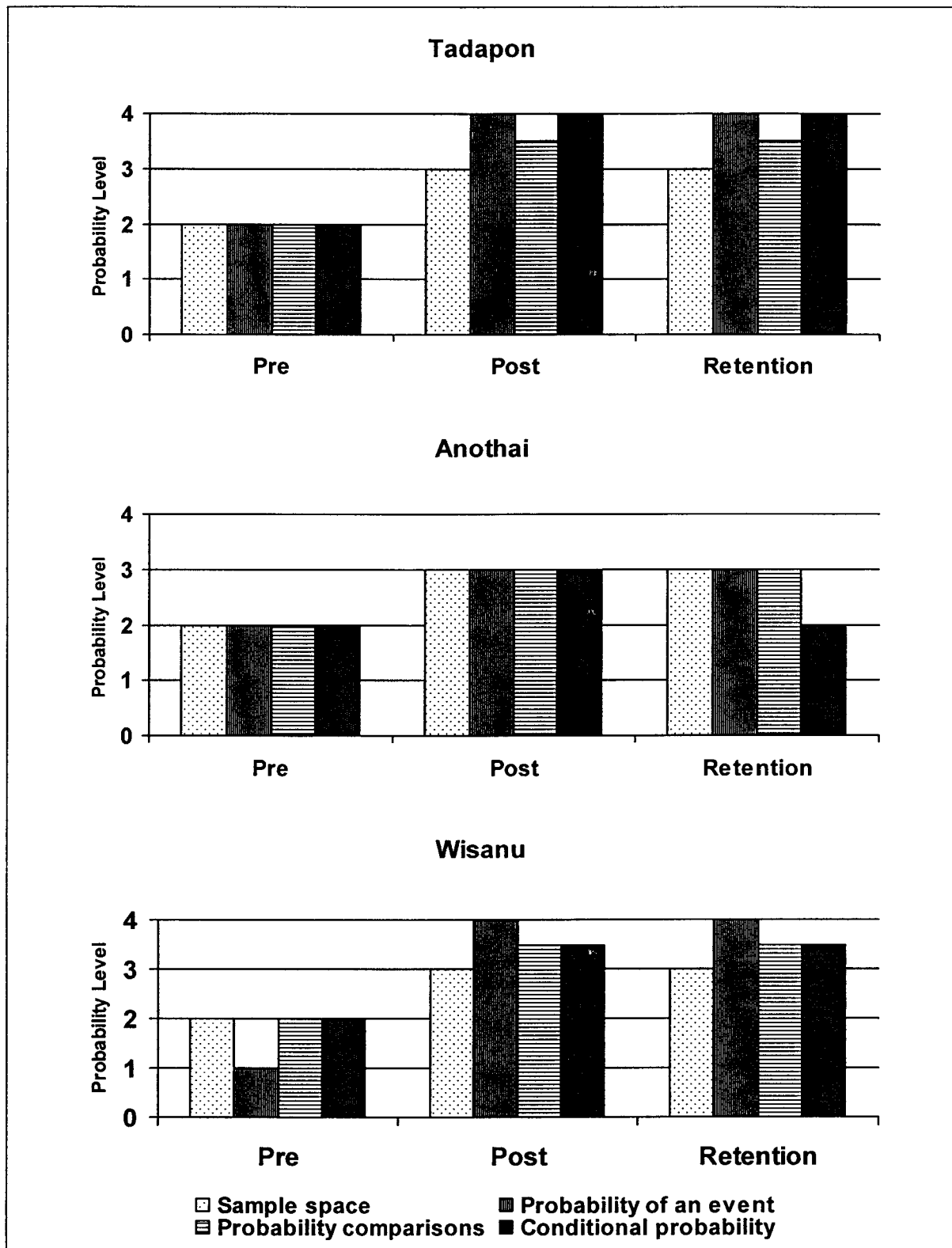


Figure 8. Profiles of probabilistic thinking patterns of target students who were in Prathomsuksa 5

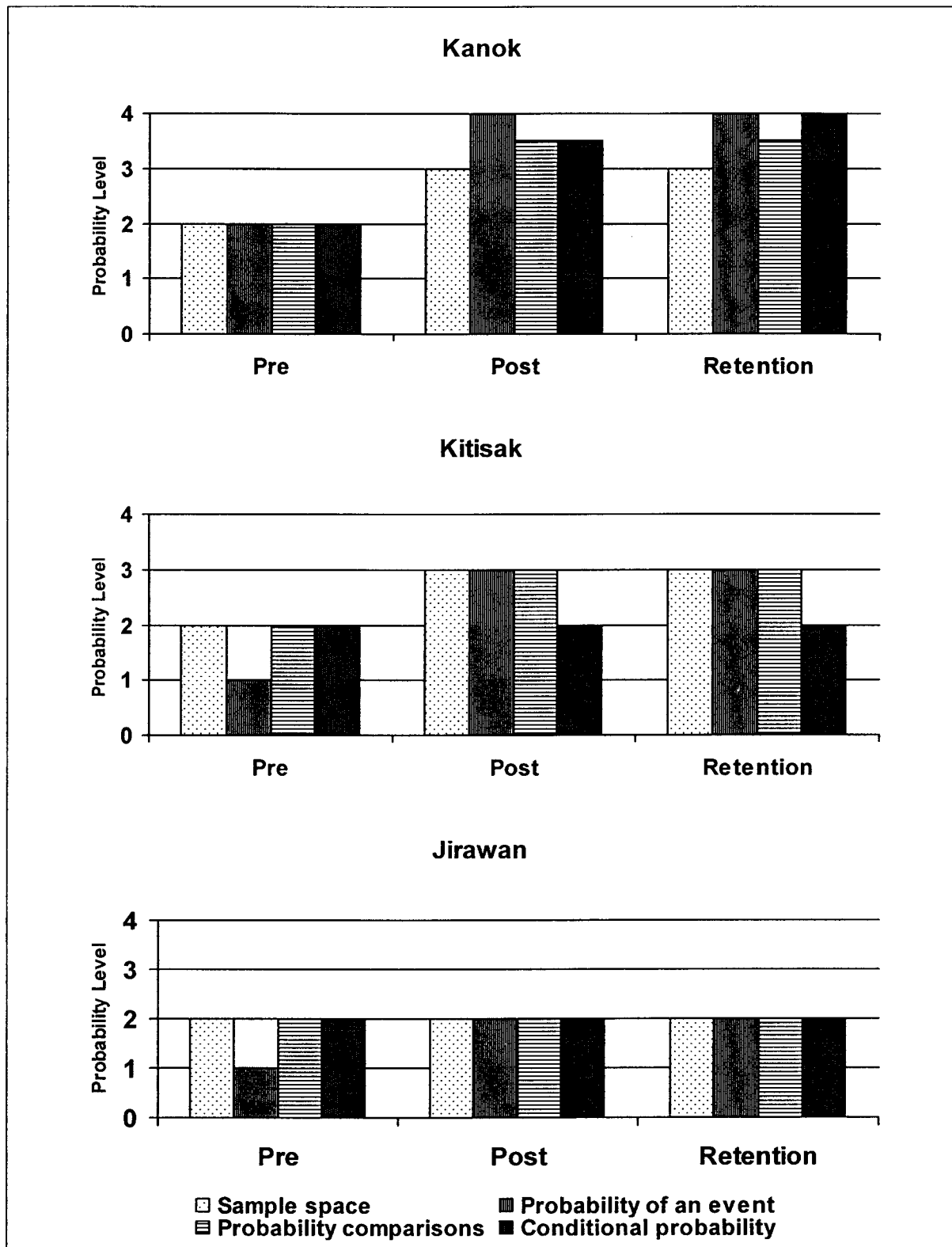


Figure 9. Profiles of probabilistic thinking patterns of target students who were in Prathomsukasa 5 (continued).

developed from Level 1 in pre-assessment to Level 2 in post- and retention assessment, whereas in the other constructs, he was at Level 2 in all assessments.

Figures 8 and 9 are the profiles of target students in Prathomsuksa 5. Similarly to target students in Prathomsuksa 3, over all constructs, the target students in Prathomsuksa 5 developed their probabilistic thinking. Tadapon and Anothai, who were in the upper quartile, and Wisanu and Kanok, who were in the middle quartile, developed their thinking from Level 2 in pre-assessment to Level 3 and 4 in post- and retention-assessments. Kitisak and Jirawan were in the lower quartile. Both students were assigned to Level 2 in most constructs (except in probability of an event, they were in Level 1) in the pre assessment. Kitisak developed his thinking to Level 3, whereas Jirawan was at Level 2 in the post-and retention-assessments.

In the analysis of target students that follows, the term *part-part reasoning* refers to thinking that involves relating of two parts to each other and *part-whole reasoning* refers to thinking that represents the relationship of a part to a whole (Jones; et. al. 1999 : 503). For example, a student was asked what is more likely to be drawn from a container that has 3 red and 2 green marbles, a red or a green? A student who examined the relationship of three red marbles (part) to two green marbles (part) represented *part-part reasoning*. A student who compared three red marbles (part) to all five marbles (whole) represented *part-whole reasoning*.

Analysis of Pirapan's and Natida's probabilistic thinking. Pirapan and Natida were in Prathomsuksa 3. Their achievement in mathematics was in the upper quartile. These students did not experience any misconceptions in thinking about sample space. Their probabilistic thinking profiles (Figure 6) show that they were assigned at Level 3 thinking about sample space at the pre-assessment point. However, there was no evidence that they could develop their thinking about sample space to Level 4. In contrast, their thinking about probability of an event at the pre-assessment was at Level 1, but they developed their thinking to Level 4 after the teaching experiment. With respect to probability comparisons and conditional probability, both students were identified at Level 2 before teaching begun. They were able to develop their thinking to Level 3 thinking about probability comparisons and Level 4 thinking about conditional probability in the post- and retention-assessment

points. The narrative below describes how these two students exhibited and developed their probabilistic thinking.

With regard to sample space, the profile (Figure 6) shows that Pirapan and Natida exhibited probabilistic thinking at Level 3 on all of the three assessments. In the pre-assessment, they were able to list the complete sets of outcomes in 1-and 2-dimensional situations, and sometimes systematically listed the outcomes of 2-dimensional probability situations. During instruction, their responses in Session 6 suggested that they seemed to move to Level 4 thinking about sample space. In the "Dress Up" problem (see Appendix B), which involved 2- and 3-dimensional situations, they were asked to systematically list all outcomes in the 2-dimensional situation (getting dressed by choosing a shirt and a pair of pants). Both students were able to list the outcome using the odometer strategy (English, 1993 : 216). In getting dressed by choosing a shirt and a pair of pants, both students started with 1 shirt and then matched it with all of pairs of pants, and then started with a new shirt again to match with all of the pairs of pants.

Pirapan was the first student who began to use this strategy, and then he explained the strategy to the class. Students' paper work provided the evidence that Natida and some students accepted this strategy to list all outcomes. Moreover, Pirapan and Natida could adapt this strategy to a 3-dimensional situation (getting dressed by choosing a shirt, a pair of pants and a pair of shoes). However they could not adopt the strategy to use in the other 2- or 3-dimensional situations, such as Situation 5 (See appendix A) in the post- and the retention-assessments. They still did not use any strategy to listing all outcomes in the situations that were different from the "Dress Up" problem.

In probability-of-an-event situations, both students were assigned at Level 1 at the pre-assessment point. Yet, they developed their thinking on this construct in the early teaching sessions. In the pre-assessment, they responded to the questions subjectively and did not consider the list of outcomes that were asked at the beginning of each situation of the assessment. For example, in Situation 1 they were asked to choose the color from red and white that was more likely on a spinner (see Appendix A). Pirapan said "Red because I like it." Natida seemed to answer more quantitatively than Pirapan. She said "I think they are

the same chance.” However, when the researcher asked her to compare the chance of getting red and yellow she said “Yellow because the first time it pointed at yellow.”

In contrast, during instruction both students showed that they did not use subjective judgments for reasoning in all probability situations. During the class discussion of the “Spinner Problem” (see Appendix B) both of them used a part-part relationship to reason why the white was going to win. They compared 5 sectors of white (one part) with 3 sectors of red (the other part). However, they developed their thinking to part-whole reasoning when the teacher asked the students to use numbers for describing the chance of white. Pirapan, who was always the leader of the discussion, stated, “I think we can say 5 from 8 because there are 8 sectors on the spinner.” This response also illustrated the existence of an *invented informal language* and suggested that Pirapan had begun to have the concept of part-whole relationship. Moreover, Pirapan consistently used part-whole relationships to describe probabilities in all situations that were asked in the later sessions, and represented his part-whole schema by using fractions. Natida also had the idea of using part-whole reasoning.

Natida’s responses in Session 4, during the class discussion of the problem “Which One is Better” (see Appendix B) suggested that she had developed part-whole schema. Natida explained, “I know they are the same chance because the chance of a coin is $\frac{1}{2}$ and the chance of a dice is $\frac{3}{6}$. I know $\frac{1}{2}$ is equal to $\frac{3}{6}$.” Natida consistently used a fraction to describe the chance of a probability-of-an-event situation after Session 5. Because of the consistency of their using part-whole reasoning to describe the chance of probability-of-an-event situations, Pirapan and Natida were assigned at the Level 4 thinking about probability of an event at both post- and retention-assessment points.

With regarded to probability comparisons, at the pre-assessment point, both students exhibited at Level 2 instead of Level 1 as they were assigned in probability of an event. Although they always reasoned the probability-of-an-event situations using subjective judgments, they compared probabilities using part-part schema in probability comparisons. In Situations 6 and 7 of the pre-assessment (see Appendix A), both students were able to choose the box correctly that had the best chance to get a green marble or a red marble. Nevertheless, they could not elaborate on their thinking. For instance, in

Situation 6 Natida said, "The box A is good for green and the box B is good for red." When asked to use numbers to describe her thinking, she said, "I don't know how."

As mentioned earlier Pirapan and Natida exhibited part-whole schema in Session 2 and 3 of the instruction. This schema enabled them to develop their thinking in probability-comparison situations to Level 3-4. One piece of the evidence was their demonstration in Session 5 (see Appendix B). They compared the chance of getting red and green on both spinners by using part-whole reasoning. Pirapan used fractions to represent the chance of probability-of-an-event situations by saying, "Spinner 1 is best for the red car because the chance of it is $\frac{3}{4}$ but just $\frac{1}{4}$ for Spinner 2." In a similar way, Natida said "Spinner 1 is best for red because both spinners have 4 sectors but there are 3 sectors of red in Spinner 1 but just 1 in Spinner 2." Moreover, in the last two assessments, they could compare probabilities across sample spaces (Situation 6 and 7, see Appendix A) by using fractions. Although, their thinking about probability comparisons seemed to be at Level 4, they were identified between Level 3 and 4 because they could not elaborate on their thinking in a contiguous situation.

In Session 14, in the discussion of "Which Spinner " problem (see Appendix B), both students believed that spinner B was best for red. Playing the game by using both spinners, the class discussion showed that both spinners gave the same results. Yet, they were still confident that what they thought was true if they were to spin themselves (This point will be discussed later in the part about students' beliefs). When asked to assign a number for the chance of getting red on each spinner, they answered for spinner A but could not answer for spinner B. The students argued that they did not know how to use a number because the sectors were not the same size.

In conditional-probability situations, when faced with a non-replacement task, Pirapan and Natida were able to recognize that the probability of all events changed. Their thinking was stable across all of the three assessments points. However, they both were assigned at Level 2 of conditional-probability thinking in the pre-assessment point. Similar to probability-comparison situations, they could not elaborate on their thinking and did not respond quantitatively to demonstrate their thinking prior to the teaching experiment. When asked to use numbers for explaining why the chance of getting a green marble changed

after the researcher had picked a yellow marble out (see Situation 9 in Appendix A), Pirapan said "Because the number of the yellow marbles has decreased. It makes the green marbles have more chance, but I don't know how to explain." Natida responded similarly.

However, both students were assigned at Level 4 of conditional-probability thinking at the post- and retention-assessment points. The key feature of this development came from their abilities at describing probability-of-an-event questions. During discussion of the "Tiger or Gold" Problem (see Session 15 Appendix B), the witness noted that Pirapan said "When we know that the third door is a tiger, the chance of getting gold is $\frac{1}{3}$ instead of $\frac{1}{4}$. It's easier to get the gold." Similar to Natida, she said "Because we already knew that the third door had a tiger. The number of the doors has decreased, just three. There is a door of gold. The chance is $\frac{1}{3}$, whereas the first time it's $\frac{1}{4}$." In the post- and retention-assessment both students consistently ordered the probabilities correctly and used fractions to describe the probabilities of non-replacement situations. In post-assessment, the researcher posed a situation that included both a replacement and a nonreplacement situation. Both students still responded correctly by using fractions. The following is the excerpt that shows how Pirapan exhibited his thinking on conditional probability in the extended situation;

Researcher: There are 3 red and 5 blue marbles in this box. If I draw a marble, can you tell me the chance of getting red?

Pirapan: $\frac{3}{8}$.

Researcher: Okay. How about blue?

Pirapan: $\frac{5}{8}$.

Researcher: I'm going to pick one. (The teacher drew a blue marble and showed it to student.) If I draw one more marble, what is the chance of getting red now?

Pirapan: $\frac{3}{7}$.

Researcher: Why?

Pirapan: There are 7 marbles now and there are 3 red.

Researcher: Compared with the first draw, is it more difficult or easier to get a red marble?

Pirapan: Easier.

Researcher: Okay,...but I need a red marble...so I return the blue marble to the box. I am going to draw a marble again. What is the chance of getting red now?

Pirapan: The same chance as the first time... $\frac{3}{8}$.

Researcher: To get a red marble easier, should I return a blue marble back in the box?

Pirapan: No, you should not. It is easier if you don't return it. When you return it, it becomes the same chance as the first time again.

In an overall sense, both Pirapan and Natida exemplified students whose mathematics achievement was in the higher quartile. They were able to improve their probabilistic thinking quickly and could develop their thinking to Level 3 or 4 across the four constructs. The key feature of their development was the use of part-whole relationships that they developed after using invented language, such as "3 from 8." In addition, their use of fractions in representing part-whole reasoning enabled them to describe, order and compare probabilities.

Analysis of Witawat's and Piyawan's probabilistic thinking. Witawat and Piyawan were in Prathomsuksa 3. Their achievement in mathematics was in the middle quartile. With respect to pre-assessment, Witawat exhibited Level 2 thinking about sample space, probability comparisons and conditional probability. He exhibited at Level 1 thinking about probability of an event. Similarly, Piyawan was assigned at Level 2 thinking for sample

space and conditional probability, and Level 1 thinking for probability of an event and probability comparisons (see their probabilistic thinking profiles, Figures 6 and 7).

Witawat and Piyawan were typical examples of students who used subjective judgments for ordering probabilities before the teaching experiment. During and after the teaching, they developed their probabilistic thinking. This development was demonstrated by their using part-part schema for describing probabilities. At the last two assessments, they were assigned at Level 2 or 3 for probabilistic thinking. The paragraphs below describe how they developed their thinking during the teaching experiment and how they responded in the post- and retention-assessments.

With regard to sample space, although Witawat and Piyawan did not experience any difficulty in listing all outcomes of 1-dimensional situations, both of them were unable to list the complete set of outcomes of 2-dimensional situations at the pre-assessment point. They listed a complete set of outcomes in Situation 1 and 2, which were 1-dimensional situations, but could not list a complete set of outcomes in 2-dimensional situations, such as Situations 3 and 4. During the teaching experiment, both students developed their thinking about sample space. Their paper work provided evidence that they could list all the outcomes of the “Dress Up” problem (see Appendix B) by using a generative strategy. The witness’s field notes confirmed their ability in listing the outcomes of a 2-dimensional situation. The witness noted: “Witawat and Piyawan began to list all outcomes systematically after the class discussion. They could adopt the strategy that Pirapan explained to the class.” However, they did not adopt the strategy for the Situations in the last two assessments. They systematically listed the outcomes in a 2-dimensional situation, Situation 3, but were nonsystematic in Situations 4 and 5 (see Appendix A). Although they could list a complete set of outcomes correctly, they were assigned at Level 3 thinking about sample space because of the inconsistencies in their strategies.

In probability-of-an-event situations, both Witawat and Piyawan used subjective reasoning to describe probability of an event at the pre-assessment point. When asked to compare the chance of getting red with the chance of getting white on the spinner in Situation 1 (see Appendix A), both students chose one color and reasoned that they liked it. Similarly, in other situations that concerned ordering probabilities they could not elaborate

on their thinking. However, after the instruction they were assigned at Level 3 thinking for probability of an event. This development was captured in an early teaching session.

Witawat and Piyawan began to use part-part relationships to order probabilities in the first teaching sessions. In the discussion of the "Spinner Problem" (see Appendix B), they compared the chance of getting white and getting red by using the number of sectors of each number (3 and 5). Moreover, both students began to use fractions for describing probability at Session 3. They both used fractions to represent the probabilities in the problem "Which One Is Better" (see Appendix B). Although they used fractions to represent probabilities, they used a fraction as "*a notation of probability*". They did not consider fractions they used as numbers that could be computed and compared. This evidence was revealed at the end of this session. The teacher asked students to choose a dice or a coin that had more chance of winning in the problem "Which One Is Better." In his journal, Witawat wrote, "A dice has a chance of $\frac{3}{6}$ and a coin has $\frac{1}{2}$ to win. The dice is better because there are three ways to win" Similarly, Piyawan wrote "The chance of a dice is $\frac{3}{6}$ and the chance of a coin is $\frac{1}{2}$. I choose the dice because it is easy to win." This evidence was interpreted as saying that they had not captured the meaning of fraction. Using fractions to represent the probability for them appeared to arise from following the class discussion.

Piyawan exhibited her misunderstanding in using fractions in the last two assessments. For instance, in Situation 6 of the post-assessment (see Appendix A), the researcher showed the box containing 9 red and 8 green marbles and asked:

Researcher: If you draw a marble from the box, which color would you be more likely to get?

Piyawan: Red.

Researcher: Why?

Piyawan: Because there are more red...9 red but 8 green.

Researcher: Can you use a number to describe the chance of getting red?

Piyawan: $1/17$.

Researcher: How about the chance of getting green?

Piyawan: $1/10$.

Researcher: Why $1/10$?

Piyawan: I don't know.

Moreover, during the teaching sessions, they always returned to compare probabilities by using part-part relationships. For example, in Session 5, when the teacher asked why spinner 1 was better for red and spinner 2 was better for green, they said that there were 3 sectors of red and 1 sector of green on spinner 1, while spinner 2 had 3 sectors of white and 1 sector of red. For them, using fractions to represent probabilities appeared when the teacher asked them to use numbers representing the chances of an event.

With regard to probability comparisons, Witawat was assigned at Level 2 prior to the teaching, whereas Piyawan was at Level 1. The difference between them was that Witawat sometimes exhibited the use of part-part relationship for comparing probabilities, while Piyawan held subjective views. For example, in Situation 6 (see Appendix A) Witawat correctly chose the box that was better for red. He said "the box A is better for red because there are more red marbles than the box B." However, he still picked box A as the box that was better for green. In the same Situation, Piyawan exhibited her subjective thinking. She said, "I think the box B is better for both red and green, because the number of marbles is less than the box A. I like the box that has a small number of marbles."

As mentioned previously, both students came to use part-part relationships to describe probability during instruction. They also used this schema in probability-comparison situations. Interestingly, both students demonstrated the use of part-part relationship that could be used to compare probabilities across sample spaces.

Witawat's and Piyawan's consistent use of part-part schemas seemed to be typical of students who had what Polaki (Polaki, 2000 : 101) called the stronger part-part schema.

The interesting point of their thinking was that Witawat and Piyawan were able to compare probabilities across sample spaces. For example, in Situations 6 and 7 in the post-assessment (see Appendix A), they were able to indicate which box was better for red and which box was better for green. When asked why they chose these boxes Witawat said “the box A is better for red because there are more red marbles, but the box B has the same numbers.” The researcher continued asking him to use numbers in his explanation. He explained, “The box A has 9 red and just 8 green marbles. It is easier to pick a red marble. But the box B has 5 red and 5 green marbles, getting red and green are the same chances.” Similarly, Piyawan also demonstrated her thinking in this way. Their responses in these situations were interpreted as indicating that both of them held *extreme part-part reasoning* to describe probability. According to the way that students demonstrated their use of part-part schema, this schema enabled students to demonstrate higher levels of reasoning in ordering or comparing probabilities. A student who held extreme part-part reasoning could use a part-part schema in more flexible and meaningful ways and sometimes could use this schema to compare probability across sample spaces.

Witawat and Piyawan were at Level 2 thinking about conditional probability at the pre-assessment point. They sometimes recognized that the probability of an event changed in non-replacement situations but sometimes they did not. In Situation 11 (see Appendix A) both of them demonstrated that they knew the chance of getting a milk candy and a fruit candy had changed after the researcher drew a milk candy. In contrast to Situation 9, they did not recognize that the chance of getting green changed after the researcher had drawn a yellow marble. In this latter situation, they said that it was the same chance because the number of green marbles was still the same and it was still a lot of red marbles.

Even at the post- and retention-assessment, Witawat struggled with this thinking. For him, recognizing a change in the probability for a nonreplacement situation, depended on his ability to compare probabilities, that is on his part-part schema. For example, in Situation 11 of the post-assessment (see Appendix A), there were 2 flower stickers and 3 animal stickers in a box. He knew that the probability of getting a flower sticker had changed after the researcher drew an animal sticker out. The following excerpt describes Witawat's responses in this Situation.

Researcher: If I let you draw a sticker now, do you think you can get the animal sticker easier or harder than the first time?

Witawat: Easier.

Researcher: Why?

Witawat: Now, both kinds of sticker have 2 stickers, the same number.

Researcher: How about the numbers of them before I drew it?

Witawat: Flower stickers were more than animal stickers. It was hard to get an animal sticker.

Researcher: How about the flower stickers. If I let you draw a sticker now, do you think you can get a flower sticker easier or harder than the first time?

Witawat: Harder.

Researcher: Why?

Witawat: The number of flower stickers reduces. The first time, it's more than animal but now it's the same.

This response provided another example of an extreme part-part schema in comparing the probability of each kind of sticker in the boxes. However, this schema did not enable him to determine the probability in a situation that needed part-whole relationship. For instance, in Situation 9 (see Appendix A, Post-assessment), there were 5 green, 4 blue and 2 white marbles in the box. After the teacher had drawn a green marble out and asked, "If I let you draw a marble now, do you think you can get a white marble easier or harder than the first time?" He responded, "It's still hard. No change. Because the number of white marbles is still the same and there's still a lot of the other marbles." This response was interpreted that he could not recognize the difference between $\frac{2}{9}$ and $\frac{2}{8}$.

In contrast to Witawat, Piyawan recognized that the probability had changed in this nonreplacement situation. However, her part-part schema did not help her to elaborate on her thinking clearly. For instance, in Situation 9 of the post-assessment, after the researcher drew a green marble and asked the chance of getting a white marble in the second draw, she said that the chance of getting a white marble had increased because the number of other marbles had decreased. Consequently, Piyawan was assigned at Level 3 thinking about conditional probability after the teaching, whereas Witawat was at Level 2.

In summary, probabilistic thinking of Witawat and Piyawan developed over the duration of the teaching experiment. In an overall sense, they grew from Level 1 or 2 across all constructs to Level 3. The key feature of their development was the use of extreme part-part reasoning that enabled them to describe and quantify probability.

Analysis of Kamol's and Jiyani's probabilistic thinking. Kamol and Jiyani were in Prathomsuksa 3. Their mathematics performance was recorded in the lower quartile. Both students exemplified students who experienced difficulty in quantifying probabilities. At the pre-assessment point, Kamol was assigned at Level 1 thinking about probability of an event and at Level 2 thinking about each of the other constructs. Jiyani exhibited Level 2 thinking about sample space and Level 1 thinking about each of the other three constructs (see their probabilistic thinking profiles, Figure 7). Both students did not have difficulty listing all the outcomes of 1-dimensional situations but they experienced difficulty in listing all outcomes of 2-dimensional situations. With respect to probability of an event, probability comparisons and conditional probability, they reasoned subjectively in most probability situations. After the teaching experiment, their responses on post- and retention-assessment suggested that their probabilistic thinking had moved to Level 2 or 3 on all constructs.

As mentioned previously, Kamol and Jiyani were at Level 2 on sample space at the pre-assessment point. They were still at Level 2 at the two points of assessment after the teaching experiment, post- and retention-assessment. They could list the outcomes of 1-dimensional tasks such as Situation 1 and 2 (see Appendix A) in the pre-assessment but could not list the outcomes of 2-dimension situations such as Situations 3, 4 and 5. Jiyani was still not able to develop her thinking on sample space during the teaching session. In Sessions 6 and 7, when students were asked to list the outcome of the "Dress Up" problem

(See Appendix B), she drew pictures to match pants and shirt but she could not draw all of the matches. The classroom discussion of how to list the outcomes, where some students could demonstrate the odometer strategy, did not help her to successfully list 2-dimensional situations. She still faced the difficulty of listing all the outcomes of 2-dimensional situations during the teaching and at the last two assessment points.

Kamol was slightly different. He could sometimes list the complete set of outcomes of 2-dimension situations during the instruction. In Session 6 of the teaching, he demonstrated in his paper work that he could list the outcomes of the "Dress Up" problem. However, in Session 7, when the researcher posed a further situation that asked students to list the outcome of 3-dimensional situations; he failed to list a complete set of outcomes for the 3-dimensional situations. In the assessments after the teaching experiment, both students' responses revealed that they were unable to list consistently the outcomes of 2-dimensional situations. Although they could list the complete set of outcomes of Situation 4, they could not list the complete set of outcomes of Situations 3 and 5. Consequently, they were assigned at Level 2 thinking about sample space in both post- and retention-assessment points.

With regard to probability of an event, Kamol and Jiyani were prone to use subjectivity when judging or ordering the probabilities at the pre-assessment point and in the early teaching sessions. In each Situation of the pre-assessment where they were asked to order or compare probabilities, they could not elaborate on their thinking. They ended up their answers by saying, "I don't know, I just guess" or "because I like it."

Kamol demonstrated that he quantitatively ordered probabilities at session 3 of the teaching experiment. When asked to choose the dice or the coin in "Which One Is Better", he said, "I choose the dice because there are three ways to win but there is just one way to win if I choose the coin." Although his answer was wrong, it showed that he exhibited part-part reasoning to describe probability. This response was interpreted as showing that he had developed his thinking about probability of an event from Level 1 to Level 2. He consistently ordered probabilities by using part-part reasoning through the rest of the teaching sessions including the post- and retention-assessment points. For instance, in Situation 3 of the post-assessment (see Appendix A), although he could not list all the

outcomes (he listed 11 outcomes from 12), he still exhibited part-part reasoning in quantifying probabilities. The excerpt below shows his thinking.

Researcher: I'm playing with them and I think I'll get RED-3. Do you think I'm right or wrong?

Kamol: You are probably wrong.

Researcher: Why?

Kamol: There are many ways that are not RED-3.

Researcher: How many ways?

Kamol: 10.

Researcher: How many ways are RED-3?

Kamol: 1.

Researcher: Can you use numbers to explain the chance that I'm right?

Kamol: ...I don't know how to use numbers.

In addition, this response was interpreted as saying Kamol could connect sample space composition to quantifying probability. Unfortunately, he was unable to list all outcomes in 2-dimensional situations.

Unlike Kamol, Jiyani used a fraction notation to represent the probability of an event in 1-dimensional situations but not in 2-dimensional situations. The witness's field note provided the evidence that Jiyani used fractions to represent the chances of events during Session 5. The witness wrote, "Jiyani said that the chance of red in the first spinner is $\frac{3}{4}$ but in the second spinner is $\frac{1}{4}$." This ability seemed to be stable in her thinking until the post- and retention assessment points. In 2-dimensional situations, she was unable to use numbers for representing the chance of an event. She revealed that she was not able to

relate sample space composition and probability measure. Her response to Situation 4 in the post-assessment is an example of her confusion. Although she could list all outcomes of the 2-dimensional situation correctly, she still could not relate the components of the outcomes to quantify probability. The excerpt below provides evidence of how Jiyani used an incorrect fraction to represent the probability of the chance of getting a 5 in Situation 4 (see Appendix A) of the post-assessment.

Researcher: If I toss them, what number would I be likely to get?

Jiyani: 5.

Researcher: Why?

Jiyani: Because there are two 5s.

Researcher: Can you use numbers to explain the chance of getting a 5?

Jiyani: $2/5$.

Researcher: Where is 5 from?

Jiyani: You ask me about the chance of getting 5.

Kamol and Jiyani always used part-part reasoning to describe not only probability-of-an-event situations but also in probability-comparison and conditional-probability situations. After the teaching experiment, Kamol and Jiyani were assigned at Level 2 thinking about probability comparisons and conditional probability (see their probabilistic thinking profiles, Figure 7). With respect to probability comparisons, their part-part schema did not enable them to compare probabilities in any situations that needed part-whole reasoning. For instance, in Situations 6 and 7 in the post-assessment (see Appendix A) their responses showed that Jiyani failed to choose the box which was better for getting a red and failed to choose the box which was better for getting a green marble. In contrast, Kamol appeared to have more idea but could not justify his thinking. When asked to compare the chance of getting red from boxes A and B (Situation 6), Jiyani could not answer correctly. She said "Box A is better to get red because there are more red marbles

than box B.” Kamol’s thinking was slightly different. He could choose the box correctly. Nevertheless, he was not able to elaborate on his thinking and could not relate it with numbers. He said he knew that box B was better to get red but he did not know how to explain it.

In conditional probability, Jiyani was at Level 1 but Kamol was at Level 2 at the pre-assessment point. Whereas Kamol sometimes recognized that the probabilities of all events had changed in a nonreplacement situation, Jiyani always gave a response that was based on subjective judgments. For example, in Situation 11 (see Appendix A) Kamol knew that the chance of getting a fruit candy was increased while the chance of getting a milk candy was decreased after a milk candy had been drawn. However he could not justify his thinking. During the instruction, Jiyani’s responses in conditional-probability situations indicated that she had moved beyond Level 1. The witness’s field notes provided this evidence. In Session 15 in discussion of the “Tiger or Gold” problem, the witness noted, “Jiyani said when we know the third door is a tiger, we have more chance to get the gold because there are 3 doors left.” Consistently, Jiyani’s responses in the assessments after teaching indicated that she recognized that the probabilities of all events changed in a nonreplacement situation. Nevertheless, Kamol and Jiyani were assigned at Level 2 in the post- and retention-assessment. Although they were able to realize that the probability of all events changes in a nonreplacement situation, they could not elaborate on their thinking.

Kamol and Jiyani exemplified students who develop part-part schema. They consistently used this schema in probability-of-an-event, probability-comparison and conditional-probability situations. The way they used this schema exhibited what Polaki called (Polaki, 2000 :101) the *stronger part-part schema*. Although both of them always used part-part reasoning to describe probability, they never could compare probabilities across sample spaces as Witawat and Piyawan could. Their use of part-part schema demonstrated lower levels of reasoning than the *extreme part-part schema* of Witawat and Piyawan..

In probability-of-an-event situations, Kamol had a concept of connecting sample space composition and probability measure. Yet, he still could not describe probability in 2-dimensional situations. The reason was that he had difficulty listing the sample space. For

Jiyani, using a fraction to describe probability was a convenient “notation” of a chance. She did not consider the fraction as a number that could be computed or compared.

Analysis of Tadapon’s and Anothai’s probabilistic thinking. Tadapon and Anothai were in Prathomsuksa 5. Their achievement in mathematics was recorded in the higher quartile. They demonstrated that they used part-part relationships to describe probability prior to the teaching experiment. Consequently, they were assigned at Level 2 across the 4 constructs at the pre-assessment point (see their probabilistic thinking profiles, Figure 8). During the teaching, Tadapon always was the leader of discussions. Tadapon developed his thinking early in the teaching sessions. His using part-whole schema to describe probability appeared in Session 2 and he consistently used this schema. Anothai was slightly different. Although she revealed that she had developed part-whole reasoning during the teaching, she used this schema inconsistently and did not use it at all in probability-comparison and conditional-probability situations. In these constructs she used part-part schema. At the last two assessment points, she was assigned at Level 3 across all constructs, whereas Tadapon was assigned at Level 4 thinking for probability of an event and conditional probability, Level 3 thinking for sample space and Level 3-4 thinking for probability comparisons (see Figure 8).

With respect to sample space, both students were at Level 2 prior to the teaching. In the pre-assessment, they could list all outcomes in 1-and 2-dimensional situations. However, they did not list the outcomes of 2-dimensional situations systematically. They both began to use a generative strategy to list shirt- pants pairs in the “Dress Up” problem (see Appendix B). Tadapon was the first student who had this idea. He explained this strategy to the class. Tadapon and Anothai listed all the outcomes of the 2-dimensional situation correctly and systematically. However, they could not list all the outcomes of the 3-dimensional situation. Consequently, they were assigned at Level 3 thinking about sample space.

In probability-of-an-event situations, Tadapon demonstrated that he had the idea of a part-whole relationship in Session 2 (discussion of the Spinner Problem). When asked to use number to represent the chance of getting red in the Spinner Problem, at first he invented the phrase “3 in 8”, and then used the fraction ($\frac{3}{8}$) to describe the chance of

getting red. Moreover, he revealed that he always made the connection between sample space composition and probability measure. This enhanced his ability in describing probability of an event in 2-dimensional situations. In Session 8 during discussion of the "Racing Game" problem (see Appendix B), he said that he did not know which horse was the most likely to win. He needed to know how many ways the dice could come up. While other students chose a horse, he tried to list the outcome of tossing two dice. Finally, he came up with the right answer. While Tadapon developed his thinking to use part-whole reasoning, Anothai showed that she still used part-part reasoning. Although, she used a fraction to describe the chance of an event, she did not realize the meaning or the value of the fraction. Like Witawat and Jiyani, in Prathomsuksa 3, in the "Which One Is Better" problem, she also said "I know that the chance of a dice to win is $\frac{3}{6}$ and the chance of the coin to win is $\frac{1}{2}$. But I think the dice is going to win because there are three ways to win." However, when asked to order probability in a 2-dimensional situation, she could make a connection between sample space composition and probability measure. For example, in the post-assessment Situation 3, she responded as depicted in the excerpt below.

Researcher: I am going to toss the coin and the dice and I guess H-3 will come up. Do you think I'm right or wrong?

Anothai: You could be wrong.

Researcher: Why?

Anothai: There are many ways that you are wrong.

Researcher: Could you tell me the chance that I am right?

Anothai: $\frac{1}{12}$.

Researcher: How about the chance that I am wrong?

Anothai: $\frac{11}{12}$.

Researcher: Could you tell me again why I am going to be wrong?

Anothai: You chose just 1 way and there are 11 ways that you will be wrong.

Although this response showed that Anothai used fractions to describe her thinking, she did not think of fractions as numbers that she could compare. When she had to explain probability she would hold on to a part-part relationship. At the last two assessment points, Tadapon was at Level 4 thinking about probability of an event while Anothai was at Level 3.

With regard to probability comparisons, Tadapon did not experience difficulty in comparing probability across sample spaces, since he consistently held the part-whole reasoning. Moreover, he was able to use fractions for comparing the chances of two events. Because of his ability to understand fractions, it was not difficult for him to explain why he chose Box A as better for getting a red marble and chose Box B as better for getting a green marble in Situation 6 in the post-assessment (see Appendix A). The excerpt below shows how he responded in this situation.

Researcher: If you want a red marble, which box A or B you are going to pick?

Tadapon: Box A.

Researcher: Why?

Tadapon: The chance of getting a red marble from box A is more than the chance from box B?

Researcher: Can you use numbers to explain?

Tadapon: From A is $\frac{9}{17}$ but from B is $\frac{5}{10}$.

Researcher: Do you mean $\frac{9}{17}$ is more than $\frac{5}{10}$?

Tadapon: Yes, because $9/17$ is more than a half but $5/10$ is a half.

From this response, Tadapon exhibited his thinking about probability comparisons at Level 4. However, he was assigned between Level 3-4, since he could not incorporate noncontiguous and contiguous outcomes in determining probabilities. In the "Which Spinner" problem (Session 14, see Appendix B), although the class discussion concluded that both spinners gave the same chance of getting red, Tadapon still could not represent the probability of getting a red from spinner 2. He said, "I think it cannot tell because the sectors are not the same size."

As mentioned earlier, Anothai always used part-part relationship as she did at the pre-assessment point. During instruction, she demonstrated that her thinking was still based on a part-part schema. For example, in "Which One Is Better" Problem (Session 3, see Appendix B), she chose the dice. The witness noted, "Anothai choose the dice. She said that the dice had three ways to win but the coin had just one." She also used part-part reasoning in the last two assessments. For instance, in Situations 6 and 7 of the post-assessment (see Appendix A) she chose the boxes that had more marbles. Consequently, Anothai was assigned at Level 3 thinking about probability comparisons.

Although Anothai used only a part-part schema to quantify probability, she could recognize that changes in probabilities occurred in nonreplacement situations. At the pre-assessment point, she was at Level 2 thinking about conditional probability because she could not justify her thinking. She was unable to describe how and why probabilities changed after the researcher had drawn a yellow marble out of the box (Situation 9, see Appendix A). When the researcher asked why the chance of green changed she said, "The number of yellow marbles decreases. It makes the green marbles have more chance. But I don't know how to explain." However, at the last two assessment points, she was assigned at Level 3. This development was supported by her elaboration, which still held part-part reasoning. The excerpt below is Anothai's responses in Situation 9 of the post-assessment. It provides the evidence that she could elaborate on her thinking by using part-part relationships.

Researcher: Why do you think the chance of getting a green marble decreases in the second draw? (First draw the teacher had drawn a green marble.)

Anothai: Because now they are the same number as the blue marbles, four marbles.

Researcher: How about the first time?

Anothai: The number of the green marble is more than the blue marbles.

Researcher: How about the chance of getting a white marble in the second draw. Does it increase or decrease?

Anothai: Increases.

Researcher: Why?

Anothai: The first time there are 5 green marbles and 2 white marbles. But in the second draw, there are 4 green marbles and 2 white marbles.

Anothai's response in the above excerpt showed that she had developed her part-part schema. Although she used part-part relationships in a stable manner to compare probabilities, she demonstrated the flexibility of using part-part schema. This appeared in her use of an *extreme part-part schema*. This schema sometimes enabled students to compare probability across sample spaces.

Tadapon was assigned at Level 4 thinking for conditional probability at the last two assessment points. He not only recognized the changes of probability in nonreplacement situations but also described these probabilities by using fractions. For example, in Situation 9 in the post-assessment, he could compare the probability of getting a white marble in the first draw and the second draw. He said "the chance of getting white in the first is $\frac{2}{11}$. But when you drew a yellow marble out, its chance becomes $\frac{2}{10}$. And $\frac{2}{11}$ is less than $\frac{2}{10}$."

In an overall sense, although, both Tadapon and Anothai were recorded as high mathematics achievers, their probabilistic thinking developments were different. Prior to the teaching experiment, both students used part-part reasoning to describe probability situations. With more experience doing probability problems, Tadapon had developed part-whole schema, whereas Anothai still held on to part-part reasoning. However Anothai developed *extreme part-part schema*. That enabled her to compare probabilities across sample spaces sometimes.

Analysis of Wisanu's and Kanok's probabilistic thinking. Wisanu and Kanok were in Prathomsuksa 5. Their achievements in mathematics were in the middle quartile. Both students exhibited Level 2 across the four constructs at the pre-assessment point. During instruction, they developed their probabilistic thinking to Level 3 or 4 on all constructs (see their probabilistic thinking profiles, Figure 7 and 8). The paragraphs below provide the narrative of their development.

With respect to sample space, the pre-assessment revealed that Wisanu and Kanok experienced difficulty in listing all outcomes in 2-dimensional situations but not in 1-dimensional situations. Both students could not list the complete set of sample space outcomes in Situations 2, 3 and 5 which involved 2-dimensional situations (see Appendix A). During instruction, there was evidence that the class discussion in Sessions 6 and 7 enabled them to systematically list all outcomes in 2-dimensional situations. This ability, listing all outcomes in 2-dimensional situations, was still stable through the last two assessments. Consequently, they were assigned at Level 3 thinking about sample space at post- and retention- assessment points.

In probability-of-an-event situations, they were at Level 2 prior to instruction. Their responses in the pre-assessment provided evidence that they sometimes quantified probabilities using part-part reasoning. In contrast, they ordered probabilities using subjective judgments. For example, in Situation 1, Wisanu said that the spinner would land on a white because the sector of white was under the sector of red. In contrast, when asked to order probabilities in Situation 2, both Wisanu and Kanok were prone to describe probability using part-part schema. Nevertheless, their thinking showed that they could make a connection between probability measure and sample space composition in a 1-

dimensional situation. The following excerpt is Kanok's responses in Situation 2 of the pre-assessment. This excerpt provided evidence that she made a connection between sample space composition and probability measure prior to instruction.

Researcher: What could come up when we toss a dice?

Kanok: 1, 2, 3, 4, 5 and 6.

Researcher: If you toss the dice, I guess you will get the number that is more than 4. Do you think my guess will be right or wrong?

Kanok: Should be wrong.

Researcher: Why?

Kanok: There are many ways that are not more than 4.

Researcher: How many?

Kanok: 3 ways.

Researcher: What are they?

Kanok: 1, 2 and 3.

Researcher: How about 4? Is it more than 4?

Kanok: No. There are 4 ways that you will be wrong.

Researcher: Okay, there are 4 ways that I will be wrong. But why do you think I will be wrong?

Kanok: Because the number of ways you will be right is less than the ways you will be wrong.

Researcher: How many ways will I be right?

Kanok: 2.

Wisanu responded in a similar way. With regard to 2-dimensional situations, both students were unable to connect the set of outcomes for describing probability. They were prone to subjectivity when justifying the probability of an event in 2-dimensional situations.

During instruction, they exhibited the development of their thinking about probability of an event. There was evidence of them using part-whole reasoning and fractions to describe and represent probability. In the problem “Which One Is Better” of Session 3 (see Appendix B), at first both of them chose the dice. After the class discussion, they changed their thinking. The witness noted: “Wisanu said the coin and the dice have the same chance to win because 3 in 6 and 1 in 2 are equal.” Similarly, the witness noted: “Kanok used the fraction to represent the chance of the coin and the dice to win. And she knew that $\frac{1}{2}$ (coin) and $\frac{3}{6}$ (dice) were equal.” In 2-dimensional situations, both students demonstrated that they could relate the sample space composition to probability measure. For example, in Session 6, “Playing 2 Coins” (see Appendix B), Witsanu and Kanok were asked to choose Nok or Toa. At first they chose Nok. Wisanu said “Nok should win because there are two ways for him but Toa has just one.” After the teacher asked students to list all outcomes of tossing two coins, both students changed their thinking and said that Nok and Toa had the same chance. In later sessions, when asked to order or compare probabilities, they listed all outcomes and then ordered or compared probabilities.

The ability of Wisanu and Kanok to use fractions to represent probability of an event and their understanding of the connection between sample space composition and probability measure appeared consistently at the last two assessment points. They were assigned at level 4 thinking for probability of an event.

With regard to probability comparisons, prior to instruction Kanok seemed to exhibit an *extreme part-part schema*. The evidence came from her responses in Situations 6 and 7 in the pre-assessment (see Appendix A). Although, in Situation 6 she correctly chose the boxes that had more chance of getting a red marble or more chance of getting a green marble, she was not able to justify her choice. In contrast, in Situation 7, she compared the number of the red marbles with the number of the green marbles in each box,

and then compared the chances of getting a red marble in each box and getting a green marble in each box.. In the same situation, Wisanu showed that he held a *stable part-part schema* in his thinking for probability comparisons. He compared the probability of getting a marble in each box by comparing the number of each kind of marbles across the boxes. For example, he chose box A for having more chance for a green marble because in box A there were more green marbles than in box B.

As mentioned early, Wisanu and Kanok developed their thinking by using part-whole relationships during instruction. They also reasoned in probability comparisons by using this schema, for example in the "Which One Is Better" game, involving the dice and the coin. Wisanu and Kanok were able to explain that the chance of the dice winning was $\frac{3}{6}$ and the chance of the coin winning was $\frac{1}{2}$, and they said that they were the same chance. For another example, in Session 5, the "Car Racing" game (see Appendix B), they used a part-whole relationship for describing the probabilities of spinner 1 and 2. At the beginning they both said that spinner 1 was better for the red car because there were more red sectors than on spinner 2. After the discussion about how to use numbers for describing the chance in each spinner, they used fractions $\frac{3}{4}$ for the chance of the red car on spinner 1 and $\frac{1}{4}$ for the chance of the red car on spinner 2.

At the last assessment points, both Wisanu and Kanok were identified at Level 3-4 thinking about probability comparisons. Similar to Tadapon, they faced the difficulty of comparing the probability of a noncontiguous situation. For example, in Situation 8 of the post-assessment, they believed that spinner 2 was better for getting red than spinner 1. They could describe the chance of getting red from spinner 1 ($\frac{3}{8}$) but could not describe the chance from spinner 2.

With regard to conditional probability, although Wisanu and Kanok demonstrated prior to the instruction that they recognized the probability of all events changed in a nonreplacement situations, they were assigned at Level 2. At the pre-assessment point, they were unable to justify their thinking. For example, in Situation 9 (see Appendix A), at the second draw, they said they knew the chance of getting a green marble increased but they did not know how to explain it. At the last two assessment points both students used

fractions to describe the changing of probability in nonreplacement situations. The excerpt below is Kanok's responses in Situation 9 in the post assessment (see Appendix A).

Researcher: Why do you think the chance of getting a green marble decreases in the second draw? (First draw the teacher drew a green marble.)

Kanok: Because now they are the same number as the blue marbles.

Researcher: Can you use numbers to compare the chance of getting a green marble between the first draw and the second draw?

Kanok: First draw is $5/11$. Second draw is $4/10$.

Researcher: Which one is greater, $5/11$ or $4/10$?

Kanok: ... (silence)... But, the first time, the number of the green marble is more than the blue marbles. At second time they are the same number. Its chance decreases.

Researcher: How about the chance of getting a white marble in the second draw. Does it increase or decrease?

Kanok: Increases.

Researcher: Why? Can you use numbers to explain?

Kanok: The first draw is $2/11$ and the second draw is $2/10$.

Researcher: Which one is greater, $2/10$ or $2/11$.

Kanok: $2/10$.

Researcher: Why do you know?

Kanok: Because two in eleven is harder to get than two in ten.

The other interesting point of this response was that Kanok had difficulty in comparing the fractions, $5/11$ and $4/10$. She still demonstrated her thinking by changing to use part-part relationship. Wisanu was slightly different. There was evidence that when he faced the difficulty of comparing fractions, he struggled and could not elaborate on his thinking. The next excerpt provides this evidence.

Researcher: Why do you think the chance of getting a green marble decreases in the second draw? (First draw the researcher drew a green marble).

Wisanu: Because they are the same number as the blue marbles. At first draw they are more than blue."

Researcher: Can you use numbers to compare the chance of getting a green marble between the first draw and the second draw?

Wisanu: First draw is $5/11$. Second draw is $4/10$.

Researcher: Which one is greater, $5/11$ or $4/10$?

Wisanu: ...I don't know.

Researcher: Do you still think that the chance of getting a green marble decreases in the second draw?

Wisanu: Yes, I do.

According to their ability to describe the change of probability in nonreplacement situations, both students were assigned at Level 3 thinking about conditional probability at the post-assessment point. At the retention-assessment point, Kanok moved to Level 4 whereas Wisanu was still at Level 3.

In conclusion, Wisanu and Kanok developed their probabilistic thinking from Level 2 to Level 3-4. The key feature of their development was using both part-part and part-whole relationships to describe probability. In addition, they used fractions to represent and quantify probability but they sometimes had difficulty in comparing fractions.

Analysis of Kitisak's and Jirawan's probabilistic thinking. Kitisak and Jirawan were in Prathomsuksa 5. Their achievement in mathematics was in the lower quartile. Their thinking at the pre-assessment point was assigned at Level 1 or 2 across four constructs. At the last two assessment points, they exhibited thinking at Level 2 or 3 (see their probabilistic thinking profiles, Figure 9). The paragraphs below describe how they responded to the assessments and how their thinking matured during instruction.

With respect to sample space, Kitisak and Jirawan demonstrated that they were able to list all outcomes in 1-dimensional situations at the pre assessment point. In contrast, they experienced difficulty listing all the outcomes in 2-dimensional situations. Thus, they were assigned at Level 2 thinking for sample space. However, Kitisak developed his thinking about sample space to Level 3 in the post- and retention-assessments points. This development was captured during the teaching. For example, when asked to list all the outcomes of 2-dimensional situations in Sessions 6 and 7, his paper work provided the evidence that Kitisak systematically listed all the outcomes. In contrast, Jirawan did not use any strategy to list all outcomes in those problems. However, she could list all outcomes correctly. In post- and retention-assessments, both students could list the complete set of outcomes in 1- and 2-dimensional situations. Jirawan did not use any system to list all outcomes, whereas Kitisak systematically listed the outcomes in some 2-dimensional situations, such as Situation 3 (see Appendix A). Consequently, in the last two assessment points, Kitisak was assigned at Level 3 thinking for sample space, whereas Jirawan was at Level 2.

In probability-of-an-event situations, Kitisak and Jirawan were typical students who held subjective reasoning prior to instruction. Both students ordered probabilities subjectively in the pre-assessment. For example in Situation 1, Jirawan said that red was better than white because it was beautiful. Similarly, Kitisak said that white was better without using reasoning. "I just guessed" he said. At this point, they were assigned at Level 1 thinking about probability of an event. At post- and retention-assessment points, Jirawan and Kitisak exhibited their thinking at Level 2 and Level 3 respectively. This development was also captured during the teaching. For instance, in the "Spinner Problem" (see Appendix B, Session 1), they could describe why they chose white by comparing the

number of sectors of white (5) and red (3). They both said “there are more white than red.” Consistently, in Session 5 (see Appendix B), they were able to choose the spinner that was better for each car. The witness’s field notes provided this evidence. The witness noted, “Jirawan said spinner 1 was better for the red car because there were more red than green, and spinner 2 was better for the green car because there were more green than red.” Similarly, the witness noted Kitisak’s responses: “Kitisak said that he chose spinner 1 for the red car because there were 3 red and 1 green, and chose spinner 2 for the green car because there were 3 green and 1 white.” These responses were interpreted that Jirawan seemed to develop from the Level 1 to the Level 2, whereas Kitisak seemed to develop to the Level 3 and the use of part-part schema. There was no evidence that either student held subjective judgments during teaching.

As mentioned, at the post-assessment point, Jirawan was assigned at Level 2 thinking about probability of an event. She exhibited her thinking consistently by using the word “more than” or “less than” to order probabilities at this point. She also found it was difficult to order probabilities in 2-dimensional situations. Although she was able to list all the outcomes of 2-dimensional situations, she did not consider the sample space composition for quantifying probability. For example, in Situation 3 (see Appendix B), when asked to compare the chance of getting a Red-3 and not getting a Red-3, she could not elaborate on her thinking as to why the chance of getting Red-3 was more than not getting a Red-3.

In the same situation, Kitisak responded differently. He consistently used part-part reasoning, which he exhibited during the teaching, to describe probabilities in both 1- and 2-dimensional situations. In Situation 3 of the post-assessment, when asked to compare the chance of getting a RED-3 and not getting a RED-3, he said that 11 ways were not RED-3 and just 1 way was RED-3. In view of this kind of response and his consistency in using part-part reasoning in other situations, he was assigned at Level 3 thinking for probability of an event at the last two assessment points.

With regard to probability comparisons, both students were at Level 2 prior to the teaching. They demonstrated that they were able to compare probabilities in two different sample spaces by considering the number of the components in each sample space. For

example, in Situation 6 of the pre assessment (see Appendix A), Jirawan chose Box A to have more chance for getting both a red marble and a green marble. This response was interpreted that she compared the number of each kind of marbles in each box. Kitiskak seemed to have more idea in probability-comparison situations. He was able to choose the boxes correctly, both for a red marble and a green marble. Unfortunately, he could not elaborate on his thinking. However, after the teaching, he was able to develop his thinking to Level 3, whereas Jirawan still struggled at Level 2.

During the instruction, in Session 3, Kitiskak chose the dice for the “Which One Is Better” Problem (see Appendix B). He said: “The dice is easy to win because there are three ways. The coin has just one.” This response was an example of part-part schema that he consistently used to compare probabilities. Moreover, in Session 10 of post-assessment, he demonstrated that his part-part schema was an *extreme part-part schema*. In the problem “Pick Up A Marble” (see Appendix B) he was able to choose the box which was better for a blue marble and the box which was better for a red marble. He said: “Box 1 is better for red because blue and red are the same number while Box 2 has more blue than red. Box 2 is better for blue because there are more blue than red but Box 1 has the same number.” This response was interpreted that he compared the number of each kind of marbles in each box. Then he compared the probabilities across the boxes. Moreover, he consistently used this schema in the last two assessments.

For Jirawan, she was still at Level 2 thinking about probability comparisons in the last two assessment points. As mentioned about her thinking in probability-of-an-event situations, she continued to use part-part schema for probability comparisons during instruction and the last two assessments. Moreover, she was unable to compare probability across sample spaces. The way she used part-part schema to compare probabilities was interpreted as she used a *stronger part-part schema*.

In thinking about conditional probability, Kitiskak and Jirawan were at Level 2 at the pre-assessment point. They were unable to recognize the change of probabilities in a nonreplacement situation. In Situation 9 of the pre-assessment (see Appendix A), they claimed that if a marble (say green) had drawn been out, the chance of the second draw of a green marble was reduced, but the other kind had the same chance as the first draw.

At the last two assessment points, Jirawan was at Level 2 the same as the pre-assessment point. Given that Jirawan always held a stable part-part reasoning to quantify probability during instruction, she was unable to compare probability using this schema in nonreplacement situations, for example the probability of the first draw and the second draw. Her response in post- and retention-assessment in conditional-probability situations was similar to what she did in the pre-assessment.

Kitisak demonstrated stronger reasoning in describing probability in nonreplacement situations. In Situation 11 of the post assessment (see Appendix A), he could explain the chance of getting a flower sticker and getting an animal sticker in the first and second draws. Nevertheless, he exhibited his thinking differently in Situation 9. He did not know that when the researcher had drawn a green marble out of the box, the probability of getting a white marble changed in the second draw. The excerpts below show how he responded in Situations 9 and 11.

Situation 9 (The students knew that a green marble has been drawn out.)

Researcher: If I draw a marble now, compared with the first draw, does the chance of getting a green marble change?

Kitisak: Yes.

Researcher: How? I mean does it increase or decrease?

Kitisak: Decrease.

Researcher: Why?

Kitisak: Its number decreases.

Researcher: How about the chance of getting a white marble, does it change, increase or decrease?

Kitisak: It does not change.

Researcher: Why?

Kitisak: Its number doesn't change.

Situation 11 (The student knew that a flower sticker has been drawn out.)

Researcher: If you draw a sticker now, compare with the first time that you do not draw, does the chance of getting an animal sticker change?

Kitisak: Yes.

Researcher: How does it change?

Kitisak: The chance of an animal sticker increases.

Researcher: Why?

Kitisak: Now the numbers of both kinds of sticker are the same. At the first time, there are more flower stickers than animal stickers.

Researcher: How about the chance of getting a flower sticker again. Does it increase, decrease or the same?

Kitisak: Decreases.

Researcher: Why?

Kitisak: The first draw its number is more than the animal sticker, but now the numbers of them are the same.

The response in Situation 9 showed that Kitisak used part-part schema to compare the number of green marbles in the box of the first draw and the second draw and did the same with the white marble. In contrast, in Situation 11, first, he compared the numbers of each kind of sticker, and then compared the chance between the first draw and the second draw. However, according to these responses, he demonstrated that he did not recognize that the probability of *all* events changes in a nonreplacement situation. Finally, he was assigned at Level 2 thinking about conditional probability.

In conclusion, Kitisak developed his probabilistic thinking to hold extreme part-part schema whereas Jirawan held a stronger part-part schema for describing probability. Their probabilistic thinking development depended on their using only part-part reasoning. Kitisak's extreme part-part schema enabled him to develop his probabilistic thinking to Level 2 or 3 across all constructs. In contrast, Jirawan could not move her probabilistic thinking beyond Level 2.

Conclusion of students' probabilistic thinking. Analyses of all target students' thinking described in the previous sections revealed the existence of these patterns: (a) a *stronger part-part schema* that enabled students to reason with greater consistency when listing the complete set of outcomes and when ordering probabilities, but did not enable students to compare probability across sample spaces; (b) an *extreme part-part schema* that is effective in enabling students to both identify and order, and sometimes compare probabilities across sample space; (c) a *part-whole schema* that students developed after the emergence of *invented informal language*.

With regard to the pattern (a), a student who exhibited a *stronger part-part schema* quantified probability situations by comparing part to part. The student also compared the number of the same components across sample space. This ability did not enable him/her to compare probabilities across sample spaces. Jiyani, Kamol, in Prathomsuksa 3, and Jirawan, in Prathomsuksa 5, exemplified this thinking.

With respect to the pattern (b), a student who exhibited *extreme part-part schema* not only used part-part reasoning to quantify probability situations but also used part-part schema in flexible and meaningful ways. This ability enabled him/her to quantify probability beyond the ability of *stable part-part schema*. He/she sometimes could compare probability across sample spaces. Witawat and Piyawan, in Prathomsuksa 3, and Kitisak, in Prathomsuksa 5, exemplified this thinking.

With regard to pattern (c), a student who exhibited part-whole schema could describe probability situations beyond a student who held part-part schema. In probability-comparison situations, this student compared probabilities across sample space by considering the chance of the event in each sample space, and then compared those

chances. Pirapan and Natida, in Prathomsuksa 3, Tadapon, Wisanu and Kanok, in Prathomsuksa 5, exemplified this thinking.

Analysis of Students' Beliefs about Probability

A purpose of this study was to examine students' beliefs as they responded in probability situations. These beliefs were captured from students' responses in all assessments, pre-, post-, and retention-assessment, and during instruction. Schoenfeld's (1985) definition of belief and those from other studies (Fischbein & Gazit. 1984; Fischbein; et.al. 1991; Fischbein & Schnarch. 1997; Garfield & Ahlgren. 1988; Piaget & Inhelder. 1975; Shaughnessy. 1992; Zimmermann. 2002) guided the analysis of students' beliefs about probability. In accord with the position taken by these aforementioned studies, a belief in this study would incorporate a disposition, an intuition and biases.

Outcome approach. Most students appeared prone to describing the probability of an event using subjective judgments at the pre-assessment point. Some students not only subjectively described their thinking but also exhibited a belief what Konold (Konold; et al. 1993 : 399-402) called *outcome approach*. Consider the excerpt below; Natida, who was in the higher quartile, described her thinking in Situation 1 in two different ways. .

(Part 1)

Researcher: If I spin the spinner, which one between red or white has more chance of winning? Or do they have the same chance?

Natida: I think they both have same chance.

Researcher: Why? Why do they have same chance?

Natida: ...(silent)... Actually, I guess.

(Part 2)

Researcher: How about red and yellow, which one has more chance?"

Natida: Yellow.

Researcher: Why?

Natida: I like yellow.

In Part 1, although Natida was not able to elaborate on her thinking, her responses showed that she was *determining probability* of red and white on the spinner. In contrast, in Part 2, she responded as if she was asked to *predict* what would come up between red and yellow. Natida exemplified many students, especially the students were above middle quartile, who responded by guessing at the early Situations in pre assessment. These students sometimes tried to guess what would happen rather than determining the probability of an outcome.

In the pre-assessment, the students were more likely to exhibit the outcome approach in situations where they thought it was difficult to make decisions, such as the situations in which all outcomes had same chance to occur or the situations where students could not list all outcomes. Analysis of 12 target students found that there are 9 students (75%) who exhibited the outcome approach misconception. All these misconceptions emerged during the pre-assessment. There was no evidence that students exhibited outcome approach misconception in the last two assessments or during instruction.

Positive and negative recency effect. In Situation 2 of the pre-assessment, the researcher asked target students if the you tossed a dice and you got a 5, would a 5 come up next time or would the dice change to another number. The students responded differently. Some said changed the others said not. Some students believed that it would change because it was difficult to repeat, whereas others believed that it would repeat again. All these reasons were interpreted as saying that these students held either the negative recency effect or positive recency effect (Greeno.1991 : 322-323, Borovcnik & Bentz. 1991 : 269-270). In the pre-assessment 3 target students (25%) exhibited positive recency effect, whereas the rest (75%) of them held negative recency effect. Some students, such as, Wisanu in Prathomsuksa 5 and Natida and Witawat in Prathomsuksa 3,

exhibited the negative or positive recency effect more than one time at the pre-assessment period. The excerpts below illustrate Natida's responses in Situation 2 of the pre-assessment. Natida described a probability of an event by holding a negative recency effect. In contrast, she exhibited positive recency effect in Situation 10.

(Situation 2)

Researcher: What number do you get?

Natida: 2.

Researcher: Okay, if you toss it again, do you think you will get 2 again or you will it change?

Natida: It will change.

Researcher: Why?

Natida: When you toss a dice again it will not repeat.

(Situation 10)

Researcher: I already tossed this coin 4 times and it's all heads. If I toss this coin again, what side will come up, head or tail or they have the same chance?

Natida: Head.

Researcher: Why?

Natida: Head came up for many times. It should be head again.

Natida was the only student who exhibited both negative and positive recency effects, whereas the others exhibited either negative or positive recency effect.

During instruction in Session 6, Situation 10 in the pre-assessment was posed to the classes. About 60% of students responded that the coin would change, about 25% chose that it would not change, and the rest (about 15%) said that it had the same chance of head and tail.

In the post- and retention assessment points, the target students showed a decrease in their use of positive and negative recency effects to describe probability situations. Nevertheless, there were 50% of students who still exhibited negative recency effects at the last two assessment points.

Representativeness. Students who are identified as holding a representativeness heuristic, would predict that an outcome which illustrates a more even distribution is more likely to occur than one that does not (Kahneman & Tversky. 1972 : 444-445). In this study, at the pre-assessment point, 80% of target students showed that they held the representativeness heuristic. When asked to guess what number would be written in a guessing game between 537 and 555, there were 9 students who chose 537. They all thought that the outcome 537 was more likely to occur than 555.

During the instruction, some evidence suggested that the majority of students in both classes exhibited the representativeness heuristic. In session 7, the researcher posed a problem about tossing a fair coin 4 times and asked what would more be likely to occur the outcome TTTT or the outcome THTH. Most students (about 90%) chose THTH. Some students said the outcome TTTT was difficult to occur because they were all tails, whereas others said the outcome should be both H and T. During the class discussion, students listed all outcomes, and then most students concluded that both TTTT and THTH had the same chance ($1/16$).

At the last two assessment points, there were about 50% of target students still who exhibited the representativeness heuristic. Most students who did had been assigned at level 2 thinking about probability of an event or sample space. Since they were unable to make connections between sample space composition and probability measure, they did not consider sample space composition in considering probability.

Belief about a non-contiguous probability generator. In Situation 8 of pre-assessment, target students were asked to determine probability of getting a red from two

spinners (see Appendix A). Most students (75%) believed that spinner 2 is better for getting a red than spinner 1. Actually, the probabilities of getting a red of both spinners were the same. Although these students knew that the area of red on both spinner were equal, they believed that the chance of getting a red on both spinner were not the same. A student said, " I think one big sector is better than many separate sectors." From these responses, students seemed to consider single event for red rather than a combined event for red (Jones. 1974). They were not convinced that all the red together on Spinner 1 had the same area of red as on Spinner 2. They were not willing to combine all the red to produce the complete event of red.

During instruction, Session 14, this problem was posed to the classes. Before playing the game, most students (95%) chose spinner 2 for winning the game. Although the results indicated that both spinners have the same chance of winning, most students still believed that spinner 2 was better than spinner 1. The following excerpt shows how a student in Prathomsuksa 5 explained why he chose the spinner 2.

Researcher: How does the game turn? Which spinner is better?

Pongsak: Both spinners have the same chance.

Researcher: Okay, if we play, which spinner do you want to use, Pongsak?

Pongsak: I choose spinner 2.

Researcher: Can I have spinner 2?

Pongsak: No, I chose it first.

Researcher: Why did you choose spinner 2?

Pongsak: I think it's easy to get red because the sector is bigger.

Researcher: But spinner 1 has 4 sectors of red.

Pongsak: They are all small. I like big sector.

Researcher: You just said that both spinners had the same chance of winning.
Do you still think that?

Pongsak: Yes, but if we play I like spinner 2.

Researcher: Who would choose spinner 2?" (The researcher asked the class.)
More than 80% of students raised their hands.)

In the post- and retention-assessment, most students (75%) in the target group still chose spinner 2 to play the game. Although, 25% of the students in this group responded that both spinners were equivalent, they were still not able to quantify probabilities in spinner 2.

Sample space misconception. In the study of Jones; et al. (1999), some students exhibited a sample space misconception—students did not recognize the possibility that all outcomes could occur and responded as the asked to predict probability. In contrast, there was no evidence of the sample space misconception in this study and all target students were assigned beyond Level 1 thinking about sample space prior to the instruction

All target students seemed to understand sample-space situations prior to the instruction. When asked to list all outcomes in pre-assessment point, most students usually asked, "Do I have to list all?" while they were listing the outcomes. Some of them said, "It can be any colors," when asked to list all the outcomes of the spinner in Situation 1 (see Appendix A). At this point, the students could list all outcomes of 1-dimensional situations.

In summary, there were a number of students' beliefs about probability emerged during instruction and in the assessments. These beliefs decreased after the instruction. The outcome approach, for example, was not identified again after the instruction, whereas the other misconceptions still persisted but the number of students who held these misconceptions decreased. For those students who still held some misconceptions after instruction, these misconceptions were deep seated in their thinking and difficult to change (Zimmerman. 2002 : 140).

Sociomathematical norms and classroom mathematical practices

According to one of the purposes of this study, the teaching experiment was designed to examine the emergence of sociomathematical norms and classroom mathematical practices. Figure 10 shows a summary of the sociomathematical norms and their corresponding classroom mathematical practices that emerged during the teaching experiment.

Sociomathematical Norms	Classroom Mathematical Practices
What counts as a valid way to describe and predict probability	• Using invented informal language (e.g., 3 in 8 or 3 from 8) • Connecting sample space composition and probability prediction
What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations	• Listing all outcomes systematically (e.g., odometer strategy)

Figure 10 Sociomathematical Norms and Classroom Mathematical Practices

Sociomathematical norms. There were two sociomathematical norms that emerged in the teaching experiment: (a) What counts as a valid way to describe and predict probability; and (b) what counts as an efficient way to list all outcomes of 2- and 3-dimensional situations. These norms will be described when and how they emerged during instruction in the following paragraphs.

What counts as a valid way to describe and predict probability. The sociomathematical norm of “What counts as a valid way to describe and predict probability” always appeared during the instruction. For example, in the first session, the researcher posed the “Spinner Problem” (see Appendix B) and asked students to describe the probability of getting a red and the probability of getting a white. In both classes, students discussed what numbers were involved on the spinner. At first, most students said these

numbers were 3 and 5 (number of each color sectors). Students also ordered the probabilities of red and white by using these numbers. When the researcher showed another spinner that had 6 red sectors and 10 white sectors (same as the first spinner but each sector was divided in half), some students decided that the number of sectors on the first spinner (3 of red and 5 of white) was not the right probability of red and of white. The issue they discussed was why the chances of red on both spinners were not the same.

In session 3, students discussed the chance of getting an H on a coin and an even number on a dice. There were two groups of students who had different ideas. The first group said the chance of getting an H was less than the chance of getting an even number on a dice. They reasoned that there were three even numbers on a dice but just one H on a coin. The other group (a few students in each class) said they both had the same chance. Nevertheless, the students in this group could not elaborate on their thinking.

Another example was that students were involved in discussion of the "Racing Game" problem, Session 9 (see Appendix B). The issue in the discussion was which horses had more chance to win. Pirapan and Natida (in Prathomsuksa 3 class), and Tadapon (in Prathomsuksa 5 class) said that they would list all outcomes of tossing two dice first, and then they would know which horse was going to win. Some students said every horse had the same chance.

As mentioned above, in most instructional sessions students discussed the ways to describe, order and predict probability. Consequently, the sociomathematical norms "What counts as a valid way to describe and predict probability" emerged from most the sessions of the teaching experiment.

What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations. The sociomathematical norms "What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations" firstly emerged during the discussion in 2-dimensional situations, in Session 6. After the researcher asked students to list all the matches of a shirt and a pair of pants (see "Dress Up" problem, Appendix B), the researcher let students check if their lists were complete or not and asked how they knew. Most students in both classes said that they just checked their list again. Some students suggested strategies to list all matches. Others agreed that the strategy make them easy to

list all the outcomes. Consequently, the issue of discussion of students became how to list all matches systematically.

In Session 7 the researcher posed a 3-dimensional situation to list all outcomes of getting dressed by choosing a shirt, a pair of pants and a pair of shoes. Students in both classes discussed how to list systematically when they had three groups of things. In each class some students presented their thinking. Natida (in Prathomsuksa 3) said that she would choose a shirt, and then matched with all pairs of pants. After that she would match with all pairs of shoes.

In session 9 the sociomathematical norm "What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations" emerged during the discussion of the "Racing Game" (see Appendix B). Tadapon (in Prathomsuksa 5) said that he needed to list all outcomes of rolling two dice because it would make it easier to know which horse was going to win. Some students in the class thought the same way and started to discuss how to list all outcomes of rolling two dice. Tadapon presented his idea that he would list by picking 1 in the first dice and match with 1-6 in the second dice, and then picked 2 in the first dice to match with 1-6 in second dice again. He would do the same until he picked 6 in the first dice to match with 1-6 of the second dice.

Classroom mathematical practices. During instruction, students' discussion led to the emergence of the sociomathematical norms. In the discussion there also appeared classroom mathematical practices that corresponded to those norms. Figure 10 shows that there are two classroom mathematical practices which correspond to the sociomathematical norm "What counts as a valid way to describe and predict probability": (a) Using invented informal language; and (b) Connecting sample space composition and probability prediction. On the other hand, the classroom mathematical practice "Listing all outcomes systematically" corresponds to the sociomathematical norm "What counts as an efficient way to list all outcomes of 2- and 3- dimensional situation.'

Using invented informal language. In session 2, the researcher posed the question how to describe the chance of getting red and white on the spinner in the "Spinner Problem" game (see Appendix B). At first, students counted the red sectors to compare with the number of white sectors. Most of them said that white should win because there

were more white sectors than red sectors. After the researcher asked students to explain by using numbers, students were able to invent *informal language* for describing the chance of getting red and white. In Prathomsuksa 5 class, Tadapon was the first one who grasped the concept of part-whole relationship. At first, he said, "Chance of red is 3 and chance of white is 5." The discussion in the class seemed to agree with Tadapon. The researcher used the same spinner but divided each sector into two parts and asked students to describe the chance of getting red and white again. Another student said, "Chance of red is 6 and chance of white is 10." At this point, some students questioned why the chances of getting red and white each time were not the same. They doubted that the number of sectors of each color might be the right answer. After the researcher convinced students to find the other number that they could think of on the spinner (the 8-sector spinner), Tadapon said, "I know 8. I count all the sectors." At this point, the researcher asked students again to use 3, 5 and 8 to describe the chance of getting red and white. Tadapon was again the first one who responded, "The chance of getting red is 3 in 8 and the chance of getting white is 5 in 8." By Session 3, the phrase "... in ..." was used by the other students in this class to describe the other situations and the phrase became taken-as-shared. In Prathomsuksa 3, students also invented informal language for describing the chance of getting red and white. Pirapan was the first one who had the idea of using the phrase "5 from 8" for describing the chance of getting white. He said, "We can say 5 from 8 because there are 5 white parts from all of 8 parts." The phrase "... (number) ... from ... (number) ..." became taken-as-shared by Session 3 as was the invented phrase "1 from 2" to describe the probability of getting a head in the Coin Game (Session 3). The phrases "5 from 8" or "5 in 8" were *invented informal language*. Students who were able to use part-whole reasoning in describing probability developed their thinking after the emergence of this invented informal language.

Connecting sample space composition and probability prediction. The other classroom mathematical practice that corresponded to the sociomathematical norm of "What counts as a valid way to describe and predict probability" was *connecting sample space composition and probability prediction*. This practice of connecting between sample space composition and probability measure was very important for students to develop

probabilistic thinking. Using sample space composition for predicting probability implicitly emerged in Session 2. As described above, students began to use informal language for describing probability during Session 2. Although, the way that students used number could be interpreted as meaning they were able to relate sample space composition and probability, it still needed more evidence from other situations, such as those where other probability generators or 2-dimensional situations were used. In Session 3 (see Appendix B), during the discussion about the chance of getting a H on a coin toss and the chance of getting an even number on a dice, some students were able to describe the chance of an H by using the phrase "1 from 2" (or "1 in 2") and "3 from 6" (or "3 in 6") for describing the chance of getting an even number from a dice. Those students were asked to explain their thinking to the class. They showed that they quantified probability by considering sample space compositions (list of all outcomes).

In 2-dimensional situations, when the researcher posed the "Racing Game" problem (Session 8, see Appendix B), more than 80% of students in each class answered subjectively. Yet, there were some students, such as Tadapon and Kanok in Prathomsuksa 5, Pirapan and Natida in Prathomsuksa 3, who listed all outcomes before making a decision. These students were asked to explain why they needed to list all outcomes. About 80% of the students explained in their journals how to choose the horse with the best chance. They indicated that they would consider how many ways that the horse could move before choosing a horse. Consequently, at Session 8, connecting sample space composition and probability predicting became taken-as-shared.

Listing all outcomes systematically. The classroom mathematical practice "Listing all outcomes systematically" corresponded to the sociomathematical norm of "What counts as an efficient way to list all outcomes of 2- and 3-dimensional situations." This practice emerged during the discussion about 2-dimensional situations, such as in Sessions 6, 7, 8, 10 and 12. In Session 6, although most students in each class could list the outcomes of the "Dress Up" problem, there were a few students who systematically listed all outcomes. When asked how to check their list of outcomes, whether it was complete or not, students who did not use any system said that they checked again and again. Tadapon in Prathomsuksa 5, who systematical listed the outcomes, could explain his thinking to the

class. He said, "I started with 1 pair of pants and then matched with all of the shirts. I started with a new pair of pants and then matched with all of the shirts again. I think I have all matches." The strategy Tadapon used is the *odometer strategy* (English, 1990 : 261). Similarly, in Prathomsuksa 3 class, Pirapan described his using of the odometer strategy to the class. In Session 7, the researcher posed more of a "Dress Up" problem which was a 3-dimensional situation—getting dressed by choosing a pair from 2 different colored pairs of pants, a shirt from three different colored shirts and a pair from two different colored pairs of shoes. About 40 % of students in each class successfully listed all outcomes. Moreover, most of them used the odometer strategy. In the Racing Game, Session 8, during the discussion of winning the game, students suggested listing all the outcomes of tossing two dice. About 80% of students in Prathomsuksa 5 and about 70% of students in Prathomsuksa 3 could list a complete set of outcomes systematically. This evidence suggested that the strategy had become taken-as-shared. Consequently, using the odometer strategy that emerged during instruction was accepted as a mathematical practice in both classes.

In summary, the analyses of students' probabilistic thinking in this study were consistent in both the quantitative and qualitative approach. The difference of students' performance scores between pre- and post assessment showed that the students had more ability in describing probability after instruction. The case study analyses confirmed that these students developed part-part or part-whole reasoning that enabled them to describe and quantify probability with various degrees of understanding. The emergence of sociomathematical norms and classroom mathematical practices was described from a social perspective to capture how students' collective thinking developed. In addition, the analysis of students' beliefs showed that students' intuitions and misconceptions decreased during instruction as their conceptions of probability became more mature.

CHAPTER 5

DISCUSSION

Since probability is a new topic in the primary school curriculum of Thailand, there is a need for research that traces primary students' thinking in probability and interaction while they are doing probability. This study addressed that need situation. In particular, this study was concerned with tracing students' growth in probabilistic thinking, determining students' beliefs about probability and examining the emergence of sociomathematical norms and classroom mathematical practices during a teaching experiment. This chapter provides a summary of the study and a discussion of the findings as they relate to the current research base.

Summary of the study and its finding

Reform in the Thai curriculum has immediate implications for research on teaching and learning. One of the current key issues that confront Thai mathematics educators or teachers is what teaching might be like if it was built on a view of students' knowledge development. An approach to addressing this problem is using research-based knowledge of students' thinking to inform instruction (Carpenter & Fennema. 1998 : 501).

There is a number of the studies (e.g.,Tarr. 1997; Jones; et al. 1999 and Polaki. 2000) that have attempted to use research-based knowledge of students' thinking to inform instruction in probability. However, there are very few studies that have been conducted in non-English speaking cultures (Polaki. 2000). Especially in Thailand, there is no study in this area. Like children in any culture, Thai students have different cultural norms for thinking and interacting among themselves. The study of Amir & Williams (1999) provides evidence that culture influences children's probabilistic thinking. Therefore, there should be studies on students' thinking and interactions while they are doing probability within the Thai context, especially in primary school where this topic is just beginning to be introduced.

Purpose of the study. This study carried out a teaching experiment and was designed to address the following purposes:

1. Trace the emergence and evolution of features of primary students' growth in probabilistic thinking;
2. Examine the beliefs that students exhibit as they reason in probability situations.
3. Identify and describe the evolution of sociomathematical norms and classroom mathematical practices that the students develop collectively as they tackle probability problems.

Methodology. The participants of this study consisted of students and witnesses. The students were from two classes of primary school in Chonburi, one class from each level of Prathomsuksa 3 and 5 respectively. In each class, 6 students were purposefully sampled in order to create a case study. One of the witnesses was a lecturer in the mathematics department in a university and the other was a primary school teacher.

Two weeks before the teaching started the researcher observed the mathematics classes of Prathomsuksa 3 and 5. Then, 6 students from each class were chosen to be target students. For purposefully sampling students from each grade level, the researcher attempted to select students that represented the following proficiency categories in mathematics: (a) lower quartile (2 students), (b) middle quartile (2 students), and (c) upper quartile (2 students). The levels of proficiency were based on students' school records of performance in mathematics. These target students were *interviewed* to measure the performance scores and levels of probabilistic thinking (pre-assessment). During the teaching experiment in each class, the researcher taught twice a week for eight weeks (45 minutes per period). The witness observed the classes. After the teaching sessions, the target students were assessed again two times, one week after the last session (post-assessment) and four weeks after the post-assessment (retention-assessment).

Data for the study was from the following sources: (a) pre-, post-, and retention assessments; (b) videotapes of classroom events; (c) students' written work, and (d) witness's field notes and reflections. There were two parts to the data analysis. The first part was concerned with students' responses in the interview assessment protocols. A double-coding procedure (Miles & Huberman, 1994 : 60) was used to code students' responses from those assessments and to generate students' performance scores and probabilistic

thinking levels. The second part was concerned with analysis of the whole-class teaching experiment. This analysis focused on students' individual and collective thinking and beliefs related to probability; it also identified the classroom sociomathematical norms and classroom practices that emerged.

Result of the study

Quantitative analysis. Using the Wilcoxon Signed Ranked Test (WSRT), the analyses of students' performance scores revealed that there was a significant difference between pre- and post-assessment scores of all target students. On the other hand there was no significant difference between post- and retention performance scores. For Prathomsuksa 3, the WART indicated a significant difference between pre- and post-assessment scores but no significant difference between post- and retention-assessment scores. Similarly, for Prathomsuksa 5, the WART indicated a significant difference between pre- and post-assessment scores but no significant difference between post- and retention-assessment scores.

Qualitative analysis. Data generated from the assessments, students' work, and from the researcher's and witness's field notes were used to trace the development of the target students' probabilistic thinking. Analyses of all these data sources revealed the existence of these patterns: (a) a *stronger part-part schema* (Polaki. 2000 : 101) that enabled students to reason with consistency when listing the complete set of outcomes of a probability situation and when ordering probabilities; however, this schema did not enable students to compare probability across sample spaces; (b) an *extreme part-part schema* that was effective in enabling students to identify, order, and sometimes compare probabilities across sample spaces; (c) a *part-whole schema* that students developed after the emergence of *invented informal language*, for describing probability measures.

The students' beliefs about probability that were captured during the assessments and instruction were concerned with representativeness (Kahnemam & Tversky. 1972 : 444-445), outcome approach (Konold; et al. 1993 : 398), negative and positive recency effect (Borovcnik & Bentz. 1991 : 269-270) and belief about a non-contiguous probability generator (Jones. 1974).

Sociomathematical norms and their corresponding classroom mathematical practices emerged during the teaching experiment. The emergence of the sociomathematical norm "What counts as a valid way to describe and predict probability" was linked to the classroom mathematical practices "Using invented informal language (e.g., 3 in 8 or 3 from 8)" and "Connecting sample space composition and probability prediction." A second sociomathematical norm "What counts as an efficient way to list all outcomes of 2- and 3- dimensional situations" generated the classroom mathematical practice "Listing all outcomes systematically."

Discussion of the study

Regarding to students' probabilistic thinking performance, the finding of this study revealed that students in each class developed their probabilistic thinking as indicated by significant differences between pre- and post-assessment scores. Although the purposes of the study did not investigate the effectiveness of the instructional program, the results showed that the activities and tasks used in the instructional program impacted students' development in probabilistic thinking. The teaching experiment of this study was based on a cognitively guided instruction (Fennemar. 1993 : 302-305), the hypothetical learning trajectory (Simon. 1995 : 135) and the Probabilistic Thinking Framework (Jones; et al. 1997 : 111). The successful use of the Probabilistic Thinking Framework to inform the instruction in this study suggests that it can be used more extensively in Thailand as a basis for designing instruction.

With regard to students' thinking about sample space, a mathematical practice that emerged was the odometer strategy (English. 1990 : 261). The strategy emerged when students were asked to determine all possible ways in which one could get dressed given 2 different pairs of pants and 3 different-colored shirts. Actually, there was evidences that students could systematically list all outcomes in the pre-assessment period; however, did was not widespread.

The students who were the participants in this study showed that their probabilistic thinking seemed to be different from other students in other cultures, such as students in North America (Jones; et. al. 1999) and Lesotho (Polaki. 2000). According to these studies,

there were students that exhibited *sample space misconception* (Jones; et al. 1999 : 504)—students do not list all the outcomes of a sample-space question and respond by stating what outcome is more likely to occur (even a 1-dimensional random situation). With respect to the sample space misconception, Jones; et. al. claimed that students in their placed a degree of certainty in a situation in which certainty does not exist (Jones; et. al. 1999 : 504); they attributed this misconception to dealing with uncertainty rather than dealing with language involving the use of “could” (Piaget & Inhelder. 1975 : 89-122). In this present study all target students were assigned beyond Level 1 thinking about sample space prior to the instruction and there was no evidence of the sample space misconception or of students placing a degree of certainty in a situation in which certainty does not exist. According to the study of Amir & Williams (1999 : 105), culture and language affect students’ probabilistic thinking. This study was conducted in a Thai context, which is different from the cultural and language contexts used in the studies of Jones; et al. and Polaki.

In this study, the questions in the assessments were translated from English to Thai. In sample space situations, the use of “could” and a “plural noun” were translated in Thai language by giving students more information. The questions in Thai language might affect students’ thinking and guide students to list all the outcomes.

Analyses of the target students’ learning during the instruction suggested that students developed their probabilistic thinking by exhibiting part-part schema or part-whole schemata. This result was consistent with the result of the study in North America Jones; et al. (1999 : 502) where it was claimed that these schemata played a crucial role in enabling students to quantify probabilities. In my study part-whole schema enabled students to develop their probabilistic thinking to Levels 3 or 4 whereas students who held part-part schemata were able to develop their thinking to Level 2 or 3. There were 5 out of 12 students that appeared to exhibit part-whole reasoning from the outset of the teaching experiment. These students exhibited part-part reasoning in the early teaching sessions and then developed a part-whole schemas in association with invented informal language, such as “1 in 2” or “1 from 2”, for describing probability measures. This phenomenon was also consistent with the study of Jones; et al. (1999 : 502-511). Jones; et al. claimed that

the emergence of invented language and students' experiences during instruction enabled students to automate their part-part thinking so that they capacitate to accommodate a part-whole schema

The students who held part-whole reasoning represent the probability of an event by using a *fraction notation*. For them, using a fraction notation to represent probability seemed to be an easy way for describing their part-whole reasoning. Moreover, these students could use fractions as a tool for comparing or ordering probabilities and they essentially used the fraction as a true representation of a rational number. In the study of Polaki (2000 : 119), students showed the benefit of fraction scaffolding in enabling them to represent probability measures especially in nonreplacement situations. Polaki mentions that fractions are not only a vehicle for expressing a part-whole schema but also serve to enhance and sharpen that schema. In essence, fractions may be used as mathematical tools that make the ordering of probabilities more transparent and efficient. Tadapon and Pirapan exemplified those students; especially, Tadapon who always compared or ordered probability by comparing the fractions which represented those probabilities. For instance, when asked to describe the chance of getting a white marble in the first draw and in the second draw of Situation 9 of post-assessment, he compared $\frac{2}{11}$ and $\frac{2}{10}$, the probabilities of getting a white marble in the first draw and the second draw respectively.

Although there were students who successfully represented their part-whole reasoning using fractions, some students used fractions as "*notation of probability*" and did not use these fractions as rational number. They were merely a notation in the form of a fraction. Piyawan and Jiyani exemplified these students. Both students always used a fraction to represent the chance of an event but their use of fractions was not concerned with comparing or ordering probabilities and sometimes they were confused about how to use a fraction to represent a probability especially in 2-dimensional situations. When asked to describe their thinking, their responses suggested that they used fraction to convey part-part reasoning. Their ideas of using fractions seemed to come from the class discussions where some students used fractions to represented rational numbers but other students used "fractions" merely as a label to describe probability. This evidence pointed to the problematic nature of using a fraction represented probability. As suggested by Jones; et

al. (1999 : 517), it may be better to concentrate on the use of part-whole schema rather than the use of fractions especially if fractions are not well known by the students.

Using a fraction to represent probability of an event for students such as Tadapon and Jiyani produced contrasting results. For Tadapon using fractions drove him to higher levels of reasoning in describing, ordering and comparing probabilities. On the other hand, the use of fractions did not help Jiyani in describing probability, especially in 2-dimensional situations. In the post-assessment, she used $\frac{2}{5}$ to represent the chance of getting a 5 on the dice. And she never compared fractions. The differences between these two students seemed to result from their understanding of fractions and their use of part-part schema in the case of Jiyani and part-whole schema in the case of Tadapon. Although this study did not examine students' understanding of fractions, their responses in using fractions could be interpreted as saying that Tadapon seemed to understand fractions more than Jiyani did. Consequently, this study claims that using fractions for describing probabilities benefits students who understand fraction and use part-whole reasoning. For students who use part-part reasoning and lack on understanding of fractions, the fractions they represented are only a "notation" for representing probabilities. These latter students did not use fractions as numbers that can be compared or computed and hence the fractions were not helpful to them in ordering or comparing probabilities.

As noted earlier, invented informal language emerged during instruction; however the majority of students (58 % of target students) still could not develop a part-whole schema. This result is consistent the finding of Polaki (2000 : 116), who claimed that the emergence of invented informal language, in the Basotho context, was not often linked to a part-whole schema because some students seemed to focus on all parts separately: the part, the complement, and the whole. The study in other areas of mathematics, such as ratio and proportion, also showed the same result. Singer & Resnick (Singer & Resnick. 1992 : 231-246) found that, in the context of ratio and proportion, children in North America used strategies that were premised on a part-part schema rather than a part-whole schema.

At the pre-assessment point most case-study students quantified probability using subjective judgments. Yet, these students were able to develop their thinking using part-part or part-whole schema during instruction. Interestingly, in the study of Polaki (Polaki.

2000 : 101), students used two different types of part-part schemata: (a) weak part-part schema, and (b) stronger or more stable part-part schema. However, in this study there was another type of part-part schema that students exhibited. I have labeled this an ***extreme part-part schema***. Anothai, Witawat, Piyawan and Kitisak exemplified the students who demonstrated that they held this schema for describing probability. These students consistently used this part-part schema for quantifying probability and its use was not only stable but also flexible and meaningful. In addition, the extreme part-part schema sometimes be used to compare probability across sample spaces.

The ***extreme part-part schema*** may have been another aspect of Polaki's stronger part-part schema (Polaki. 2000 : 101). However, Polaki did not focus on this kind of part-part reasoning. In my study students applied part-part reasoning in different ways. For example, in Situation 6 in the post assessment, students were asked to compare probability of getting a green marble from box A and B. A student might say that box B was better for green because there were more green marbles in box B (compare 8 green marbles in Box B with 5 green marbles in box A). Another might say that box A was better for green because in box A the number of red marbles and green marbles were the same (compare 5 red and 5 green marbles) but in box B the number of red marbles more than green marbles (compare 9 red and 8 green marble). Although both students used a part-part schema, the latter student seemed to use the higher level of reasoning. For this reason I have introduced the term "***extreme part-part schema***."

Students who held an extreme part-part schema were able to compare probabilities across sample spaces because they compared *part to part* in each sample space but did not compare *part to part* across sample spaces. For example, in Situation 7 of the post-assessment (see Appendix A) students did not compare the number of green marbles or the number of red marbles across the boxes. They compared the number of green marbles and the number of red marbles in the same box, and then ordered probabilities of getting a green marble and getting a red marble. They said that box A would be better to get a red marble and box B would be better to get a green marble. They reasoned that there were more the red marbles than the green marbles in box A; hence the chance of getting a red marble was more than the chance of getting a green marble. In box

B the chances of getting a red marble and getting a green marble were the same because the number of the red marbles and the green marbles were the same.

These students could not compare probabilities that need a part-whole reasoning. For example, in the nonreplacement situation of Situation 9 (see Appendix A, post-assessment) they could not compare the probabilities of getting a white marble in the first draw and the second draw after a green marble had been drawn. Although some of them knew that the probability of the second draw had changed from the first draw, they could not make the necessary comparison using the extreme part-part schema.

In this study, there were 3 target students who exhibited a stronger part-part schema. Polaki (2000 :120) mentions that holding stronger part-part reasoning does not enable students to order or predict probabilities in 2-dimensional situations and the analysis of target students in this study revealed the same result. The students who exhibited a stronger part-part schema, such as Jiyani, Kamol, Jirawan were unable to order probabilities in 2-dimensional situations as Polaki mentioned. However, the analyses of this study indicated that a stronger part-part schema was not the main problem in students' ability to order or predict probabilities in 2-dimensional situations. Kamol's and Jirawan's responses, for example, suggested that ordering probabilities or making predictions in 2-dimensional problems required two kinds of ability: (1) ability of list all of the outcomes of 2-dimensional situations, and (2) ability of making the connection between sample space composition and probability measure.

In Situation 3 in post-assessment (see Appendix A) Kamol could not list all the outcomes correctly (listed 11 outcomes from 12 outcomes). Even so, he still ordered probabilities by considering sample space composition and by using a part-part relationship. He explained, "There are 10 ways that are not Red-3 and only 1 way is Red-3. The chance of getting Red-3 is less than the chance of not getting Red-3." From this response if he had listed all outcomes correctly, he would have ordered probability correctly using a part-part relationship. In contrast, Jirawan's response in Situation 4 in the post assessment (see Appendix A) showed that she was able to list all the outcomes, but she failed to order probability. She said that the sum "3" is most likely to get without giving reason. She did not consider all the outcomes she listed to predict probability.

Although this study did not examine how different between probabilistic thinking development of students in Prathomsuksa 3 and Prathomsuksa 5 were, the analyses of students' probabilistic thinking in both classes showed some aspects of these development. Comparing the development of target students in both classes, the development of the students who were in higher quartile and lower quartile were not different but the development of the students who were in medium quartile were different. In both classes the students who were in higher quartile developed their thinking to Level 3-4, whereas students who were in lower quartile developed their thinking to Level 2 across four constructs. On the other hand, the students who were in medium quartile of Prathomsuksa 5 developed their thinking beyond students who were in medium quartile of Prathomsuksa 3. In Prathomsuksa 5, these students were assigned at Level 3-4 across the four constructs at the last two assessments. In contrast, the target students in Prathomsuksa 3 were assigned at Level 2-3.

In contrast to the studies in Lesotho (Polaki. 2000) and North America (Jones; et al. 1999), there were no target students in this study that exhibited subjective judgments and a weak part-part schema for describing probability after instruction. Although most target students used subjective judgments and weak part-part reasoning at the pre-assessment point, they used only part-part reasoning in the early sessions of the teaching. The majority of students in each class seemed to exhibit the part-part reasoning by the end of the second session of the instruction. In contrast, at the pre-assessment point, most students responded in the assessment subjectively and tried to guess what would happen rather than think about probability. Vygotsky (1986) suggested that when students are confronted with a new situation, they often use low-level generalized that are characterization by vague discrimination between mathematically relevant and irrelevant features. Yet, there were some students, such as Jiyani, Jirawan, that consistently tried to guess what would occur in probability situations, especially in the pre-assessment. This phenomenon was interpreted as indicating that those students were exhibiting the outcome approach heuristic (Konold. 1989: 59-98). However, the students' use of the outcome approach misconception decreased during the instruction and there was no evidence that students exhibited the outcome approach misconception in the last two assessments. Similarly, negative and

positive recency effects, and representativeness that had appeared at the pre-assessment point decreased after the instruction. This provided the evidence that instruction can decrease students' misconceptions (Zimmerman. 2002 : 140).

There were numbers of sociomathematical norms and classroom mathematical practices that emerged during instruction. The classroom mathematical practice "using informal invented language" emerged in early teaching session. After the emergence of this practice some students became to use fractions to represent probabilities. Although most students used fractions to describe probabilities, majority of students used fractions as the symbols and did not realize fractions as irrational numbers. Consequently, using fractions to describe probability was not sufficiently understood in a collective sense to become taken-as-shared.

The results of this study show a number of aspects that Thai students' probabilistic thinking is different from students in other countries. In each part of Thailand, people have their own cultures and it is possible that these different cultures within Thailand may also affect students' probabilistic thinking. Further research is needed to trace thinking and beliefs in probability in other parts of Thailand. The results from such studies would provide broader landscape of Thai students' probabilistic thinking and would benefit teaching and mathematics curriculum development.

The instructional program of this study was informed by a probabilistic thinking framework and the results show that student were able to develop stronger probabilistic thinking during and after instruction. This makes a strong case for conducting research on students' thinking in other areas (e.g., statistical thinking or algebraic thinking) by using research-based knowledge to inform instruction in the context of Thailand (Fennema; et. al. 1993 : 556). The results may well have similar benefits for teaching and learning in those areas.

BIBLIOGRAPHY

BIBLIOGRAPHY

- Acredolo, C.; et al. (1989, August). Children's Ability to Make Probability Estimates: Skills Revealed through Application of Anderson's Functional Measurement Methodology. *Child Development*. 60(1) : 933-945.
- Amir, G.S.& Williams, J.S.(1999, January). Cultural Influences on Children's Probabilistic Thinking. *Journal of Mathematical Behavior*, 18(1) : 85-107.
- Bauersfeld, H. Krummheuer, G., & Voigt, J. (1988). Interactional Theory of Learning and Teaching Mathematics and Related Microethnographical Studies. in *Foundations and Methodology of the Discipline of Mathematics Education*. H. G. Steiner & A. Vermandel, editors. pp. 174-188. Antwerp, Belgium: Proceedings of the Theory of Mathematics Education Conference.
- Bednarz, N., Dufour-Janvier, B., Poirier, L., & Bacon, L. (1993, March). Socioconstructivist Viewpoint on the Use of Symbolism in Mathematics Education. *Alberta Journal of Educational Research*. 39(1) : 41-58.
- Benson, B. T. (2000). *Assessing Students' Thinking in Modeling Probability Contexts*. Dissertation (Mathematics Education). Normal: Illinois State University. Photocopies.
- Bentz, H.J., & Borovcnik, M.G. (1984). Some Remarks on Empirical Investigations on Probability and Statistical Concepts. in *Theory, Research & Practice in Mathematics Education: Working Group Reports and Collected Papers*. A.Bell, B. Low; & J. Kilpatrick, editors. pp. 266-275. Adelaide, South Australia: University of Nottingham, U.K.
- Biggs, J. B.; & Collis, K. F. (1991). Multimodal Learning and the Quality of Intelligent Behavior. in *Intelligence: Reconceptualization and Measurement*. H. A. H. Rowe, editor. pp. 57-76. Hillsdale, NJ: Erlbaum .
- Borovcnik, M. G.; & Bentz, H. J. (1991). Empirical Research in Understanding Probability. in *Chance encounters: Probability in education*. R. Kapadia; & M. Borovcnik, editors. pp. 73-105. Dordrecht, The Netherlands: Kluwer.

- Carpenter, T. P.; et al. (1989, Winter). Using Knowledge of Children's Mathematics Thinking in Classroom Teaching: An Experimental Study. *American Educational Research Journal*. 26 : 499-531.
- Carpenter, T.P.; & Fennema, E. (1998). Research and Cognitively Guided Instruction. in *Integrating Research on Teaching and Learning Mathematics*. E. Fennema; & T. P. Carpenter, editors. pp.2-17. Wisconsin: Wisconsin Center for Education Research.
- Carpenter, T. P.; et al. (1999). *Children Mathematics: Cognitively Guided Instruction*. NH: Heinemann.
- Cobb, P. (1995, October). Continuing the Conversation: A response to Smith. *Educational Researcher*. 24(5) : 25–27.
- Cobb, P. (1999, January). Individual and Collective Mathematical Development: The Case of Statistical Data Analysis. *Mathematical Thinking and Learning*. 1(1) : 5-43.
- Cobb, P..(2000). Conducting Teaching Experiments in Collaboration with Teachers. in *Handbook of Research Design in Mathematics and Science Education*. A. E. Kelly; & R. A. Lesh, editors. pp. 307-333. Mahwah, NJ: Lawrence Erlbaum
- Cobb, P.; et. al. (1991, January). Assessment of a Problem-Centered Second-Grade Mathematics Project. *Journal for Research in Mathematics Education*. 22(1) : 3-29.
- Cobb, P.; & Bauersfeld, H. (1995). *Emergence of Mathematical Meaning: Interaction in Classroom Cultures*. Hillsdale, NJ: Lawrence Erlbaum Associates
- Cobb, P.; Yackel, E.; & Wood, T. (1989). Young Children's Emotional Acts While Doing Mathematical Problem Solving. in *Affect and Mathematical Problem Solving: A New Perspective*. D. B. McLeod ;& V. M. Adams, editors. pp. 117–148. New York: Springer–Verlag.
- Cobb, P.; Yackel, E.; & Wood, T. (1993). Theoretical Orientation. in *Rethinking Elementary School Mathematics: Insights and Issues*. *Journal for Research in Mathematics Education Monograph Series, Number 6*. T. Wood; P. Cobb; E. Yackel; & D. Dillon, editors. pp. 21-32. Reston, VA: National Council of

Teachers of Mathematics.

- Collis, K. F.; & Biggs, J. B. (1991). Developmental Determinants of Qualitative Aspects of School Learning. in *Learning and Teaching cognitive Skills*. G. Evans, editor. pp. 185-207. Melbourne: Australian Council for Educational Research.
- Cohen, L.; & Manion, L. (1989). *Research Methods in Education*. 3rd. ed. New York: Routledge.
- Confrey, J. (1998). Building Mathematical Structure Within a Conjecture Driven Teaching Experiment on Splitting. in. *Proceedings of the Twentieth Annual Meeting of the North America Chapter of the International Group for Psychology of Mathematics Education*. S. B.; et al., editors. pp. 39-50. Columbus: Clearing House for Science Mathematics, and Environment Education.
- DelMas, R. C.; & Bart, W. M. (1989, Summer). The Role of an Evaluation Exercise in Resolution of Misconceptions of Probability. *Fogus on Learning Problems in Mathematics*. 11(2) : 39-53.
- Dorfler, W. (1995, September). Means for Meaning. Paper presented at the Symposium on Symbolizing, Communication, and Mathematizing. Nashville: TN
- English, L. D. (1993, May). Children's Strategies for Solving Two- and Three-Dimensional Combinatorial Problems. *Journal for Research in Mathematics Education*, 24(3) : 255-273.
- Erickson, F. (1986). Qualitative Methods in Research on Teaching. in. *The Handbook of Research on Teaching*. M. C. Wittrock, editor. pp. 119-161. New York: Macmillan.
- Falk, R. (1983). Children's Choice Behavior in Probabilistic Situations. in. *Proceedings of the First International Conference on Teaching Statistics*. D. R. Grey; et al., editors. pp. 714-726. Sheffield, UK: Teaching Statuctics Trust.
- (1987). Conditional Probabilities: Insights and Difficulties. in *Proceedings: The Second International Conference on Teaching Statistics*. R. Davidson & J. Swift, editors. pp. 195-221. Victoria, Canada: University of Victoria.
- Falk, R.; & Konold, C. (1992). The Psychology of Learning Probability. in *Statistics for the Twenty-First Century*. F. Gordon; & S. Gordon, editors. pp. 151-164.

- MAA Notes Number 26. Washington, D.C.: Mathematics Association of America.
- Falk, R.; & Levin, J. (1980, May). A Potential for Learning Probability in Young Children. *Educational Study in Mathematics*. 11(2) : 181-204.
- Fast, G. R. (1999, May). Analogies and Reconstruction of Probability Knowledge. *School Science and Mathematic*. 99(5) : 230-240.
- Fennema, E.; et.al. (1993, Fall). Using Children's Mathematical Knowledge in Instruction. *American Educational Research Journal*. 30(3) : 555-583.
- (1996, July). A Longitudinal Study of Learning to Use Children's Thinking in Mathematics Instruction. *Journal for Research in Mathematics Education*. 27(4) : 403-434.
- Fischbein, E. (1975). *The Intuitive Source of Probabilistic Thinking in Children*. Dordrecht.: The Netherlands.
- (1987). *Intuition in Science and Mathematics*. Dordrecht: The Netherland.
- Fischbiein, E; & Gazit, A. (1984, February). Does the Probability Teaching Improve Probabilistic Ituition? *Educational Study in Mathematics*. 15(1) : 1-24, .
- Fischbein, E.; Nello, M. S.; & Marino, M. S. (1991, December). Factors Affecting Probabilistic Judgements in Children and Adolescents. *Educational Studies in Mathematics*. 22(6) : 523-549.
- Fischbein, E.; & Schnarch, D. (1997, January). The Evolution with Age of Probabilistic, Intuitively Based Misconceptions. *Journal for Research in Mathematics Education*. 28(1) : 96-105.
- Fuys, D.; Geddes; D.; & Tischler, R. (1988). The van Hiele Model of Thinking in Geometry Among Adolescents. *Journal for Research in Mathematics Education, Monograph Series, Number 3*. Reston, VA: National Council of Teachers of Mathematics.
- Garfield, J. B.; & Ahlgren, A. (1988, January). Difficulty in Learning Basic Concepts in Probability and Statistics: Implication for Research. *Journal for Research in Mathematics Education*. 19(1) : 44-63.

- Garfield, J. B.; & del Mas, R. (1989). Reasoning about Chance Events: Assessing and Changing Students' Conception of Probability. in *The Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for Psychology of Mathematics Education*. C. Maher; G. Goldin; & B. Davis, editors. pp. 203-209. Rutgers, NJ: Rutgers University Press.
- (1991). Students' Conceptions of Probability. in *Proceedings of the Third International Conference on Teaching Statistics, Volume 1*. D. Vere-Jones, editor. pp. 340-349. Voorburg, The Netherlands: International Statistical Institute.
- Green, D. R. (1983). A Survey of Probability Concepts in 3000 Pupils Age 11-16. in. *Proceedings of the First International Conference on Teaching Statistics*. D. R. Grey; et al., editors. pp. 784-801. Sheffield, UK: Teaching Statistics Trust.
- (1988). Children's Understanding of Randomness: Report of a Survey of 1600 Children Aged 7-11 Years. in. *Proceedings: The Second International Conference on Teaching Statistics*. R. Davidson; & J. Swift, editors. pp. 287-291. Victoria, B.C.: University of Victoria.
- Greeno, J. (1991, May). Number Sense as Situated Knowing in a Conceptual Domain. *Journal for Research in Mathematics Education*. 22(3) : 170-218
- Hogg, R. V.; & Tanis, E. A. (1997). *Probability and Statistical Inference*. Upper Saddle River, NJ: Prentice Hall.
- Hovarth, J. K.; & Lehrer, R. (1998). A Model-Based Perspective on the Development of Children's Understanding of Chance and Uncertainty. in. *Reflections on Statistics: Learning, Teaching and Assessment in Grade k-12*. S. P. Lajoie, editor. Pp. 121-148. New Jersey: Erlaum.
- Jaworski, B. (1997, May). The Centrality of the Researcher: Rigor in a Constructivist Inquiry into Mathematics teaching. *Journal for Research in Mathematics Education*. 22(3) : 150-176.
- Jones, G. A. (1974). *The Performance of First, Second, and Third Gradeers on Five Concepts of Probability and Effects of Grade, IQ, and Embodiments on Their*

- Performance*. Dissertation (Mathematics Education). Bloomington: Indiana University. Photocopies.
- Jones, G. A.; Langrall, C.W.; & Thornton, C. A. (1997, February). A Framework for Assessing and Nurturing Young Children's Thinking in Probability. *Educational Studies in Mathematics*. 32(2) : 101-125.
- Jones, G.A.; et. al. (1999, November). Using Students' Probabilistic Thinking in Instruction. *Journal for Research in Mathematics Education*. 30(5) : 487-519.
- Kahneman, D.; & Tversky, A. (1972, July). Subject Probability: A Judgment of Representativeness. *Cognitive Psychology*. 3(5) : 430-454.
- Kahneman, D.; & Tversky, A. (1973, September). Availability: A heuristic for Judging Frequency and Probability. *Cognitive Psychology*. 5(2) : 207-232.
- Kapadia, R. (1988). Didactical Phenomenology of Probability. in *The Proceedings of the Second International Conference on Teaching Statistics*. R. Davidson; & J. Swift, editors. pp. 260-264. Victoria, B.C.: University of Victoria.
- Konold, C. (1989a, Winter). Informal Conceptions of Probability. *Cognition and Instruction*. 6(1) : 59-98.
- Konold, C (1989b) An Outbreak of Belief in independence? in *Proceedings of the 11th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 2*. C. Maher; G. Goldin; & B. Davis, editors. pp. 203-209. Rutgers NJ: Rutgers University Press.
- (1991a) Understanding Students' Beliefs about Probability. in. *Radical Constructivism in Mathematics Education*. E. von Glasersfeld, editor. pp. 139-156. Holland: Kluwer.
- (1991b). The Origin of Inconsistencies in Probabilistic Reasoning of Novices. in *Proceedings of the Third International Conference on Teaching Statistics, Volume 1*. D. Vere-Jones, editor. pp. 89-96. Voorburg, The Netherlands: International Statistical Institute.
- Konold, C.; et al. (1993, November). Inconsistencies in Students' Reasoning about Probability. *Journal for Research in Mathematics Education*. 24(5) : 392-414.
- Lampert, M. (1990, Spring). When the Problem Is Not the Question and the Solution Is

- Not the Answer: Mathematical Knowing and Teaching. *American Educational Research Journal*. 27(1) : 29–63.
- Lamon, S. J. (1993). Ratio and Proportion: Connecting Content and Children's Thinking. in. *Journal for Research in Mathematics Education, Monograph Series, Number3*. pp. 41-61. Reston, VA: National Council of Teachers of Mathematics.
- Lave, J. (1988). *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life*. Cambridge, MA: Cambridge University Press.
- Lecoutre, M. P. (1992, December) Cognitive Models and Problem Space in "Purely Random" Situations. *Educational Studies in Mathematics*. 23 : 557-568.
- Mack, N. K. (1995, November). Confounding Whole-Number and Fraction Concepts when building on Informal Knowledge. *Journal for Research in Mathematics Education*. 26(5) : 422-441.
- Miles, M. B.; & Huberman, A. M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook* 2nd ed. London: Sage.
- National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.
- Piaget, J.; & Inhelder, B. (1975). The Origin of the Idea of Chance in Children (L. Leake, Jr., P. Burrell, & H. D. Fischbein, Trans.). New York: W. W. Norton.
- Polaki, M.V.(2000). *Using instruction to trace Basotho elementary students' growth in probabilistic thinking*. Dissertation (Mathematics Education). Normal: Illinois State University. Photocopies.
- Pollatsek, A.; et al. (1987, October) Understanding Conditional Probabilities. *Organizational Behavior and Human Decision Processes*. 40(2) : 255- 269.
- Saxe, G. B. (1991). *Culture and Cognitive Development: Studies in Mathematical Understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.
- Shaughnessy, J. M. (1977, August). Misconceptions of Probability: An Experiment with a Small- Group Activity-Based, Model Building Approach to Introductory

- Probability at the College Level. *Educational Studies in Mathematics*. 8(3) : 285-316.
- Shaughnessy, J. M. (1983). Misconceptions of Probability, Systematic and Otherwise: Teaching Probability and Statistics so as to Overcome Some Misconceptions. in *Proceedings of the First International Conference on Teaching Statistics, Vol. II*. D. R. Grey; et al., editors. pp. 784-801. England: University of Sheffield.
- (1992). Research in Probability and Statistics: Reflections and Directions. in *Handbook of Research on Mathematics Teaching and Learning*. D.A. Grouws, editor pp. 465-494. New York: Macmillan,
- Singer, J. A.; & Resnick, L. B. (1992, August). Representations of Proportional Relationships: Are Children Part-Part or Part-Whole Reasoners? *Educational Studies in Mathematics*. 23(3) : 231-246.
- Simon, M. (1995, March). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*. 26 (2) : 114-145.
- Stake, R. E. (1998) Case Studies. In. *Handbook of Qualitative Research*. N. K. Denzin; & Y. Lincoln, editors. pp. 435-450. California: Sage Publication.
- Steffe, L. P.; & Thompson, P.W. (2000). Teaching Experiment Methodology: Underlying Principles and Essential Elements. in *Handbook of Research Design in Mathematics and Science Education*. A. E. Kelly; & R. A. Lesh, editors. pp. 276-306. Mahwah, NJ: Lawrence Erlbaum.
- Tarr, E.J.(1997). *Using Knowledge of Middle School Students' Thinking in Conditional Probability and Independence to Inform Instruction*. Dissertation (Mathematics Education). Normal: Illinois State University. Photocopies.
- Tarr, J. E.; & Jones, G. A. (1997, May). A Framework for Assessing Middle School Students' Thinking in Conditional Probability and Independence. *Mathematics Education Research Journal*. 9(1) : 39-59.
- Tzur, R. (1999, July). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting that Learning. *Journal for Research in Mathematics Education*. 30(4) : 390-416.
- Vygotsky, J. A. (1986). *Thought and Language* Cambridge, MA: MIT Press

- Watson, J. M.; Collis, K. F.; & Moritz, J. B. (1997, May). The Development of Chance Measurement. *Mathematics Education Research Journal*. 9(1) : 60-82.
- Way, J. (1996). Children' s Strategies for Comparing two Types of Random Generators. in. Proceedings of the 20th Conference of International Group for Phycology of Mathematics Education. L. Puig; & A. Gutierrez, editors. pp. 419-426. Valencia, Spain: University of Valencia.
- Yackel, E.; & Cobb, P. (1996, July). Sociomath Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education*. 27 (4) : 458-477.
- Zimmermann, G. (2002). *Students' Reasoning about Probability Simulations During Instruction*. Dissertation (Mathematics Education). Normal: Illinois State University. Photocopies.

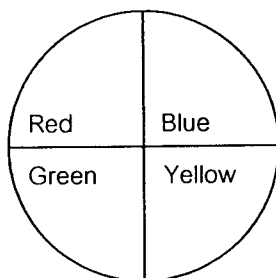
APPENDIX

APPENDIX A
ASSESSMENT PROTOCOLS

PRE-ASSESSMENT PROTOCOL

SITUATION 1

Material: A spinner as shown



Researcher spins the spinner then asks:

- What color came up on the spinner?

Researcher allows the student to spin spinner one time and ask:

- What color came up on the spinner?
- What colors could you get by spinning this spinner? Please write them down. Can you explain that to me?
- Is the chance of getting red more than or less than the chance of getting white, or are they the same chance?
- Can you use number to explain to chance of getting red? How?
- Can you use number to explain to chance of getting white? How?
- If we play the game by choosing a color on the spinner. You choose a color. I choose a color. Is this a fair game? Why?

SITUATION 2

Material: A six-sided dice.

The researcher allows student to roll a dice then asks:

- What did you get?
- What could you get if you tossed the dice? Write them down.
- If you toss the dice again, will you get the same number as the first time or a different number? Why?

- If you roll the dice and I guess you will get the number that is more than 4, do you think I am going to be right or wrong? Can you use numbers to explain the chance of getting the number that is more than 4.
- We are going to play a game by choosing a number on the dice. You and I choose a number. Is this game fair? Can you use a number to describe the chance that I would win the game? How?

SITUATION 3

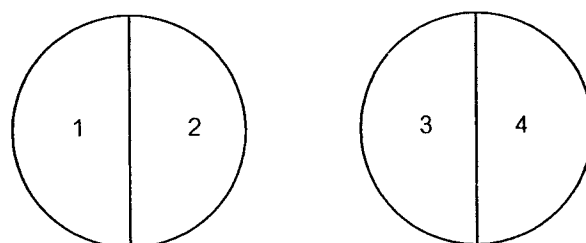
Materials: A six-sided dice and a coin

The researcher allows the student to toss the dice and the coin at the same time then asks:

- What did you get?
- Please toss them again. What did you get this time?
- If you toss them again and again for many times, what could you get? List all the outcomes that you could get.
- Please explain your answer.
- If you are going to toss them again and I guess you will get H-3, am I going to be right or wrong? Why?
- Can you use numbers to explain the chance that my guess is right? How?

SITUATION 4

Materials: Two spinners as shown.



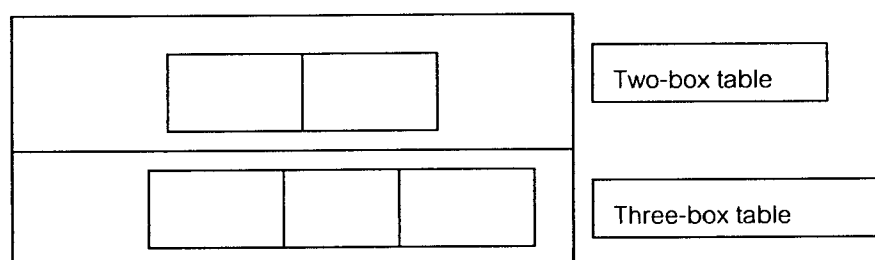
The researcher spins one of the spinners and lets the student spin the other, then asks:

- That sum did we get?

- What sums could we get if we spin both spinners again? Write them down and explain your answer.
- If you spin both spinners, what sum are you most likely to get? How do you know?
- If I spin them, can I get the sum "1"? Why?
- If I tell you that I will get the sum "10", do you believe me? Why?

SITUATION 5

Material: A figure as shown



The researcher asks these questions:

- If you write 1 or 2 in the two-box table, what are all the numbers you could get? (For example, if you write 1 in the first box and 2 in the second box, you will get "twelve.") Explain your answer.
- If you put 1 or 2 in the three-box table, what are all the numbers you could get? For example, if you put 1 in the first box, 2 in the second box and 1 in the third box, you will get "One hundred and twenty-one"). Explain your answer.
- If I let your friend write 1 or 2 in both tables and let you guess what number your friend wrote (for example 12 in the two-box table and 121 in the three-box table). Which box is easier to guess the two-box table or the three-box table?
- If I let your friend write 1 or 2 or 3 in the three-box table, we both guess what number he/she wrote. If I guess 111 and you guess 132, whose guess had more chance to be right? Why?

SITUATION 6

Materials: Box A: (3 red marbles and 3 green marbles)

Box B: (6 red marbles and 5 green marbles)

The researcher allows the student to look in the containers and lets him/her remember the number of the marbles of each color in each box.

- I am going to give you a prize if you draw one marble from the container and get a green marble. Which container would you like to draw from ,A or B? Why?
- I am going to give you a prize if you draw one marble from the container and get a red marble. Which container would you like to draw from, A or B? Why?
- If you draw a marble from box A, how could you use numbers to explain the chance of getting a green marble?
- If you draw a marble from box B, how could you use numbers to explain the chance of getting a green marble?

SITUATION 7

Materials: Box A: (6 red marbles and 5 green marbles)

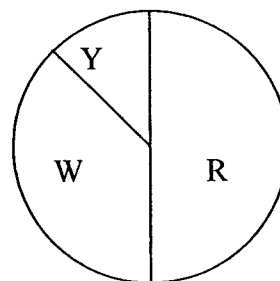
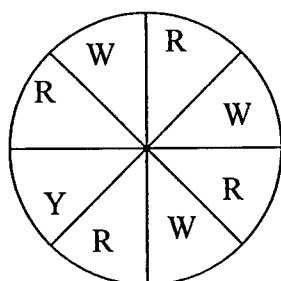
Box B: (1 red marbles and 2 green marbles)

The researcher allows the student to look in the containers and lets him/her remember the number of marbles of each color in each box.

- I am going to give you a prize if you draw one marble from the container and get a green marble. Which container would you like to draw from, A or B? Why?
- I am going to give you a prize if you draw one marble from the container and get a red marble. Which container would you like to draw from, A or B? Why?

SITUATION 8

Materials: Two spinners as shown



The researcher let the student consider the spinners.

- If you spin the spinner and the pointer lands on red, you will get a prize. Which spinner would be the best for you?
- If we play a game where the winner is the one who gets a white on a spin. I spin the first spinner and you spin the second spinner. Who is going to win this game? Why? Is this a fair game?
- Can you use numbers to explain the chance of getting red on both spinners?

SITUATION 9

Materials: Container: (4 yellow marbles, 3 red marbles and 2 green marbles)

The researcher allows the student to look in the container and lets him/her remember the number of marbles of each color.

- If you draw a marble from the container, what color will you have the best chance of getting? Why?

(The researcher draws a yellow marble and asks the student to observe that the yellow marble is not replaced.)

- If you draw a marble now (we will call it the second draw), does the chance of getting a yellow marble (or other color that the student answered in the prior question) change or not? Why?
- Does the chance of getting a green marble in the second draw change from the first draw or not? Why? Can you use numbers to explain?

SITUATION 10

Material: A coin.

The researcher asks students to close his/her eyes, and then tosses the coin 3 times.

The researcher allows the student to open his/her eyes and asks:

- I tossed the coin 3 times and a HEAD came up all the time. If I toss it again, do you think HEAD or TAIL has more chance to come up, or have they the same chance? Why?

SITUATION 11

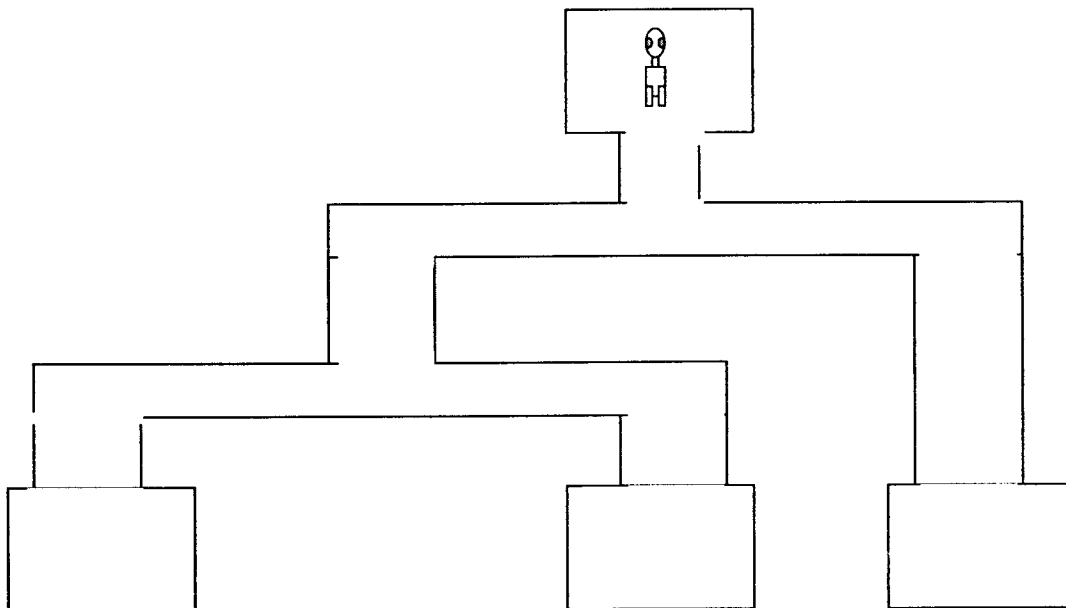
Materials: A container (3 milk candies and 2 fruit candies)

The researcher allows the student to observe the container, then asks the questions:

- I'll let you draw a candy but you have to eat any kind of candy you get. Suppose you like fruit candies but on the first draw you get a milk candy. So you have to eat it then. If I allow you to draw a candy again, do you think you have more chance or less chance to get a fruit candy in this draw? Why? Can you use numbers to explain?
- Does the chance of getting a milk candy in the second draw change from the first draw or not? Why?

SITUATION 12

Materials: A figure as shown:



The researcher explains and asks the question:

A robot travels from the starting point along the hallway. He can go straight, turn left and turn right but cannot turn back. Which room has more chance that the robot will stop?

POST-ASSESSMENT PROTOCOL

SITUATION 1

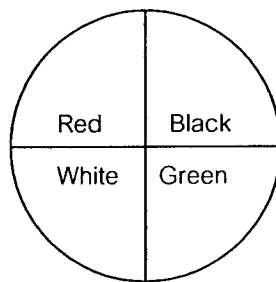
Materials: A box (6 paper cards that are numbered 1 to 6) .

The researcher lets the student observe the box and allows the student to draw a card then asks the questions:

- What did you get?
- What could you get if you draw a card? Write them down.
- If you draw a card again, will you get the same number as the first time or a different number? Why?
- You can draw a card and I guess you will get the number that is more than 4. Do you think I am going to be right or wrong? Can you use numbers to explain the chance of getting a number that is more than 4.
- We are going to play a game by choosing a number. You and I choose a number and I let someone else draw a card. Is this game fair? Can you use numbers to describe the chance that I would win the game? How?

SITUATION 2

Material : A spinner as shown



Researcher spins the spinner then asks:

- What color came up on the spinner?

Researcher allows the student to spin spinner one time and asks:

- What color came up on the spinner?

- What colors could you get by spinning this spinner? Please write them down. Can you explain that to me?
- Is the chance of getting green more than or less than the chance of getting black, or are they the same chance?
- Can you use numbers to explain the chance of getting green? How?
- Can you use numbers to explain the chance of getting black? How?
- We play the game by choosing a color on the spinner. You choose a color. I choose a color. Is this a fair game? Why?

SITUATION 3

Materials: A six-sided die and a coin that has one side red and the other side green.

The researcher allows the student to toss the dice and the coin at the same time then asks:

- What did you get?
- Please toss them again. What did you get this time?
- If you toss them again and again many times, what could you get? List all the outcomes that you could get.
- Please explain your answer.
- If you are going to toss them again and I guess you will get Red-3, am I going to be right or wrong? Why?
- Can you use numbers to explain the chance that my guess is right? How?

SITUATION 4

Materials: Two plastic coins. The first coin has "1" on one side and "3" on the other. The second coin has "2" on one side and "4" on the other.

The researcher tosses both coins at the same time, and then asks the questions:

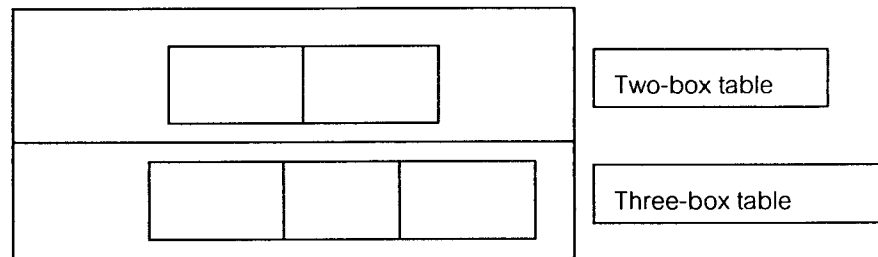
- What sum did we get?
- What sums could we get if we toss both coins again? Write them down and explain your answer.
- If you toss both coins, what sum are you most likely to get? How do you know?

- If I toss them, can I get the sum "1"? Why?
- If I tell you that I will get the sum "10", do you believe me? Why?

SITUATION 5

Materials: Nine small cards, 3 cards have A on them. 3 cards have B on them and 3 cards have C on them.

A figure as shown



The researcher asks these questions:

- If you put A or B in the two-box table, what could you get? (For example, if you put A in the first box and B in the second box, you will get "AB.") Explain your answer.
- If you put A or B in the three-box table, what could you get? (For example, if you put A in the first box, B in the second box and A in the third box, you will get "ABA"). Explain your answer.
- If I let your friend write A or B in both tables and let you guess what letters your friend wrote (for example AB in two-box table and ABA in three-box table). Which box is easier to guess, two-box table or three-box table?
- If I let your friend write A or B or C in three-box table, we both guess what number he/she wrote. If I guess AAA and you guess ACB, whose guess have more chance to be right? Why?

SITUATION 6

Materials: Box A: (5 red marbles and 5 green marbles)

Box B: (9 red marbles and 8 green marbles)

The researcher allows the student to look in the boxes and let him/her remember the number of marbles of each color in each box.

- I am going to give you a prize if you draw one marble from the box and get a green marble. Which box would you like to draw from, A or B? Why?
- I am going to give you a prize if you draw one marble from the box and get a red marble. Which box would you like to draw from, A or B? Why?
- If you draw a marble from box A, how could you use numbers to explain the chance of getting a green marble?
- If you draw a marble from box B, how could you use numbers to explain the chance of getting a green marble?

SITUATION 7

Materials: Box A: (8 red marbles and 7 green marbles)

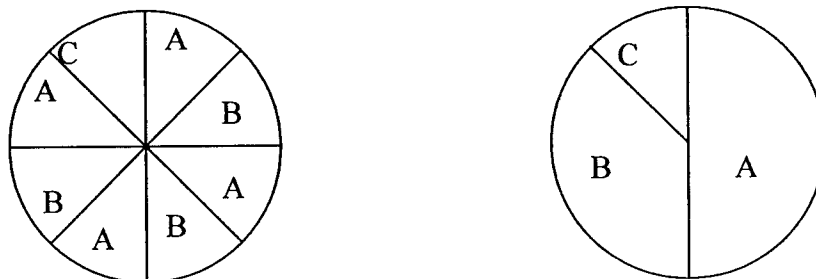
Box B: (1 red marbles and 2 green marbles)

The researcher allows the student to look in the boxes and lets him/her remember the number of marbles of each color in each box.

- I am going to give you a prize if you draw one marble from the box and get a green marble. Which box would you like to draw from, A or B? Why?
- I am going to give you a prize if you draw one marble from the box and get a red marble. Which box would you like to draw from, A or B? Why?

SITUATION 8

Materials: Two spinners as shown



The researcher let the student consider the spinners.

- If you spin the spinner and the pointer land on A, you will get a prize. Which spinner would be the best for you?

- If we play a game that the winner is the one who spins on the B sector. I spin the first spinner and you spin the second spinner. Who is going to win this game? Why? Is this a fair game?
- Can you use numbers to explain the chances of getting A on both spinners?

SITUATION 9

Materials: Box: (2 white marbles, 4 blue marbles and 5 green marbles)

The researcher allows the student to look in the box and lets him/her remember the number of marbles of each color.

- If you draw a marble from the box, what color will you have the best chance of getting? Why?

(The researcher draws a green marble and asks the student to observe that the green marble is not replaced.)

- If you draw a marble now (we will call it the second draw), does the chance of getting a green marble (or other color that the student answered in the prior question) change or not? Why?
- Does the chance of getting a white marble in the second draw change from the first draw or not? Why? Can you use numbers to explain?

SITUATION 10

Material: A coin with one red side and one green side.

The researcher asks students to close his/her eyes, and then tosses the coin 3 times.

The researcher allows the student to open his/her eyes and asks:

- I tossed the coin 3 times and RED came up all the time. If I toss it again, do you think RED or GREEN has more chance to come up, or they have the same chance? Why?

SITUATION 11

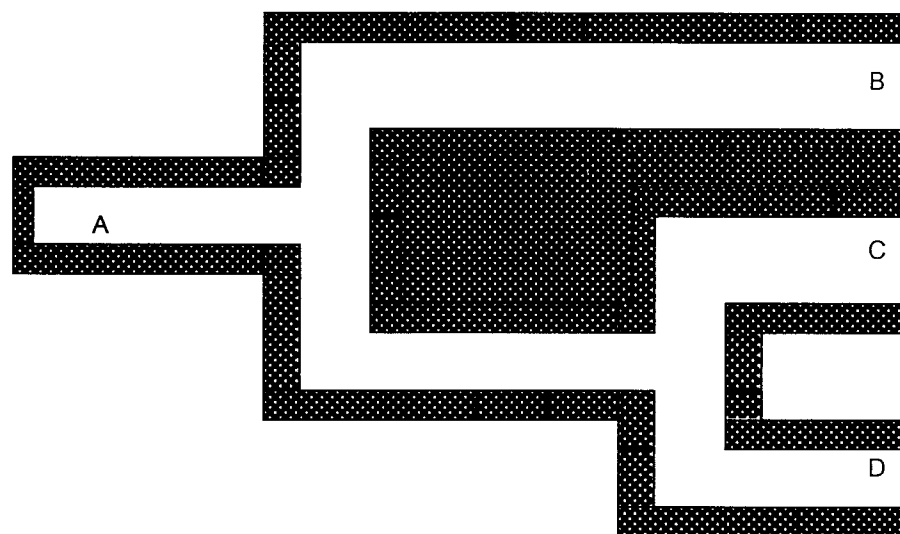
Material: A box (3 flower stickers and 2 animal stickers)

The researcher allows the student to observe the box, and then asks the questions:

- If I let you draw a sticker, you have to take any kind of sticker you get. Suppose you like animal stickers but on the first draw you get a flower sticker. So you have to take it. If I allow you to draw a sticker again, do you think you have more chance or less chance to get an animal sticker in this draw? Why? Can you use numbers to explain?
- The chance of getting a flower sticker in the second draw changes from the first draw or not? Why?

SITUATION 12 [CP-1]

Materials: A figure as shown:



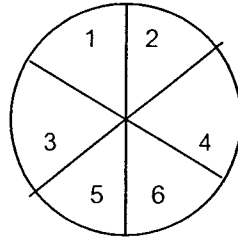
The researcher explains and asks the question:

An ant travels from point A to eat sugar at point B C and D. Which point has more chance that the ant will go to eat sugar.

RETENTION-ASSESSMENT PROTOCOL

SITUATION 1

Materials: A spinner as shown



Researcher spins the spinner then asks:

- What color came up on the spinner?
- What colors could you get by spinning this spinner? Please write them down. Can you explain that to me?
- If you spin the spinner again, will you get the same number as the first time or a different number? Why?
- You can spin the spinner again and I guess you will get the number that is more than 4. Do you think I am going to be right or wrong? Can you use numbers to explain the chance of getting a number that is more than 4.
- We are going to play a game by choosing a number on the spinner. You and I choose a number and I let someone else spin the spinner. Is this game fair? Can you use numbers to describe the chance that I would win the game? How?

SITUATION 2

Materials: A six-sided dice (red two sides, blue two sides and yellow two sides).

The researcher roll the dice then asks:

- What did you get?

The researcher allows student to roll the dice then asks:

- What did you get?
- What colors could you get when you roll the dice? Please explain you answer.
- Is the chance of getting red more than or less than the chance of getting white, or are they the same chance?

- Can you use number to explain to chance of getting red? How?
- Can you use number to explain to chance of getting white? How?
- If we play the game by choosing a color on the spinner. You choose a color. I choose a color. Is this a fair game? Why?

SITUATION 3

Materials: A six-sided dice and a coin

The researcher allows the student to toss the dice and the coin at the same time then asks:

- What did you get?
- Please toss them again. What did you get this time?
- If you toss them again and again for many times, what could you get? List all the outcomes that you could get.
- Please explain your answer.
- If you are going to toss them again and I guess you will get H-3, am I going to be right or wrong? Why?
- Can you use numbers to explain the chance that my guess is right? How?

SITUATION 4

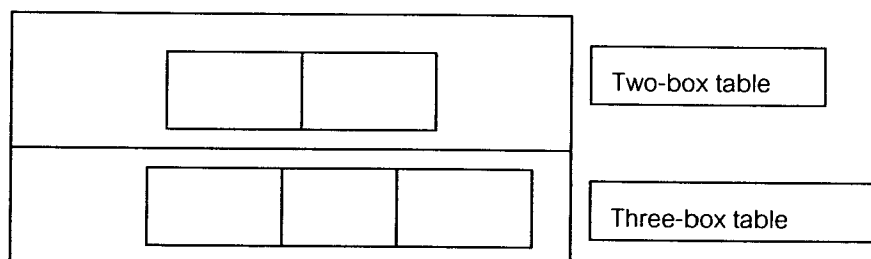
Materials: Two plastic coins. The first coin has "1" on one side and "2" on the other. The second coin has "3" on one side and "4" on the other.

The researcher tosses a coin and the student tosses the other coin at the same time, and then asks the questions:

- What sum did we get?
- What sums could we get if we toss both coins again? Write them down and explain your answer.
- If you toss both coins, what sum are you most likely to get? How do you know?
- If I toss them, can I get the sum "1"? Why?
- If I tell you that I will get the sum "10", do you believe me? Why?

SITUATION 5

Material: A figure as shown



The researcher asks these questions:

- If you write 3 or 5 in the two-box table, what are all the numbers you could get? (For example, if you write 1 in the first box and 2 in the second box, you will get “twelve.”) Explain your answer.
- If you put 3 or 5 in the three-box table, what are all the numbers you could get? For example, if you put 3 in the first box, 5 in the second box and 3 in the third box, you will get “three hundred and fifty-three”). Explain your answer.
- If I let your friend write 3 or 5 in both tables and let you guess what number your friend wrote (for example 35 in the two-box table and 353 in the three-box table). Which box is easier to guess the two-box table or the three-box table?
- If I let your friend write 3 or 5 or 7 in the three-box table, we both guess what number he/she wrote. If I guess 555 and you guess 375, whose guess had more chance to be right? Why?

SITUATION 6

Materials: Box A: (5 red pastic coins and 5 green pastic coins)

Box B: (9 red pastic coins and 8 green pastic coins)

The researcher allows the student to look in the boxes and let him/her remember the number of coins of each color in each box.

- I am going to give you a prize if you draw one coin from the box and get a green coin. Which box would you like to draw from, A or B? Why?

- I am going to give you a prize if you draw one coin from the box and get a red coin.
Which box would you like to draw from, A or B? Why?
- If you draw a coin from box A, how could you use numbers to explain the chance of getting a green coin?
- If you draw a coin from box B, how could you use numbers to explain the chance of getting a green coin?

SITUATION 7

Materials: Box A: (6 red plastic coins and 5 green plastic coins)

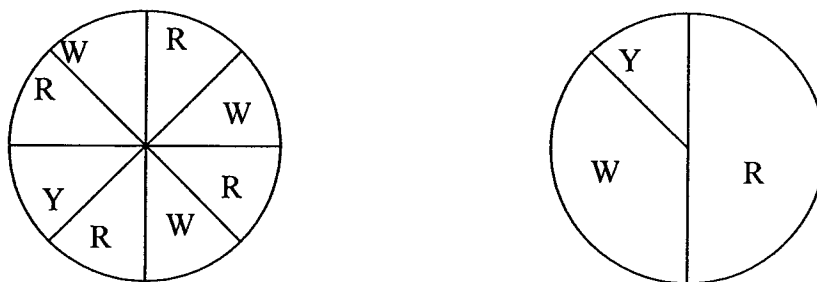
Box B: (1 red plastic coins and 2 green plastic coins)

The researcher allows the student to look in the boxes and lets him/her remember the number of coins of each color in each box.

- I am going to give you a prize if you draw one coin from the box and get a green coin. Which box would you like to draw from, A or B? Why?
- I am going to give you a prize if you draw one coin from the box and get a red coin. Which box would you like to draw from, A or B? Why?

SITUATION 8

Materials: Two spinners as shown



The researcher let the student consider the spinners.

- If you spin the spinner and the pointer lands on red, you will get a prize. Which spinner would be the best for you?

- If we play a game where the winner is the one who gets a white on a spin. I spin the first spinner and you spin the second spinner. Who is going to win this game? Why? Is this a fair game?
- Can you use numbers to explain the chance of getting red on both spinners?

SITUATION 9

Materials: A box: (2 white marbles, 4 blue marbles and 5 green marbles)

The researcher allows the student to look in the box and lets him/her remember the number of marbles of each color.

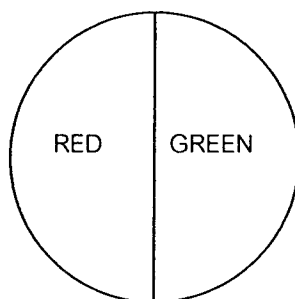
- If you draw a marble from the box, what color will you have the best chance of getting? Why?

(The researcher draws a green marble and asks the student to observe that the green marble is not replaced.)

- If you draw a marble now (we will call it the second draw), does the chance of getting a green marble (or other color that the student answered in the prior question) change or not? Why?
- Does the chance of getting a white marble in the second draw change from the first draw or not? Why? Can you use numbers to explain?

SITUATION 10

Material: A spinner as shown



The researcher asks students to close his/her eyes and then spin the spinner 3 times.

The researcher allows the student to open his/her eyes and asks:

- I spin the spinner 3 times and I got RED all the time. If I spin it again, do you think I have more chance to get RED or GREEN, or they have the same chance? Why?

SITUATION 11

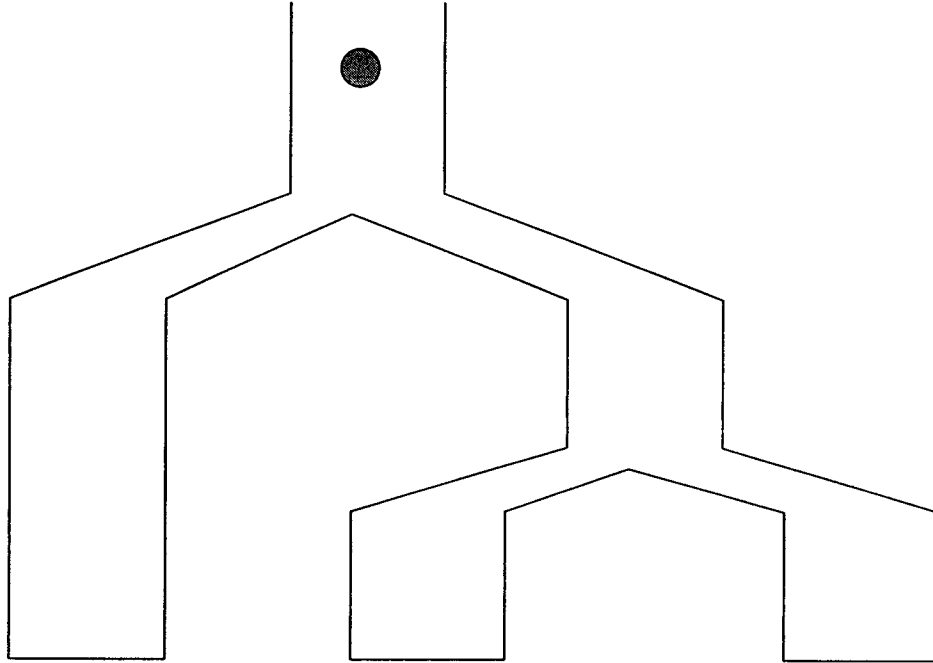
Materials: A container (3 lemon candies and 2 mint candies)

The researcher allows the student to observe the container, then asks the questions:

- I'll let you draw a candy but you have to eat any kind of candy you get. Suppose you like mint candies but on the first draw you get a lemon candy. So you have to eat it then. If I allow you to draw a candy again, do you think you have more chance or less chance to get a mint candy in this draw? Why? Can you use numbers to explain?
- Does the chance of getting a lemon candy in the second draw change from the first draw or not? Why?

SITUATION 12

Material: A figure as shown



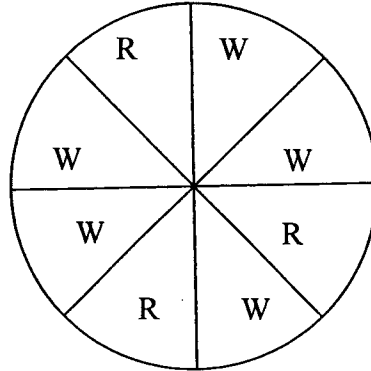
The researcher explains and asks the question:

If we let the ball drops down the pipe, which pipe has more chance that the ball will stop?

APPENDIX B
EXAMPLE TASKS USED IN INSTRUCTION

EXAMPLE TASKS USED IN INSTRUCTION

SPINNER PROBLEM (Session 1-2)



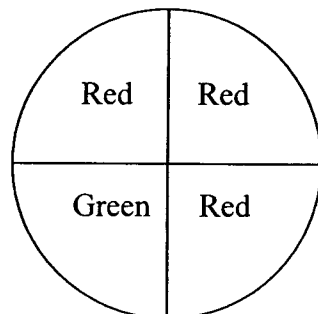
Question: If you spin the spinner, what color would you choose? Why?

WHICH ONE IS BETTER (Session 3-4)

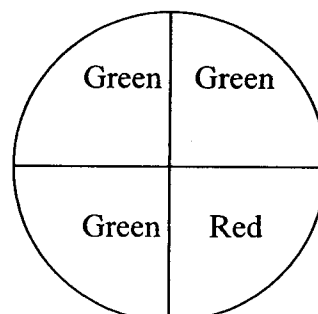
If you and your friend play tossing a coin and rolling a dice, whoever chooses a coin will get a point when the coin turns HEAD. Whoever choose a dice will get a point when the dice shows an even number. We will play 20 rounds. If you want to win the game, would you choose a coin or a dice?

RACING GAME (Session 5)

In this racing game, there are two cars, red and white, and two spinners as shown. The car can move one box if the spinner lands on the same color as the car. Which spinner is the best for the red car? Why?



SPINNER 1



SPINNER 2

PICK UP A MARBLE (Session 10-11)

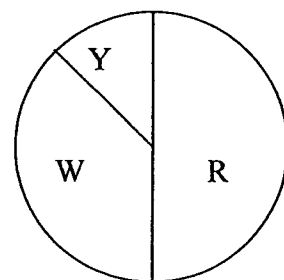
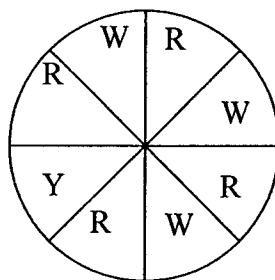
There are 2 boxes. Box 1 contains 3 blue and 3 red marbles. Box 2 contains 5 blue and 4 red marbles. If you pick a marble without looking and you need a red marble, which box are you going to pick? And if you need a blue marble, which box are you going to pick?

PLAY WITH A COINS (Session 12-13)

Four students play with a plastic coin which is red on one side and white on the other side. Three students take turns to toss the coin. If the coin turns red every time, the 1st student will get a point. If the coin turns red a 2 time, the 2nd student will get a point. If the coin turns red 1 time, the 3rd student will get a point. And if the coin never turns red, the 4th student will get a point. Play 5 rounds. Is this game fair? Why?

WHICH SPINNER (Session 14)

Which spinner is the best for getting red?

**TIGER OR GOLD** (Session 15-16)

There are 4 doors. There is a pile of gold behind one door and a tiger behind the rest of the doors. If your friend opened the 3rd door and found the tiger, he/she closed it. He/she told you that you should not open the 3rd door. If you want the gold, do you have more chance to get the gold without taking your friend's advice?

APPENDIX C
HYPOTHETICAL LEARNING TRAJECTORY

SESSION 1

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can order probabilities.

Learning activities

Activity 1 Students discuss and answer the following question.

In the soccer game events of a school, there are 2 teams, Blue and Red. In the last 5 years, the Blue team won 2 times and Red team won 3 times. This year, these two teams are going to play again. Which team do you think is going to win? Why?

Activity 2 "SPINNER PROBLEM" (see Appendix B)

The teacher poses the following questions to students and lets them discuss these questions:

1. If I spin the spinner, what colors could come up?
2. Which color will the pointer land on? Why?
3. Is the chance of getting red more than or less than the chance of getting white (or they are the same chance)? Why?

Students match with a partner and play the game. The teacher explains the direction of playing the game to students:

1. Every player chooses a color (white or red).
2. Both players take turn to spin the spinner. If the pointer lands on the color the student chose, he/she will get 1 point.
3. Each player spins the spinner 20 times. Who gets more points will win the game.

The teacher poses the following questions to the class:

1. Do both players have the same chance of winning?
2. What color should you choose if you want to win the game? Why?

Activity 3 Students play the game described in Activity 2, and record the points of each student gets from playing the game.

Activity 4 Students answer the following questions in paper work.

In playing "SPINNER PROBLEM" game, which color should you choose and which color you should not? Why?

Hypothetical learning process

1. To reason the questions in *Activity 1*, students may not base on probability measure. Students will present the variety of aspects in their reasoning.

2. Most students correctly choose the color on the spinner in "SPINNER PROBLEM." Most of them cannot elaborate their thinking.

SESSION 2

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities
4. Students can distinguish a fair game from an unfair game.

Learning activities

Activity 1. The teacher gathers the result in Activity 3 of Session 1 from every group. Students make a bar graph of the score of red and white. Students discuss to answer the following question.

1. What can you conclude from the bar graph?
2. If you want to win the game, what color should you choose and what color should not choose?

Activity 2. The teacher groups students by 4. In each group, students discuss these questions:

1. If you want to win the "SPINNER PROBLEM" game, how do you choose the color on the spinner?

2. What colors could come up on the spinner?

3. Do the numbers of the sectors of each color on the spinner affect the chance of winning?

4. Can you use numbers to describe the chance that the pointer lands on red?

Why do you use those numbers?

5. Can you use numbers to describe the chance that the pointer lands on white?

Why do you use those numbers?

A student in each group reports to the class.

Activity 3. Class discusses the following questions.

1. Is "SPINNER PROBLEM" a fair game? Why?

2. If the game is unfair, how can you make a new spinner that makes the game fair?

Activity 4. Students answer the question: What do you have learned from the "SPINNER PROBLEM" game?

Hypothetical learning process

1. After the discussion of Activity 1, students may reason what color they should choose by considering the score of red and score of white. For example, some may say "I choose red because from the game red got more points than white"

2. In Activity 2, most students may be able to correctly choose the color in "SPINNER PROBLEM." Some students may be able to consider the number of the sectors of each color for describing probability. The class discussion helps students to use numbers in measuring probability.

SESSION 3

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can order probabilities.

Learning activities

Activity 1. Students discuss to answer the following question:

There are two boxes. The first box contains 3 red and 3 blue balls. The second contains 5 red and 6 blue balls. We are going to play a game "Picking a Ball." We take turn to draw a ball from a box. If I get a blue ball, I will win. If you get a red ball, you will win.

1. Which box, first or second, would be best for you to pick a ball?
2. How to use numbers to describe the chance of getting a red ball in each box?
3. How to use numbers to describe the chance of getting a blue ball in each box?

Activity 2 "WHICH ONE IS BETTER" (see Appendix B)

If you and your friend play tossing a coin and rolling a dice, whoever chooses a coin will get a point when the coin turns HEAD. Whoever chooses a dice will get a point when the dice shows an even number. We will play 20 rounds. If you want to win the game, would you choose a coin or a dice?

Activity 3. The teacher groups students by 3. In each group, one student rolls the dice. Another tosses the coin. The last student records the point of winning of the coin and the dice.

Activity 4. Students answer the following questions in a piece of paper.

1. Does the game turn as you expected?
2. Does the dice or the coin have more chance to win, or do they have the same chance? Why?

Hypothetical learning process

1. In Activity 1 most students may not use number to describe probability, but they start to consider and think about numbers to describe probability.

2. In Activity 2, most students may not answer by considering sample space composition. Some students will begin to make a connection between sample space composition and probability measure.

SESSION 4

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities
4. Students can distinguish a fair game from unfair game.

Learning activities

Activity 1. Students gather data from Activity 3 in Session 3, and then draw a bar graph of the scores of the coin and the dice.

Activity 2. Students discuss to answer the following questions:

1. What can you conclude from the graph?
2. What should you choose, the dice or the coin, to win the game? Why?

Activity 3. The teacher groups students by 4. Students in each group discuss to answer these questions:

1. What could come up in tossing a coin?
2. What could come up in rolling a dice and what numbers are even numbers?
3. If you want to win the game, before playing the game, how do you choose to play? Why?
4. How can you use numbers to describe the chance of winning of the dice? Why?
5. How can you use numbers to describe the chance of winning of the coin? Why?

A student from each group reports to the class.

Activity 4 Class discusses to answer these questions:

1. Is the game "WHICH ONE IS BETTER" a fair game?
2. If your friend who chose the dice gets 100, how many points would you get (you chose the coin)?

Activity 5. Students answer the following question in paper work.

What do you learn from the game "WHICH ONE IS BETTER"?

Hypothetical learning process

1. In the discussion of Activity 2, students may give the reason by considering the result from the game.

2. In the discussion of Activity 3, some students may not make a connection between sample space composition and probability measure. The class discussion may help these students.

SESSION 5

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1 (Review) The teacher poses the following questions:

1. What is the chance of getting red on the spinner in "SPINNER PROBLEM"? Why?
2. What is the chance of getting red on the spinner in "SPINNER PROBLEM"? Why?

Activity 2 "CAR RACING" (see Appendix B)

The teacher pairs students. Students play the game by using Spinner 1 at the first time and change to play with Spinner 2 at the second time. The teacher poses these questions after students finish playing the game.

1. Does the game turn as you expected?
2. Which spinner is best for red car and which spinner is best for the green car?

Why?

Activity 4. Class discusses to answer these questions:

1. What could come up when we spin the spinners?
2. In spinner 1, what color is that the pointer has more chance to stop? Why?
3. In spinner 2, what color is that the pointer has more chance to land on? Why?
4. Which spinner is best for the red car and which spinner is the best for the green car? Why?

Hypothetical learning process

1. In Activity 1, there will be some students that can use numbers to describe probability and explain to the class.
2. In Activity 3, most students will choose the spinner that is best for each car but they cannot use numbers to compare probability.
3. In Activity 4, there will be some students that can compare the chance of winning of each car when use spinner 1 or spinner 2. These students can describe to the class.

SESSION 6

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1 (Review) The teacher poses the following questions:

1. A box contains 5 red and 6 blue marbles. What could you get in drawing a marble from the box? Is the chance of getting a red marble the same as the chance of getting a blue marble? How do you use numbers to describe the chance of getting a red marble? How to use numbers to describe the chance of getting a blue marble?
2. In rolling a dice, what could come up on a dice? How you use numbers to describe the chance of getting an even number?

Activity 2. "DRESS UP PROBLEM" (see Appendix B)

Activity 3 "PLAYING 2 COINS"

Toa and Nok are going to play PLAYING 2 COINS game. They take turn to toss two coins. If the coins show the different sides, Nok will win the game. On the other hand, if the coins show the same sides, Toa will win the game. Toa says that the game was unfair because he thought he has less chance to win. In contrast, Nok says that the game was fair.

How do you think? Is the game fair or unfair? Why?

Activity 4. Students answer the question in paper work:

Are "DRESS UP PROBLEM" and "SPINNER PROBLEM" different? How are they different?

Hypothetical learning process

1. In Activity 1, most students may correctly use numbers to describe probability in situations that they have learned in the prior Sessions.
2. In Activity 2, most students may not list the outcomes systematically.
3. In Activity 3, most students will be unable to elaborate their thinking.
4. Most students may not distinguish between 1-dimensional and 2-dimensional situations.

SESSION 7

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. Class discusses how to use number in describing probability.

1. How to use number to describe probability of getting a number that less than 3 in rolling-a-dice situation.
2. How to use numbers to describe probability of getting H in tossing-a-coin situation.
3. How to use number to describe probability of getting a number that more than 6 in rolling-a-dice situation.

Activity 2. The teacher groups students by 6. Each group discusses to answer the questions:

1. In "DRESS UP" problem, how do you list all matches of a pair of pants and a shirt? How do you know that it is complete?

2. In "DRESS UP" problem, if Nat has two pair of shoes, one pair is black, the other is white, can you list all the ways that Nat can pick a shirt, a pair of pants and a pair of shoes?

Activity 3. The teacher groups students by 3 to play "PLAYING 2 COINS" game. One student pretends to be Toa and another pretends to be Nok. The last student records scores. Toa and Nok take turn to toss the coins and play 30 rounds.

The teacher poses these problems:

1. Does the game turn as you expected?
2. Is the game fair?

Activity 4. Students discuss to answer these questions:

1. Before playing a game, you are thinking if the game is fair or unfair. How do you decide that the game is fair or unfair?

2. How to use numbers to describe the chance that Toa would win the game and the chance that Nok would win the game?

Activity 5. Class discusses this situation:

In tossing a coin 4 times, can you list all the outcomes? Which event, TTTT or THHT, is more likely to occur?

Hypothetical Learning Process

1. Most students will use numbers to describe probability of the situations that they have learned.

2. In Activity 2, most students may not systematically list all outcomes. Some of them will list using some systems because they are familiar with the problem from Session 6.

3. In Activity 3 and 4, some students may subjectively judge that the game is a fair game or an unfair game.

SESSION 8

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. The teacher poses these questions:

1. In rolling a dice, how many ways could the dice come up? How can you use numbers to describe the chance of getting a 4?
2. In tossing two coins, what could the coins come up? Do you have any strategies to list them?
3. How do you use numbers to describe the chance of getting the same face in tossing two coins?
4. How do you use numbers to describe the chance of getting the different faces in tossing two coins?

Activity 2 "RACING GAME" (see Appendix B)

Activity 3. The teacher groups students by 11 to play RACING GAME. In each group, each student chooses a horse (Number 2-12). Students take turn to roll the dice until a student wins the game. Students change the horse number and play the game again.

After each group has the winner, the teacher asks these questions:

1. Does the game turn as you expected?
2. Which horse has the biggest chance of winning?

Hypothetical learning process

1. In activity 1, most students may be able to use numbers to describe probability in 1-dimensional situations but cannot in 2-dimensional situations.
2. In Activities 2 and 3, there will be students who cannot compare the chance of winning of each horse. Most of them use subjective judgment.

SESSION 9

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can connect sample space composition to probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. The teacher gathers the results from each group in Activity 3 (SESSION 8) and poses the question:

1. From all results from every group, how do you conclude?

Activity 2. Analysis of "RACING GAME"

The teacher poses these questions for class discussion:

1. If you want to win the game, how do think which horse has more chance to win?

Why?

2. How do you list all outcomes of rolling two dice? How many outcomes do you get?

3. Does every horse have the same chance of winning? Why?

4. Which horse has the biggest chance of winning? How do you know?

5. Which horse has the smallest chance of winning? How do you know?

6. How to use number to describe the chance of winning of the horse number 12?

Why?

7. How to use number to describe the chance of winning of the horse number 7?

Why?

8. How to use number to describe the chance of winning of the horse that has the biggest chance of winning? Why?

9. Which horses have the same chance of winning? Why?

10. Is the game fair?

Activity 3. Students answer the following question in a piece of paper:

In "RACING GAME" how do you choose the horse?

Hypothetical learning process

1. In activity 1, students may not realize the difference of the chance of the horse number 6, 7 and 8.

2. In Activity 2, some students may be able to list all outcomes of rolling two dice and a few of them list systematically.

3. Some students who can list all outcomes of rolling two dice may be able to use numbers to describe probability.

SESSION 10

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. The teacher poses these questions for class discussion:

A box contains 5 blue, 2 red and 4 yellow marbles. I draw a marble and I get a blue marble.

1. After I picked a blue marble, how are numbers of each kind of marbles in the box?

2. Does the chance of getting a blue marble decrease, increase or the same?

Why?

3. Does the chance of getting a red marble decrease, increase or the same? Why?

4. Does the chance of getting a yellow marble decrease, increase or the same?

Why?

Activity 2 "PICK UP A MARBLE" (see Appendix B)

Activity 3. The teacher groups students by 3. In each group, two students take turn to draw a marble from each box 20 times. The other student records the result and counts how many times they get red and how many times they get blue of each box.

The teacher asks the question: Do you get the results as you expected?

Hypothetical learning process

1. In Activity 1, students may be unable to answer correctly. (The question is for presenting a nonreplacement situation.)
2. In Activity 2, some students may be able to describe their thinking but some may not elaborate their thinking.

SESSION 11

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. The teacher poses these questions:

Tay has 3 shirts, 1 white, 1 red and 1 blue. She also has 4 skirts, 1 green, 1 red and 2 blue.

1. If Tay picks a shirt and a skirt without looking, how many different ways can she get dress?
2. What are the different dresses that Tay can put on? List all of them.
3. Which dress has the biggest chance that she would put it on? Why? Can you use numbers to describe its chance?
4. What is the chance that she would put on a white shirt and a blue skirt?

Activity 2. (Analysis of "PICK UP A MARBLE")

The teacher poses these questions:

1. What is the result of the box 1?
2. What is the result of the box 2?
3. If you need a red marble, which box are you going to draw? If you need a blue marble, which box are you going to draw?

4. How do you use numbers to describe the chance of getting a blue marble in each box?
5. How do you use numbers to describe the chance of getting a red marble in each box?

Hypothetical learning process

1. In Activity 1, students who can list all the outcomes and use the numbers to describe probability of 2-dimensional situations from Session 6-10 may be able to answer the questions correctly.
2. In Activity 2, some students may be able to choose the box that is best for red and the box that is best for blue correctly but may not elaborate their thinking.

SESSION 12

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can connect sample space composition to probability measure.
3. Students can compare and order probabilities.

Learning activities

Activity 1. The teacher poses this question for class discussion:

I am going to roll a dice and let you guess what number I will get. I roll it but I do not let you see yet. If I tell you that I got an even number, do you have more chance to guess the number correctly than that I do not tell you? Why?

During the discussion, the teacher asks these questions:

1. What could come up in rolling a dice? What is the chance of your guess?
2. After I tell you that the number is an even number, what numbers can it be?

What is the chance of your guess?

Activity 2. (Analysis of Activity 1 in Session 11)

The teacher poses the following questions for class discussion:

1. If Tay pick a shirt and a skirt without looking, how many different ways can she put on the dress (a shirt and a skirt)?

2. Can you list all the ways that she can put on the dress?
3. Which dress is more likely to be picked?
4. What is the chance that Tay put on a white shirt and a blue skirt?

Activity 3. The teacher groups students by 4 to play the “PLAYING WITH A COIN”
(see Appendix B)

Hypothetical learning process

1. In Activity 1, most students may use subjective judgment to answer the question.
2. In Activity 2, students who could answer 2-dimensional situations in prior Sessions may be able to answer the questions. The other students experience more 2-dimensional situations.
3. Some students may respond the questions in Activity 3 by using subjective judgment.

SESSION 13

Learning goals

1. Students can list all the outcomes of a 2-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities.
4. Students can compare probability in nonreplacement situations.

Learning activities

Activity 1. The teacher gathers the results of “PLAYING WITH A COIN” from every group. Students discuss about the way that has more chance of winning.

Activity 2. (Analysis of “PLAYING WITH A COIN”)

The teacher groups students by 4. Each group discusses to answer these questions:

1. How many outcomes of tossing a coin 3 times?
2. How do you list all outcomes?

3. How do you list all outcomes systematically?
4. Is the chance of winning of every student the same?
5. Which student has the biggest chance of winning?
6. Which student has the smallest chance of winning?

A student of each group presents to the class.

Activity 3. "PICK A CANDY"

A container contains 3 milk, 2 fruit and 1 cola candies. Suppose you like cola candies. If you draw a candy from the container without looking, what is the chance of getting a cola candy? What is the chance of getting a fruit candy?

Suppose first draw you do not get a cola candy. So you have to eat that candy then. If I allow you to draw a candy again, do you think you have more chance or less chance to get a cola candy in this draw? Why?

Activity 4. Students list all outcomes in the following situations in paper work.

1. Tossing 1 coin.
2. Tossing 2 coins.
3. Tossing 3 coins.

Hypothetical learning process

1. Students who systematically list all the outcomes of 2-dimensional situations may be able to list all the outcomes of 3-dimensional situations. Class discussion can help the other students to understand and use a system to list all the outcomes.
2. In Activity 2, most students may be able to describe probability of this situation.
3. In Activity 3, most students may be able to answer the first part of the question. But in the second part most students may not elaborate their reason.

SESSION 14

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.

3. Students can compare and order probabilities.
4. Students can compare probability in a nonreplacement situation.

Learning activities

Activity 1. (Repeat situations about tossing coins)

The teacher pose these questions:

1. What could come up in tossing 3 coins? How can you list all the outcomes?
2. What is the chance of getting HHH?
3. What is the chance of getting TTT?
4. What is the chance of getting 2 Hs and 1 T?

Activity 2. (Analysis of "PICK A CANDY")

The teacher groups students by 4 to discuss these questions:

1. Are the chance of getting every kind of candy the same?
2. What is the chance of getting a milk candy? What is the chance of getting a fruit candy? What is the chance of getting a cola candy?
3. Which kind of candy has the biggest chance?
4. Suppose you got a milk candy in the first draw. In the second draw, is the chance of getting a milk candy the same as the first draw? Why? Is the chance of getting a cola candy the same as the first draw? Why?
5. Suppose Nok said, "I got a milk candy in the first draw. The chance of getting a cola candy is still the same in the second draw because the number of cola candy still the same, just 1." Do you think as same as Nok? Why?

Activity 3. "WHICH SPINNER" (see Appendix B)

Activity 4. The teacher groups students by 4. In each group, The first three students play with the spinners by choosing a color on the Spinner 1. The students take turn to spin spinner 10 times. The other student records the result. Students play again by using Spinner 2.

After student finished playing the game, the teacher asks this question:

Is the result from both spinners the same or different?

Hypothetical learning process

1. In Activity 1, students who could answer in SESSION 13 may be able to answer the questions.

2. In Activity 2, students who can order probability in 1- or 2- dimensional situations may be able to compare probability in a nonreplacement situation.

3. In Activity 3, some students may be unable to distinguish a contiguous and a non-contiguous situation.

SESSION 15

Learning goals

1. Students can list all the outcomes of a 1-dimensional sample space.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare and order probabilities in a nonreplacement situation.

Learning activities

Activity 1. (Review probability comparisons and conditional probability)

The teacher poses these questions:

A box contains 10 paper cards, number 1 to 10. Suppose I let you draw a card without looking. If you get a number that is more than 7, you will get a prize.

1. Suppose you are the first student who draws a card. Is the chance of getting the prize more than or less than the chance of not getting the prize?

2. Suppose you do not get a prize. You take the card you drew out. Do the second student who comes to draw a card after you have the same chance of getting a prize as your chance in the first draw? Why?

3. Suppose the first 3 students do not get the prize. What is the chance of getting the prize of the fourth student when he/she comes to draw a card?

Activity 2. "TIGER OR GOLD" (see Appendix B)

Activity 3. (Playing the game "TIGER OR GOLD")

The teacher groups students by 3 to play the game. A student sets the door of gold and the doors of tigers. The other students take turn to guess where the gold is. When the group has the winner, a student who guessed in the first time sets the door of gold and the doors of tigers. The other students take turn to guess the door of gold again.

Activity 4. Class discusses this question:

Suppose your friend opened the 3rd door and found a tiger. Do you have more chance of getting gold more than that you do not know the 3rd door is tiger? Why?

Hypothetical learning process

1. Students who can answer the questions in Sessions 13 and 14 may be able to answer the questions in Activity 1. The other students get more idea to answer the question.
2. Most students may be able to describe that the chance of finding tiger more that the chance of getting gold.
3. Most students may be able to describe that the chance of getting the gold increases when the first student found a tiger.

SESSION 16

Learning goals

1. Students can distinguish a fair game and an unfair game.
2. Students can make a connection between sample space composition and probability measure.
3. Students can compare probabilities.
4. Students can compare and order probability in a nonreplacement situation.

Learning activities

Activity 1. Class discusses these questions:

1. What does a fair game mean? How do you know which game is a fair game?
2. Suppose 2 students are playing a game with a coin and a dice. The first student rolls a dice. If he/she get an even number he/she will get 1 point. The other student tosses a coin. He/she will get a point if he/she get an H. They play 5 rounds.
Is this game fair? Why?

Activity 2. (Analysis of "TIGER OR GOLD" problem)

Class discusses these questions:

1. In the first time of choosing a door, is the chance of meeting a tiger more than or less than the chance of not meeting a tiger?
2. Suppose the first time your friend chose the door that has a tiger. In your turn, what is the chance of getting the gold? What is the chance of meeting a tiger?
3. If you have known what the first door is, is the chance of meeting a tiger of you more than or less than that you do not know what the first door is?

Activity 3. Class discusses this question:

How is probability useful for you? How do you measure probability of an event?

Hypothetical learning process

1. Most students may be able to answer the questions in Activity 1.
2. Some students who can measure probability in 1-dimensional situations may not answer the questions in Activity 2 correctly. These students may ignore some conditions in the question.

APPENDIX D
EXAMPLES OF STUDENTS' PAPER WORK

เสื้อดำ, ผ้าเขียว กางเกงดำ, ผ้าเขียว

ดำ-ดำ

ดำ-ผ้าเขียว

ผ้า-ดำ

ผ้า-ผ้าเขียว

เขียว-ดำ

เขียว-ผ้าเขียว

เสื้อดำ, ผ้าเขียว

กางเกงดำ, ผ้าเขียว

รองเท้าดำ, ขาว

ดำ-ดำ-ดำ

ผ้า-ดำ-ดำ

ดำ-ดำ-ขาว

ผ้า-ดำ-ขาว

ดำ-ผ้าเขียว-ดำ

ผ้า-ผ้าเขียว-ดำ

ดำ-ผ้าเขียว-ขาว

ผ้า-ผ้าเขียว-ขาว

เขียว-ดำ-ดำ

เขียว-ดำ-ขาว

เขียว-ผ้าเขียว-ดำ

เขียว-ผ้าเขียว-ขาว

ស្រី ចាំ ការពេល ចាំ ស្រី ចាំ ការពេល ង៉ាវីង

ស្រី ផ្កា ការពេល ចាំ ស្រី ផ្កា ការពេល ង៉ាវីង

ស្រី ផ្កា ការពេល ចាំ ស្រី ផ្កា ការពេល ង៉ាវីង

ស្រី ចាំ, ផ្កា, ផ្កា ការពេល ង៉ាវីង, ចាំ រង្វង់ វា, ចាំ

ស្រី ចាំ ការពេល ចាំ រង្វង់ ចាំ ស្រី ចាំ ការពេល ចាំ រង្វង់ វា

ស្រី ផ្កា ចាំ ការពេល ង៉ាវីង រង្វង់ ចាំ ស្រី ចាំ ការពេល ង៉ាវីង រង្វង់ វា

ស្រី ផ្កា ផ្កា ការពេល ចាំ រង្វង់ ចាំ ស្រី ផ្កា ការពេល ចាំ រង្វង់ វា

ស្រី ផ្កា ការពេល ង៉ាវីង រង្វង់ ចាំ ស្រី ផ្កា ការពេល ង៉ាវីង រង្វង់ វា

ស្រី ផ្កា ការពេល ចាំ រង្វង់ ចាំ ស្រី ផ្កា ការពេល ចាំ រង្វង់ វា

ស្រី ផ្កា ការពេល ង៉ាវីង រង្វង់ ចាំ ស្រី ផ្កា ការពេល ង៉ាវីង រង្វង់ វា

Natida's list of all outcomes in "Dress Up" problem

ลูกเต๋า 1, 2, 3, 4, 5, 6

เหรียญ หัว, ก้อย

ลูกเต๋า มีโอกาส ชนะ: $\frac{3}{6}$

เหรียญ มีโอกาส ชนะ: $\frac{1}{2}$

เลือก ลูกเต๋า เพราะว่ามีทาง ชนะ 3 ทาง

Witawat's answer in "Which One Is Better" problem

ตอบ

เลือก ลูกเต๋า

ลูกเต๋า มีโอกาส ชนะ $\frac{3}{6}$

เหรียญ มีโอกาส ชนะ $\frac{1}{2}$

เลือก ลูกเต๋า เพราะ มีโอกาส ชนะ มากกว่า

Piyawan's answer in "Which One Is Better" problem

ดำ, ผ้า, เขียว

กางเกง ดำ, หน้า, เขียว

ดำ-ดำ

ผ้า + หน้าเขียว

เขียว - หน้าเขียว

ผ้า - ดำ

เขียว - ดำ

ดำ - หน้าเขียว

เสื้อดำ

กางเกงหน้าเขียว

รองเท้าดำ

เสื้อเขียว

กางเกงดำ

รองเท้าขาว

เสื้อผ้า

กางเกงดำ

รองเท้าดำ

เสื้อผ้า

กางเกงหน้าเขียว

รองเท้าขาว

เสื้อเขียว

กางเกง หน้าเขียว

รองเท้าดำ

เสื้อดำ

กางเกง ดำ

รองเท้าขาว

เสื้อผ้า

กางเกง หน้าเขียว

รองเท้าขาว

เสื้อเขียว

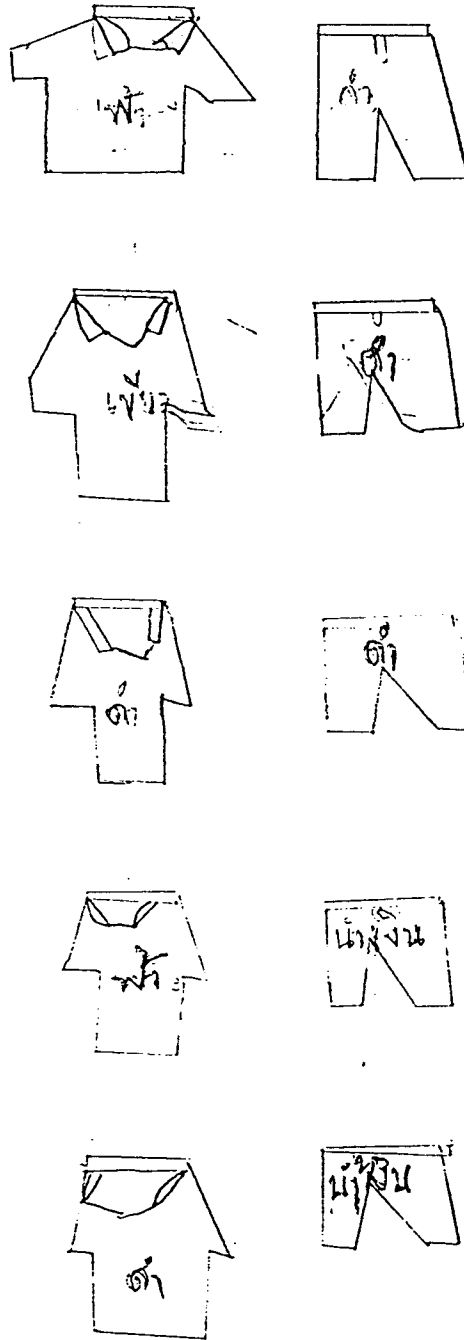
กางเกง ดำ

รองเท้าขาว

เสื้อผ้า

กางเกง หน้าเขียว

รองเท้าดำ



Jiyani's list of all outcomes in "Dress Up" problem

^๑
 เสื้อดำ ผ้าเขียว กางเกงดำ ผ้าเงิน
 กางเกงดำ เสื้อดำ
 กางเกงดำ เสื้อดำ
 กางเกงดำ เสื้อเขียว
 กางเกงผ้าเงิน เสื้อดำ
 กางเกงผ้าเงิน เสื้อดำ
 กางเกงผ้าเงิน เสื้อเขียว

Tadapon's list of all outcomes in "Dress up" problem

^๑
 ผลที่ได้
 1-1 1-2 1-3 1-4 1-5 1-6 ๗ ผล 6 6 2 2 2 2
 2-1 2-2 2-3 2-4 2-5 2-6 ๖ ผล 5 6 2 2 2 2
 3-1 3-2 3-3 3-4 3-5 3-6 ๕ ผล 5 6 2 2 2
 4-1 4-2 4-3 4-4 4-5 4-6
 5-1 5-2 5-3 5-4 5-5 5-6
 6-1 6-2 6-3 6-4 6-5 6-6

^๑ ^๒ ^๓ ^๔
 เสื้อดำดำดำดำ กางเกงผ้าเงิน กางเกงผ้าเงิน กางเกงผ้าเงิน

Tadapon's analysis in "Racing Game" problem

เหรียญ 2 เหรียญ ออกได้

หัวหัว ก้อยก้อย หัวก้อย ก้อยหัว
 เหมือนกัน 2 แบบ (หัวหัว , ก้อยก้อย)
 ไม่เหมือนกัน 2 แบบ (หัวก้อย , ก้อยหัว)
 โอกาสชนะเท่ากัน

Wisanu's analysis in "Playing Two Coins" problem

โยนเหรียญ 2 เหรียญ ออกได้
 หัวหัว หัวก้อย ก้อยหัว
 ก้อยก้อย
 เท่ากัน 2 แบบ
 ไม่เท่ากัน 2 แบบ
 โอกาสชนะเท่ากัน $\frac{2}{4}$

Kanok's analysis in "Playing Two Coins" problem

1. เสื้อดำ กับ กางเกงดำ
 2. เสื้อเขียว กับ กางเกงดำ
 3. เสื้อฟ้า กับ กางเกงดำ
 4. เสื้อดำ กับ กางเกง ~~ฟ้า~~ ขาว
 5. เสื้อเขียว กับ กางเกง ขาว
 6. เสื้อฟ้า กับ กางเกง ขาว

เสื้อมี 5 ตัว คือ เสื้อดำ เสื้อเขียว
 กางเกงมี 2 ตัว คือ กางเกงดำ กางเกงขาว

Kitisak's list of all outcomes in "Dress Up" problem

กางเกง ดำ, ขาว, เสื้อดำ, เสื้อเขียว, เสื้อฟ้า, เสื้อขาว, ดำ
 ดำดำ, ดำเขียว, ดำฟ้า, ดำขาว, ขาวดำ, ขาวเขียว, ขาวฟ้า, ขาวขาว
 ขาวดำ

Jirawan's list of all outcomes in "Dress Up" problem

VITAE

VITAE

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