366

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Some Results on Hamiltonian Walks in Graphs

สัญญาเลขที่ 043/2548

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สนับสนุนโดยทุนอุดหนุนการวิจัยจากงบประมาณเงินรายได้มหาวิทยาลัยศรีนครินทรวิโรฒ

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ทุน :

ทุนอุดหนุนการวิจัยจากงบประมาณเงินรายได้มหาวิทยาลัยศรีนครินทรวิโรฒ
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สำหรับโครงการวิจัย :

ผลจากแนวเดินแฮมิลโทเนียนของกราฟ
Some Results on Hamiltonian Walks in Graphs

กิจการที่วางแผนไว้ :

ช่วงเวลา	กิจกรรม	ผลที่ได้รับ	ผู้รับผิดชอบ
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กิจกรรมอื่นๆที่เกี่ยวข้อง

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ผลงานวิจัยที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ

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กราฟที่กำหนดด้วยอันดับและค่าแฮมิลโทเนียน

บทคัดย่อ

แนวเดินแฮมิลโทเนียนในกราฟเชื่อมโยง G ที่มีอันดับ n เป็นแนวเดินปิดแผ่ทั่วที่มีระยะสั้นที่สุดใน G สำหรับอันดับวง

$$s: v_1, v_2, \ldots, v_n, v_{n+1} = v_1 \text{ ของ } V(G)$$

กำหนดให้ $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$ เมื่อ $d(v_i, v_{i+1})$ เป็นระยะทางระหว่าง v_i และ v_{i+1} สำหรับ $1 \leq i \leq n$ เรานิยามค่าแฮมิลโทเนียน h(G) ของ G ว่า $h(G) = \min\{d(s)\}$ เมื่อเปรียบเทียบค่าน้อยสุดจากอันดับวง s ของ V(G) ทั้งหมด จากงานวิจัยพบว่า กราฟเชื่อมโยง G ใดๆที่มีอันดับ $n \geq 2$ จะมีค่าแฮมิลโทเนียนมากสุดคือ $2n-2$ และ $h(G) = 2n-2$ ก็ต่อเมื่อ G เป็นกราฟต้นไม้ ในงานวิจัยนี้เราแสดงกราฟเชื่อมโยงทั้งหมดที่มีอันดับ n และค่าแฮมิลโทเนียนเป็น $2n-3$ หรือ $2n-4$ นอกจากนั้นเราแสดงขอบเขตบนของค่าแฮมิลโทเนียนของกราฟเชื่อมโยงด้วย

Graphs with Prescribed Order and Hamiltonian Number

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ABSTRACT

A Hamiltonian walk in a connected graph G of order n is a closed spanning walk of minimum length in G. For a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$, let $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. The Hamiltonian number $h(G)$ of G is defined as $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of $V(G)$. It was known that every connected graph G of order $n \geq 2$ has Hamiltonian number at most $2n-2$ and $h(G) = 2n-2$ if and only if G is a tree. In this work, we characterize all connected graph of order n with Hamiltonian number $2n-3$ or $2n-4$. Also, we establish sharp upper bounds for the Hamiltonian number of a connected graph.

Key Words: Hamiltonian walk, Hamiltonian number.

AMS Subject Classification: 05C12, 05C45

1 Introduction

In [8] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by $h(G)$.

Therefore, for a connected graph G of order $n \geq 3$, it follows that $h(G) = n$ if and only if G is Hamiltonian. Hamiltonian walks were studied further by T. Asano, T. Nishizeki, and T. Watanabe [1, 2, 7], J. C. Bermond [3], and P. Vacek [9]. We refer to the book [6] for graph-theoretical notation and terminologies not described in this paper.

In [5] an alternative way to define the length $h(G)$ of a Hamiltonian walk in G was presented. A Hamiltonian graph G contains a spanning cycle $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$. Thus Hamiltonian graphs of order $n \geq 3$ are those graphs for which there is a cyclic ordering $v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. For a connected graph G of order $n \geq 3$ and a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, $d(s) \geq n$ for each cyclic ordering s of $V(G)$. The *Hamiltonian number* $h(G)$ of G is defined in [5] by

$$h(G) = \min\{d(s)\},$$

where the minimum is taken over all cyclic orderings s of $V(G)$. It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G. To illustrate these concepts, consider the graph $G = K_{2,3}$ of Figure 1. For the cyclic orderings s_1: v_1, v_2, v_3, v_4, v_5, v_1 and s_2: v_1, v_3, v_2, v_4, v_5, v_1 of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since G is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h(G) = 6$.

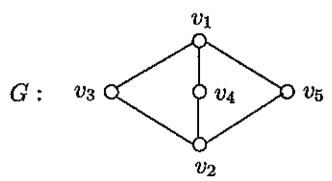

Figure 1: A graph G with $h(G) = 6$

The following results are known (see [4, 5, 8]).

Theorem A *For every connected graph G of order $n \geq 2$,*

$$n \leq h(G) \leq 2n - 2.$$

Moreover,

(1) $h(G) = 2n - 2$ *if and only if* G *is a tree and*

(2) *for every pair* n, k *of integers with* $3 \leq n \leq k \leq 2n-2$, *there exists a connected graph* G *of order* n *having* $h(G) = k$.

Theorem B *If* G *is a* k*-connected graph of order* $n \geq 2$ *having diameter* d, *then*

$$h(G) \leq 2n - \left\lfloor \frac{k}{2} \right\rfloor (2d-2) - 2.$$

Theorem C *Let* G *be a connected graph having blocks* $B_1, B_2, \ldots, B_k$. *Then the union of the edges in a Hamiltonian walk for each of the blocks* B_i *forms a Hamiltonian walk for* G *and, conversely, the edges in a Hamiltonian walk of* G *that belong to* B_i *form a Hamiltonian walk in* B_i.

The following is an immediate consequence of Theorem C.

Corollary 1.1 *If* G *is a connected graph having blocks* $B_1, B_2, \ldots, B_k$, *then*

$$h(G) = \sum_{i=1}^{k} h(B_i)$$

Theorem D *Let* $G = K_{n_1, n_2, \cdots, n_k}$ *be a complete* k*-partite graph of order* $n = n_1 + n_2 + \cdots + n_k$, *where* $n_1 \leq n_2 \leq \cdots \leq n_k$. *Then*

(a) $h(G) = n$ *if and only if* $n_1 + n_2 + \cdots + n_{k-1} \geq n_k$. *In this case,* G *is Hamiltonian, and*

(b) $h(G) = 2n_k$ *if* $n_1 + n_2 + \cdots + n_{k-1} < n_k$.

In this work, we characterize all connected graph of order n with Hamiltonian number $2n-3$ or $2n-4$ in Sections 2 and 3, respectively. In Section 4 we present two upper bounds for the Hamiltonian number of a connected graph in terms of (1) its order and clique number and (2) its order and connectivity. Since every Hamiltonian graph G of order n has $h(G) = n$, we will only consider connected graphs that are not Hamiltonian in this paper.

2 Graphs of Order n with Hamiltonian Number 2n-3

We have seen in Theorem A that every connected graph G of order $n \geq 2$ has Hamiltonian number at most $2n-2$ and $h(G) = 2n-2$ if and only if G

is a tree. In this section, we determine all connected graphs of order $n \geq 3$ with Hamiltonian number $2n - 3$. In order to do this, we first present a lemma, which is an immediate consequence of Theorem B.

Lemma 2.1 *If G is a 2-connected graph of order $n \geq 4$ with diameter $d \geq 2$, then*

$$h(G) \leq 2n - 2d \leq 2n - 4.$$

The following is a consequence of Theorems A and and Lemma 2.1.

Corollary 2.2 *If G is a 2-connected graph of order $n \geq 4$ and T is a spanning tree of G, then*

$$h(G) \leq h(T) - 2.$$

Proof. By Theorem A, $h(T) = 2n - 2$. It then follows by Lemma 2.1 that

$$h(G) \leq 2n - 4 = (2n - 2) - 2 = h(T) - 2.$$

as desired. ■

Note that if G is a 2-connected graph of order $n = 3$ and T is a spanning tree of G, then $G = K_3$ and $T = P_3$. Since $h(K_3) = 3$ and $h(P_3) = 4$, it follows that $h(G) = h(T) - 1$. This observation together with Corollary 2.2, we have the following corollary.

Corollary 2.3 *If G is a 2-connected graph of order $n \geq 3$ and T is a spanning tree of G, then*

$$h(G) \leq h(T) - 1.$$

A connected graph with exactly one cycle is called a *unicyclic graph*. Let $\mathcal{U}_T$ be the set of all unicyclic graphs containing exactly one triangle. Therefore, if $G \in \mathcal{U}_T$, then G is connected and contains exactly one cycle that is a triangle. We now present a characterization of connected graphs of order $n \geq 3$ with Hamiltonian number $2n - 3$.

Theorem 2.4 *Let G be a connected graph of order $n \geq 3$. Then*

$$h(G) = 2n - 3 \text{ if and only if } G \in \mathcal{U}_T.$$

Proof. If $n = 3$, then $G = C_3$ and $h(C_3) = 3 = 2 \cdot 3 - 3$. Thus we may assume that $n \geq 4$. Let $G \in \mathcal{U}_T$ be a unicyclic graph containing exactly one triangle C_3. Thus G has $n - 2$ blocks, $n - 3$ of which are bridges and one of which is C_3. Thus $(n - 2)$ blocks have Hamiltonian number 2 and one block has Hamiltonian number 3. By Theorem C,

$$h(G) = (n - 3) \cdot 2 + 1 \cdot 3 = 2n - 3.$$

For the converse, let G be connected graph of order $n \geq 4$ such that $h(G) = 2n - 3$. It then follows by Lemma 2.1 that G is not 2-connected and so G has cut-vertices. Assume, to the contrary, that $G \notin \mathcal{U}_T$. Let $B_1, B_2, \cdots, B_k$ be blocks of G, where $k \geq 2$. Since G is not a tree by Theorem A, it follows that G must contain some cycles and so $|V(B_i)| \geq 3$ for some i with $1 \leq i \leq k$. For each i with $1 \leq i \leq k$, let T_i be a spanning tree of B_i. Let T be the spanning tree of G with $E(T) = \bigcup_{i=1}^{k} E(T_i)$. We consider two cases.

Case 1. *There is a block of* G *whose order is at least* 4, say $|V(B_1)| \geq 4$. Notice that $h(B_1) \leq h(T_1) - 2$ by Corollary 2.2 and $h(B_i) \leq h(T_i)$ for $2 \leq i \leq k$. Therefore,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) \leq [h(T_1) - 2] + \sum_{i=2}^{k} h(T_i) \\ &= \sum_{i=1}^{k} h(T_i) - 2 = h(T) - 2 = 2(n-1) - 2 = 2n - 4, \end{aligned}$$

which is a contradiction.

Case 2. *There is no block of* G *whose order is* 4 *or more.* Thus every cycle of G is a triangle. Since $G \notin \mathcal{U}_T$ and G has no block of order 4 or more, it follows that G must contain at least two edge-disjoint triangles. This implies that G has at least two blocks of order 3, say $|V(B_1)| = |V(B_2)| = 3$. Thus $B_1 \cong C_3$ and $B_2 \cong C_3$ and $h(B_1) = h(B_2) = 3$. On the other hand, $h(T_1) = h(T_2) = 4$ by Theorem A and so $h(B_i) = h(T_i) - 1$ for $i = 1, 2$. Therefore,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) \leq [h(T_1) - 1] + [h(T_2) - 1] + \sum_{i=3}^{k} h(T_i) \\ &= \sum_{i=1}^{k} h(T_i) - 2 = h(T) - 2 = 2(n-1) - 2 = 2n - 4, \end{aligned}$$

which is a contradiction.

Therefore, $G \in \mathcal{U}_T$, as claimed. ∎

3 Graphs of Order n with Hamiltonian Number 2n-4

We have seen in Corollary 1.1 that the Hamiltonian number of a graph is the sum of the Hamiltonian numbers of its blocks. Thus, the topic of

Hamiltonian walks can be restricted to 2-connected graphs. In this section, we first present a characterization of 2-connected graphs of order $n \geq 4$ with Hamiltonian number $2n-4$.

Theorem 3.1 *Let G be a 2-connected graph of order $n \geq 4$. Then*

$$h(G) = 2n-4 \text{ if and only if } G \in \{K_4, K_{2,n-2}, K_2 + \overline{K}_{n-2}\}.$$

Proof. Observe that the only 2-connected graphs of order 4 are K_4, $C_4 = K_{2,2}$, and $K_4 - e = K_2 + \overline{K}_2$ and each of these graphs is Hamiltonian. Hence the Hamiltonian number of every 2-connected graph of order 4 is 4 and so the result holds for $n = 4$. Thus we may assume that $n \geq 5$. By Theorem D, if $G = K_{2,n-2}$ or $G = K_2 + \overline{K}_{n-2} = K_{1,1,n-2}$, then $h(G) = 2n-4$. Thus, it remains to verify the converse. Let G be a 2-connected graph of order $n \geq 5$ with $h(G) = 2n-4$. If $n = 5$, then it can be verified that $G = K_{2,3}$ or $G = K_2 + \overline{K}_3$. Thus, we may assume that $n \geq 6$. By Lemma 2.1, $\text{diam}(G) = 2$. First, we make two observations.

(b1) The graph G contains no path of length 5 or more. For otherwise, let $P : v_1, v_2, \cdots, v_6$ be a path of length 5 in G. Define a cyclic ordering s of $V(G)$ by

$$s : v_1, v_2, \cdots, v_6, v_7, v_8, \cdots, v_n, v_1,$$

where $v_7, v_8, \cdots, v_n$ is an arbitrary ordering of the vertices in $V(G) - V(P)$. Since $\text{diam}(G) = 2$, it follows that

$$d(s) \leq 5 + 2(n-5) = 2n-5 < 2n-4,$$

which is impossible.

(b2) By (b1), the graph G contains no cycle of length 5 or more.

Since G is 2-connected, every two vertices of G lie on a common cycle. In particular, every two nonadjacent vertices lie on a 4-cycle by (b2). Since $h(G) = 2n-4$, it follows that G is not complete and so G contains two nonadjacent vertices. Thus G contains a 4-cycle $C_4 : u, v, x, y, u$ such that u and x are not adjacent in G (v and y may or may not be adjacent). Figure 2 shows the 4-cycle C_4 in G, where the dish line indicates that u and x are not adjacent in G.

Let $W = V(G) - V(C_4)$ and so $|W| = n-4 \geq 2$. Since G is connected, there exists $w \in W$ such that w is adjacent to some vertex in C_4. We consider two cases, according to whether w is adjacent to a vertex in $\{u, x\}$ or to a vertex in $\{v, y\}$. Let

$$s_W : w_1, w_2, \cdots, w_{n-4} \qquad (1)$$

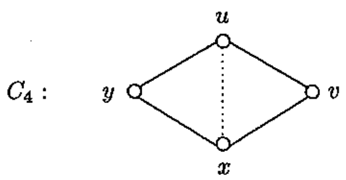

Figure 2: The 4-cycle C_4 in G

be an arbitrary ordering of the vertices in W. Then $w = w_i$ for some i with $1 \le i \le n-4$.

Case 1. *$w = w_i$ is adjacent to u or x, say $w_iu \in E(G)$.* We claim in this case that $G = K_{2,n-2}$ whose partite sets are $\{u,x\}$ and $\{v,y\} \cup W$.

We first show that W is an independent set in G. Using the ordering s_W described in (1), we define a cyclic ordering s of $V(G)$ by

$$s : w_1, w_2, \cdots, w_i, u, v, x, y, w_{i+1}, w_{i+2}, \cdots, w_{n-4}, w_1.$$

Since $d(w_i,u) = d(u,v) = d(v,x) = d(x,y) = 1$ and $\text{diam}(G) = 2$, it follows that $d(s) \le 4 + 2(n-4) = 2n-4$. Because $h(G) = 2n-4$, we have $d(s) = 2n-4$. This implies that $d(w_j, w_{j+1}) = 2$ for $1 \le j \le i-1$ and $i+1 \le j \le n-5$ and $d(w_{n-4}, w_1) = 2$. Since s_W is an arbitrary ordering of W, the distance between every two vertices of W is 2 in G. Therefore, W is an independent set in G.

Next, we show that no vertex in W is adjacent to any vertex in $\{v,y\}$. Assume, to the contrary, that there exists $w' \in W$ such that w' is adjacent to at least one of v and y, say $w'v \in E(G)$. If $w' = w$, then u, w, v, x, y, u is a 5-cycle in G, which contradicts (b2). On the other hand, if $w' \ne w$, then w', v, x, y, u, w is a path of length 6 in G, which contradicts (b1).

Since (i) W is an independent set in G, (ii) no vertex in W is adjacent to any vertex in $\{v,y\}$, and (iii) G is 2-connected, every vertex in W must be adjacent to both u and x. This implies that v and y are not adjacent; for otherwise, G contains the 5-cycle w, u, y, v, x, w, which contradicts (b2).

Therefore, $\{u,x\}$ and $\{v,y\} \cup W$ are independent sets in G and every vertex in $\{v,y\} \cup W$ is adjacent to both u and x. This implies that $G = K_{2,n-2}$ whose partite sets are $\{u,x\}$ and $\{v,y\} \cup W$, as claimed.

Case 2. *$w = w_i$ is adjacent to v or y, say $w_iv \in E(G)$.* We claim in this case that either $G = K_2 + \overline{K}_{n-2}$, where $V(K_2) = \{v,y\}$ and $V(\overline{K}_{n-2}) = \{u,x\} \cup W$ or $G = K_{2,n-2}$ whose partite sets are $\{v,y\}$ and $\{u,x\} \cup W$.

First we show W is an independent set in G. Again, let s_W be an arbitrary ordering of the vertices in W as described in (1). We define a cyclic ordering s of $V(G)$ by

$$s : w_1, w_2, \cdots, w_i, v, u, y, x, w_{i+1}, w_{i+2}, \cdots, w_{n-4}, w_1.$$

An argument similar to that in Case 1 shows that $d(s) = 2n-4$ and the distance between every two vertices of W is 2 in G, which implies that W is an independent set in G.

Next, we show that no vertex in W is adjacent to any vertex in $\{u, x\}$. Assume, to the contrary, that there exists $w' \in W$ such that w' is adjacent to at least one of u and x, say $w'u \in E(G)$. If $w' = w$, then G contains the 5-cycle x, v, w, u, y, x, a contradiction. If $w' \neq w$, then G contains the path w', u, y, x, v, w of length 6, a contradiction.

Again, since (i) W is an independent set in G, (ii) no vertex in W is adjacent to any vertex in $\{u, x\}$, and (iii) G is 2-connected, every vertex in W must be adjacent to both v and y. If $vy \in E(G)$, then $G = K_2 + \overline{K}_{n-2}$, where $V(K_2) = \{v, y\}$ and $V(\overline{K}_{n-2}) = \{u, x\} \cup W$. On the other hand, if $vy \notin E(G)$, then $G = K_{2,n-2}$ whose partite sets are $\{v, y\}$ and $\{u, x\} \cup W$.

Therefore, $G = K_2 + \overline{K}_{n-2}$ or $G = K_{2,n-2}$, as claimed. ■

Next, we present a characterization of connected graphs of order $n \geq 5$ containing cut-vertices whose Hamiltonian number is $2n-4$. In order to do this, we first present a useful lemma.

Lemma 3.2 *Let G be a connected graph with $k \geq 2$ blocks $B_1, B_2, \cdots, B_k$ such that each of the blocks $B_1, B_2, \cdots, B_p$ has order at least 3, where $1 \leq i \leq p < k$, and the $k-p$ remaining blocks of G are K_2. Then*

$$k - p = (n + p - 1) - \sum_{i=1}^{p} |V(B_i)|.$$

Proof. Let T_i be a spanning tree of B_i for $1 \leq i \leq p$ and let T be the spanning tree of G with

$$E(T) = \left(\bigcup_{i=1}^{p} E(T_i)\right) \bigcup \left(\bigcup_{i=p+1}^{k} E(B_i)\right).$$

Since $|E(T)| = n-1$, $|E(T_i)| = |V(B_i)| - 1$ for $1 \leq i \leq p$, and $|E(B_i)| = 1$ for $p+1 \leq i \leq k$, it follows that

$$n - 1 = \sum_{i=1}^{p}[|V(B_i)| - 1] + (k-p) = \left(\sum_{i=1}^{p} |V(B_i)|\right) - p + (k-p)$$

and so we have the desired result. ■

Let $\mathcal{G}_1$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly two blocks that are K_3 and each of the remaining blocks of G is K_2. Let $\mathcal{G}_2$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly one block that is one of graphs in the set

$$\mathcal{S} = \{K_4\} \cup \{K_{2,r-2}, K_2 + \overline{K}_{r-2} : 4 \leq r \leq n-1\} \tag{2}$$

and each of the remaining blocks of G is K_2.

Theorem 3.3 *Let G be a connected graph of order $n \geq 5$ with cut-vertices. Then*

$$h(G) = 2n - 4 \textit{ if and only if } G \in \mathcal{G}_1 \cup \mathcal{G}_2.$$

Proof. Let $B_1, B_2, \cdots, B_k$ be the blocks of G, where $k \geq 2$. First, we show that if $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $h(G) = 2n - 4$. Let $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ and we consider two cases, according to whether $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$.

Case 1. $G \in \mathcal{G}_1$. Thus G has exactly two blocks that are K_3 and the remaining $k-2$ blocks of G are K_2. We may assume that B_1 and B_2 are K_3. By Lemma 3.2, $k - 2 = (n+1) - 6 = n - 5$. Since $h(K_3) = 3$, it follows that

$$h(G) = \sum_{i=1}^{k} h(B_i) = 2 \cdot 3 + (k-2) \cdot 2 = 6 + 2(n-5) = 2n - 4.$$

Case 2. $G \in \mathcal{G}_2$. Thus G has exactly one block belonging to the set $\mathcal{S}$ described in (2) and the remaining $k-1$ blocks are K_2, say $B_1 \in \mathcal{S}$.

If $B_1 = K_4$, then $k - 1 = n - 4$ by Lemma 3.2. Since $h(K_4) = 4$,

$$h(G) = \sum_{i=1}^{k} h(B_i) = 4 + (k-1) \cdot 2 = 4 + 2(n-4) = 2n - 4.$$

If $B_1 = K_{2,r-2}$, then $k - 1 = n - r$ by Lemma 3.2. Since $h(K_{2,r-2}) = 2r - 4$,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) = (2r-4) + (k-1) \cdot 2 \\ &= (2r-4) + (n-r) \cdot 2 = 2n - 4. \end{aligned}$$

If $B_1 = K_2 + \overline{K}_{r-2}$, then $k - 1 = n - r$ by Lemma 3.2. Since $h(K_2 + \overline{K}_{r-2}) = 2r - 4$,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) = (2r-4) + (k-1) \cdot 2 \\ &= (2r-4) + (n-r) \cdot 2 = 2n - 4. \end{aligned}$$

Therefore, if $G \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $h(G) = 2n - 4$, as claimed.

For the converse, let G be a connected graph of order $n \geq 5$ with cut-vertices such that $h(G) = 2n-4$. We show that $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. Since G is not a tree by Theorem A, it follows that G contains cycles and so G contains at least one block of order 3 or more. First we verify the following claim.

Claim 1 The graph G has at most two blocks of order 3 or more.

Proof of Claim 1. Assume, to the contrary, that G has three blocks of order 3 or more, say B_1, B_2, and B_3 have order 3 or more. For each i with $1 \leq i \leq k$, let T_i be the spanning tree of B_i and let T be the spanning tree of G with $E(T) = \bigcup_{i=1}^{k} E(T_i)$. Thus $h(B_i) \leq h(T_i) - 1$ for $1 \leq i \leq 3$ by Corollary 2.3. Therefore,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) \leq \sum_{i=1}^{3} [h(T_i) - 1] + \sum_{i=4}^{k} h(T_i) \leq \left(\sum_{i=1}^{k} h(T_i) \right) - 3 \\ &= h(T) - 3 = (2n-2) - 3 = 2n - 5, \end{aligned}$$

which is a contradiction. Thus, G has at most two blocks of order 3 or more. This completes the proof of Claim 1.

By Claim 1, either G contains exactly one block of order 3 or more or G contains exactly two blocks of order 3 or more. We now verify the following two claims.

Claim 2 If G has exactly one block of order 3 or more, then this block belongs to $\mathcal{S}$.

Proof of Claim 2. Suppose, without loss of generality, that B_1 has order $r \geq 3$ and $B_i = K_2$ for $2 \leq i \leq k$. Assume, to the contrary, that $B_1 \notin \mathcal{S}$. Since $h(G) = 2n - 4$, it follows by Theorem 2.4 that $r \geq 4$. Moreover, the only non-separable graphs of order 4 are K_4, $K_{2,2}$, $K_2 + \overline{K}_2$, each of which belongs to $\mathcal{S}$. Since $B_1 \notin \mathcal{S}$, it follows that $r \geq 5$. By Theorem 3.1, $h(B_1) \leq 2r - 5$. Furthermore, $k - 1 = n - r$ by lemma 3.2. Therefore,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) \leq (2r-5) + (k-1) \cdot 2 \\ &= (2r-5) + (n-r) \cdot 2 = 2n - 5, \end{aligned}$$

which is a contradiction. This completes the proof of Claim 2.

Claim 3 If G has exactly two blocks of order 3 or more, then these two blocks are both K_3.

Proof of Claim 3. Suppose, without loss of generality, that B_1 and B_2 have order 3 or more. Assume, to the contrary, that at least one of B_1

and B_2 is not K_3, say $B_1 \neq K_3$. Since K_3 is the only nonseparable graph of order 3, it follows $|V(B_1)| \geq 4$. Since B_1 and B_2 are two connected, $h(B_1) \leq 2|V(B_1)| - 4$ and $h(B_2) \leq 2|V(B_2)| - 2$ by Theorem 3.1 and Theorem A. Moreover,

$$k - 2 = (n+1) - |V(B_1)| - |V(B_2)|$$

by Lemma 3.2. Therefore,

$$\begin{aligned} h(G) &= \sum_{i=1}^{k} h(B_i) \leq [2|V(B_1)| - 4] + [2|V(B_2)| - 3] + (k-2) \cdot 2 \\ &= [2|V(B_1)| + 2|V(B_2)| - 7] + 2[(n+1) - |V(B_1)| - |V(B_2)|] \\ &= 2n - 5, \end{aligned}$$

which is a contradiction. This completes the poof of Claim 3.

It then follows by Claims 1–3 that $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. ∎

4 Upper Bounds for Hamiltonian numbers

In this section, we present two upper bounds for the Hamiltonian number of a connected graph in terms of (1) its order and clique number and (2) its order and connectivity.

The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of the graph. Thus if G is a connected graph of order $n \geq 2$, then $\omega(G) = 2$ if and only if G is a tree; while $\omega(G) = n$ if and only if G is complete. Next we present an upper bound for $h(G)$ in terms of the order and clique number of a connected graph G.

Proposition 4.1 *If G is a nontrivial connected graph of order n having clique number ω, then*

$$h(G) \leq 2n - \omega.$$

Furthermore, for each integer ω with $2 \leq \omega \leq n$, there exists a connected graph F of order n having clique number ω such that $h(F) = 2n - \omega$.

Proof. By Theorem A, the result is true if G is a complete graph or G is a tree. Thus, we may assume that G is neither a complete graph nor a tree. Let K_ω be a complete subgraphs of order ω in G. Then $3 \leq \omega \leq n-1$. Furthermore, let T be a spanning tree of G and let G' be the spanning subgraph of G with $E(G') = E(K_\omega) \cup E(T)$. Observe that G' is a connected graph with $n - \omega + 1$ blocks such that exactly one block is K_ω and the remaining blocks are K_2. Let $B_1, B_2, \cdots, B_{n-\omega+1}$ be the $n-\omega+1$ blocks of G', where $B_1 = K_\omega$. Then

$$\begin{aligned} h(G') &= \sum_{i=1}^{n-\omega+1} h(B_i) = h(K_\omega) + (n-\omega)h(K_2) \\ &= \omega + 2(n-\omega) = 2n-\omega. \end{aligned}$$

Therefore, $h(G) \le h(G') = 2n - \omega$.

If $\omega = 2$, then let F be any tree of order $n \ge 2$; while if $\omega = n$, then let F be the complete graph of order n. By Theorem A, $h(F) = 2n - \omega$ in each case. Thus we may assume that $3 \le \omega \le n-1$. Now let F be the graph obtained from K_ω and the path $P: v_1, v_2, \cdots, v_{n-\omega}$ be joining v_1 to a vertex in K_ω. Then exactly one block of F is K_ω and the remaining blocks of F are K_2. By the same argument above, we have $h(F) = 2n - \omega$. ■

Let $\kappa(G)$ be the connectivity of a graph G. It is known that $\kappa(G) \le \delta(G)$ for every graph G. A graph G is *k-connected* if $\kappa(G) \ge k$.

Proposition 4.2 *Let G be a nontrivial k-connected graph of order n and diameter $d \ge 3$. If G is not Hamiltonian, then $h(G) \le 2(n-k)$.*

Proof. We have seen that the result is true for $k = 1$ or $k = 2$. Thus, we may assume that $k \ge 3$. Let $d = \mathrm{diam}(G) \ge 3$. By Theorem B,

$$h(G) \le 2n - \left\lfloor \frac{k}{2} \right\rfloor (2d-2) - 2.$$

If k is even, then

$$h(G) \le 2n - k(d-1) - 2 \le 2n - 2k - 2 < 2n - 2k;$$

while if k is odd, then

$$h(G) \le 2n - (k-1)(d-1) - 2 \le 2n - 2(k-1) - 2 = 2n - 2k.$$

In each case, $h(G) \le 2n - 2k$ for $d \ge 3$. ■

The upper bound in Theorem 4.3 is sharp for integers k and n with $1 \le k \le n/2$. For example, for $k = 1$, let G be any tree T of order $n \ge 2$. Then G is 1-connected and $h(G) = 2n - 2$. For $2 \le k \le n/2$, let $G = K_{k,n-k}$. Then G is k-connected. Since $k \le n - k$, it follows by Theorem D that $h(G) = 2(n-k) = 2n - 2k$.

Observe that if G is a non-Hamiltonian k-connected graph of order n and diameter 2, then $h(G) \le 2n - k - 1$. We conclude this paper with the following conjecture.

Conjecture 4.3 *Let G be a k-connected graph of order n and diameter 2. If G is not Hamiltonian, then $h(G) \le 2n - 2k$.*

References

[1] T. Asano, T. Nishizeki, and T. Watanabe, An upper bound on the length of a Hamiltonian walk of a maximal planar graph. *J. Graph Theory* **4** (1980) 315-336.

[2] T. Asano, T. Nishizeki, and T. Watanabe, An approximation algorithm for the Hamiltonian walk problems on maximal planar graphs. *Discrete Appl. Math.* **5** (1983) 211-222.

[3] J. C. Bermond, On Hamiltonian walks. *Congr. Numer.* **15** (1976) 41-51.

[4] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, On the Hamiltonian number of a graph. *Congr. Numer.* **165** (2003) 51-64.

[5] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, A new look at Hamiltonian walks. *Bull. Inst. Combin. Appl.* **42** (2004) 37-52.

[6] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill (2004).

[7] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs. *Congr. Numer.* (1973) 335-342.

[8] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs. *SIAM J. Comput.* **3** (1974) 214-221.

[9] P. Vacek, On open Hamiltonian walks in graphs. *Arch Math. (Brno)* **27A** (1991) 105-111.