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"EXTREMAL PROBLEM ON GRAPH PARAMETERS"

(ทุนองค์ความรู้ใหม่ที่เป็นพื้นฐานต่อการพัฒนา)

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# Extremal Problem on Graph Parameters

Abstract: Let $\mathbb{J}$ be a class of graphs or subgraphs of a graph. If there is a fixed type of transformation of a member of $\mathbb{J}$ into another, then this will provide a definition of a **graph transformation on $\mathbb{J}$**. Equivalently, a graph transformation may be defined as a relation $\rho$ from $\mathbb{J}$ to $\mathbb{J}$. In other words, $\rho \subseteq \mathbb{J} \times \mathbb{J}$. Let $\rho$ be a graph transformation on $\mathbb{J}$. We can define **the $\rho$-graph** having $\mathbb{J}$ as its vertex set and there is an edge between two vertices $G$ and $H$ if $(G, H) \in \rho$. If $\rho$ is an irreflexive and symmetric, then the $\rho$-graph is simple. Thus the $\rho$-graph arises from graphs in at least two different ways:

1. from a class of graphs, or
2. from a class of subgraphs of a fixed graph.

Harary(1983) introduced a graph transformation and called it an **edge exchange** as follows: Let $G$ be a connected graph of order $n \geq 3$. The **tree graph, $\mathbf{T}(G)$**, of $G$ is defined by specifying $V(\mathbf{T}(G))$ to be the set of all distinct spanning trees of $G$, and two vertices $T_1, T_2 \in V(\mathbf{T}(G))$ are adjacent in $\mathbf{T}(G)$ if and only if $T_1$ and $T_2$ have exactly $n-2$ edges in common. This is an example of an undirected $\rho$-graph.

Let $G$ be a graph. For the distinct vertices $a$, $b$, $c$, and $d$ in $V(G)$, let $ab$ and $cd$ be edges in $G$, given that $ac$ and $bd$ are not edges in $G$. Define $G^{\sigma(a,b;c,d)}$, simply written $G^{\sigma}$, to be the graph obtained from $G$ by deleting the edges $ab$ and $cd$ and adding the edges $ac$ and $bd$.

The transformation $\sigma(a, b; c, d)$ is called a **switching operation**.

Let $\mathbf{d}$ be a graphic degree sequence. Let $\mathbb{R}(\mathbf{d})$ and $\mathbb{CR}(\mathbf{d})$ be the sets of realizations and connected realizations of $\mathbf{d}$, respectively. Since $\sigma$ is a degree preserving transformation, it is reasonable to define $\Sigma(\mathbf{d})$ as a relation on $\mathbb{R}(\mathbf{d})$ as $(G, H) \in \Sigma(\mathbf{d})$ if $G \ncong H$ and there is a switching $\sigma$ on $G$ such that $H = G^{\sigma}$. It is easy to see that $\sigma(a, b; c, d)$ is well-defined on $G$ if and only if $\sigma(a, c; b, d)$ is well-defined on $G^{\sigma(a,b;c,d)}$. Thus the $\Sigma(\mathbf{d})$-graph is simple. The $\Sigma(\mathbf{d})$-graph was proved to be connected by Taylor in 1981. One year later, it was also proved by Taylor that the subgraph of $\Sigma(\mathbf{d})$-graph induced by $\mathbb{CR}(\mathbf{d})$ is connected.

For positive integers $m$ and $m$ with $0 \leq m \leq \binom{n}{2}$, let $\mathbb{G}(m,n)$ and $\mathbb{CG}(m,n)$ be the sets of all graphs and connected graphs, respectively, of order $n$ and size $m$. Let $G \in \mathbb{G}(m,n)$ with $e \in E(G)$ and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G-e+f)$. A transformation $t(e,f)$ is called a **jumping transformation**. Now let $\mathbb{T}(m,n)$ be a relation on $\mathbb{G}(m,n)$ defined by $(G,H) \in \mathbb{T}(m,n)$ if $G \ncong H$ and $H$ can be obtained from $G$ by a jumping transformation. Since $\mathbb{T}(m,n)$ is symmetric, it follows that $\mathbb{T}(m,n)$-graph is simple.

We proved recently that the $\mathbb{T}(m,n)$-graph and the subgraph of $\mathbb{T}(m,n)$-graph induced by $\mathbb{CG}(m,n)$ are connected.

Let $\mathbb{G}$ be the class of all graphs. A function $\pi : \mathbb{G} \to \mathbb{Z}$ is called a **graph parameter** if $\pi(G) = \pi(H)$ for all isomorphic graphs $G$ and $H$. A graph parameter $\pi$ is called an **interpolation graph parameter over $\mathbb{J} \subseteq \mathbb{G}$** if there exist integers $x$ and $y$ such that

$$\{\pi(G) : G \in \mathbb{J}\} = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

We reveiw our results on the interpolation properties of the following graph parameters:

$\chi$ = the chromatic number, $\omega$ = the clique number

$f$ = the forest number, $\phi$ = the decycling number

$\alpha_0$ = the independence number, $\alpha_1$ = the matching number

$\beta_0$ = the vertex covering number, $\beta_1$ = the edge covering number

$\gamma$ = the domination number, $\gamma'$ = the edge domination number

If $\pi$ is an interpolation graph parameter over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi,\mathbb{J}) := \min\{\pi(G) : G \in \mathbb{J}\}$ and $\max(\pi,\mathbb{J}) := \max\{\pi(G) : G \in \mathbb{J}\}$.

The problem of finding $\min(\pi,\mathbb{J})$ and $\max(\pi,\mathbb{J})$ is called the **extremal problem in graph theory**. We also reveiw our results on extremal problem on graph parameters.

# ปัญหาค่าสุดขีดบนกราฟพารามิเตอร์

**บทคัดย่อ** ให้ $\mathcal{J}$ เป็นคลาสของกราฟหรือคลาสของกราฟย่อยของกราฟหนึ่ง ถ้ามีการแปลงที่กำหนดรูปแบบคงที่ของสมาชิกหนึ่งใน $\mathcal{J}$ ไปยังอีกสมาชิกหนึ่งแล้ว เราจะเรียกว่าการแปลงกราฟบน $\mathcal{J}$ ซึ่งกล่าวได้ว่า การแปลงกราฟเป็นความสัมพันธ์ $\rho$ จาก $\mathcal{J}$ ไปยัง $\mathcal{J}$ นั่นคือ $\rho \subseteq \mathcal{J}\times\mathcal{J}$ ให้ $\rho$ เป็นการแปลงกราฟบน $\mathcal{J}$ เราสามารถนิยาม the $\rho$–graph เป็นกราฟที่มีเซตของจุดคือ $\mathcal{J}$ และมีเส้นเชื่อมระหว่างจุด 2 จุดคือกราฟ G และ H ถ้า $(G, H) \in \rho$ ถ้า $\rho$ ไม่มีสมบัติสะท้อนและมีสมบัติสมมาตรแล้ว the $\rho$ – graph เป็นกราฟเชิงเดียว ดังนั้น การแปลงกราฟสามารถสร้างได้จากกราฟใน 2 ลักษณะคือ

1. จากคลาสของกราฟ หรือ 2. จากคลาสของกราฟย่อยของกราฟที่กำหนดให้

ฮารารี (1983) แนะนำการแปลงกราฟแบบหนึ่งที่เรียกว่า “การแลกเปลี่ยนเส้น” ให้ G เป็นกราฟเชื่อมโยงที่มีขนาด $n \geq 3$ the tree graph $\mathbf{T}(G)$ ของ G ถูกกำหนดโดยให้ $V(\mathbf{T}(G))$ เป็นเซตของกราฟต้นไม้แผ่ทั่วที่แตกต่างกันทั้งหมดของ G และจุด 2 จุด $T_1$, $T_2 \in V(\mathbf{T}(G))$ ประชิดกันใน the tree graph $\mathbf{T}(G)$ ก็ต่อเมื่อ $T_1$ และ $T_2$ มีจำนวนเส้น $n-2$ เส้นเท่านั้นที่ใช้ร่วมกัน the tree graph $\mathbf{T}(G)$ นี้เป็นตัวอย่างหนึ่งของ $\rho$–graph ที่เป็นกราฟเชิงเดียว

ให้ G เป็นกราฟหนึ่ง สำหรับจุด a, b, c, d ที่แตกต่างกันใน V(G) ให้ ab และ cd เป็นเส้นใน G โดยที่ ac และ bd ไม่เป็นเส้นใน G กำหนดให้ $G^{\sigma(a, b : c, d)}$ หรือเขียนอย่างง่ายคือ $G^{\sigma}$ เป็นกราฟที่ได้จาก G โดยการตัดเส้น ab และ cd ออกและเพิ่มเส้น ac และ bd การแปลง $\sigma(a, b : c, d)$ ถูกเรียกว่า การดำเนินการสลับเส้น

ให้ $\mathbf{d}$ คือลำดับดีกรีเชิงกราฟและให้ $\mathbb{R}(\mathbf{d})$ และ $\mathbb{CR}(\mathbf{d})$ คือเซตของ realizations และ realizations เชื่อมโยงของ $\mathbf{d}$ ตามลำดับ โดยที่ $\sigma$ เป็นการแปลงที่คงสภาพดีกรี ดังนั้นเราสามารถให้นิยามของ $\Sigma(\mathbf{d})$ ว่าเป็นความสัมพันธ์บน $\mathbb{R}(\mathbf{d})$ ที่กำหนดโดย $(G, H) \in \Sigma(\mathbf{d})$ ถ้า $G \ncong H$ และมี การดำเนินการสลับเส้น $\sigma$ บน G ที่ซึ่ง $H = G^{\sigma}$ จะเห็นได้ชัดว่า $\sigma(a, b : c, d)$ แจ่มชัดบน G ก็ต่อเมื่อ $\sigma(a, c : b, d)$ แจ่มชัดบน $G^{\sigma(a, b : c, d)}$ ดังนั้น $\Sigma(\mathbf{d})$ – graph เป็นกราฟเชิงเดียว ในปี 1981 เทเลอร์ได้ พิสูจน์ว่า $\Sigma(\mathbf{d})$–graph เป็นกราฟ

เชื่อมโยง และหนึ่งปีต่อมา เขาก็ได้พิสูจน์อีกว่ากราฟย่อยของ $\Sigma(\mathbf{d})$–graph ที่อินดิวซ์โดย $\mathbb{CR}(\mathbf{d})$ เป็นกราฟเชื่อมโยง สำหรับจำนวนเต็มบวก m และ n ซึ่ง $0 \le m \le \binom{n}{2}$ ให้ G(m, n) และ $\mathbb{C}G(m, n)$ เป็นเซตของกราฟและกราฟเชื่อมโยงทั้งหมดตามลำดับที่มีอันดับ n และขนาด m ให้ $G \in G(m, n)$ ซึ่ง $e \in E(G)$ และ $f \notin E(G)$ กำหนดให้ $G^{t(e, f)}$ เป็นกราฟที่ $V(G^{t(e, f)}) = V(G)$ และ $E(G^{t(e, f)}) = E(G - e + f)$ เราเรียกการแปลง t(e, f) ว่า jumping transformation ให้ $\mathbf{T}(m, n)$ คือความสัมพันธ์บน G(m, n) กำหนดโดย $(G, H) \in \mathbf{T}(m, n)$ ถ้า $G \ncong H$ และ H ได้มาจาก G โดย jumping transformation เนื่องจาก $\mathbf{T}(m, n)$ มีสมบัติสมมาตร ดังนั้นจึงได้ว่า T(m, n)–graph เป็นกราฟเชิงเดียว

เราได้พิสูจน์มาแล้วว่า $\mathbf{T}(m, n)$–graph และกราฟย่อยของ $\mathbf{T}(m, n)$–graph ที่อินดิวซ์โดย $\mathbb{C}G(m, n)$ เป็นกราฟเชื่อมโยง

ให้ $\mathcal{G}$ เป็นคลาสของกราฟทั้งหมด เราเรียกฟังก์ชัน $\pi : \mathcal{G} \to \mathbb{Z}$ ว่า กราฟพารามิเตอร์ ถ้า $\pi(G) = \pi(H)$ สำหรับกราฟ G และ H ที่ไอโซมอฟิกกันทั้งหมดและเรียกกราฟพารามิเตอร์ $\pi$ ว่ากราฟพารามิเตอร์ที่ปรากฏค่าทุกค่าเหนือ $\mathcal{J} \subseteq \mathcal{G}$ ถ้ามีจำนวนเต็ม x และ y ที่ซึ่ง

$$\{\pi(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : x \le k \le y\}$$

เรากล่าวถึงผลงานที่เกี่ยวกับสมบัติการปรากฏค่าทุกค่าของกราฟพารามิเตอร์ต่อไปนี้

$\chi$ = จำนวนโครมาติก $\quad$ $\omega$ = จำนวนคลีก

f = จำนวนฟอร์เรส $\quad$ $\phi$ = จำนวนดีไซคลิง

$\alpha_0$ = จำนวนจุดอิสระ $\quad$ $\alpha_1$ = จำนวนการจับคู่

$\beta_0$ = จำนวนจุดปกคลุม $\quad$ $\beta_1$ = จำนวนเส้นปกคลุม

$\gamma$ = จำนวนโดมิเนชัน $\quad$ $\gamma'$ = จำนวนโดมิเนชันของเส้น

ถ้า $\pi$ คือกราฟพารามิเตอร์เหนือ $\mathcal{J}$ แล้ว $\{\pi(G) : G \in \mathcal{J}\}$ ถูกกำหนดโดย $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ และ $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$

ปัญหาของการหาค่า $\min(\pi, \mathcal{J})$ และ $\max(\pi, \mathcal{J})$ เราเรียกว่า ปัญหาค่าสุดขีดในทฤษฎีกราฟ เรากล่าวถึงผลงานที่เกี่ยวกับปัญหาค่าสุดขีดบนกราฟพารามิเตอร์

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# PREFACE

Pure mathematical concepts, such as groups, rings, fields, vector spaces, topology, measured theory, etc. were derived by abstracting special properties of some familiar objects like real number system. Similarly, the generic concept of graphs also arose by abstracting some properties common to the diagrams representing various physical system like the molecular structure of chemical compound, the interconnections in an electrical network, a railway network or road network or interrelations among a set of people. What is common to these and several similar systems is that each of them consists of a number parts or components interlinked or interrelated in specific ways.

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of the computing, social, and natural sciences, as you can see in Chapter 1.

This is well reflected by the number of books and the number of research papers that have been published in this area.

The very simple structure of a graph, as a collection of points called vertices together with a set of lines called edges such that each edge is identified with a pair of vertices, makes it suitable for modelling a wide variety of situations.

The purpose of this report is to summarize my 3-year project (2005-2008) with financial support from the **Thailand Research Fund** under the grant number **BRG4880020**. It is summarizing all of my work in graph theory emphasizing on **Extremal Problems on Graph Parameters**. No previous knowledge of graph theory is assumed. A reader should be familiar with the language of set theory and should have enough background in basic mathematics courses emphasizing on proof techniques such as Abstract Algebra, Linear Algebra and/or Number Theory. These experiences will enable the readers to well understand throughout the subject as well as to be able to initiate their own proof technique. This report is also purposed to facilitate as a material for undergraduate and graduate students, and young mathematicians who want to pursue knowledge along the line of graph theory emphasizing on interpolation theorems for graph parameters and extremal problems in graph theory. It covers all necessary basic background in graph theory as well as all necessary basic related results that will be used.

In recent years I have been particularly interested in properties of the graph of realizations. This concept was introduced and developed in a joint paper by Professor Derek Holton and Professor Roger B. Eggleton published in the Springer Lecture Notes in Mathematics, **748** (1979), 1-10. It provides a structured way to examine all the graphs which "realize" a given degree sequence. As a very simple example, the degree sequence $(2, 2, 2, 2, 2, 2)$ can be realized as $A$, one cycle of 6 vertices, and also as $B$, two disjoint cycles each of 3 ver-

tices. One can pass from either of these realizations to the other by a simple switching, an operation which replaces two edges between four vertices by a pair of "cross" edges between the same set of four vertices. Thus the graph of realizations of the degree sequence $(2,2,2,2,2,2)$ comprises two vertices of $A$ and $B$, with an edge connecting them. In general, the graph of realizations of a given degree sequence $\mathbf{d}$ has all the realizations of $\mathbf{d}$ as its vertices, and has an edge between any pair that are related by being able to be transformed into each other by a single switching. As shown in the 1979 paper, a fundamental property of the graph of realizations of any degree sequence is that it is always connected. This means that if one begins with any realization of a degree sequence $\mathbf{d}$, every possible realization can be produced from it by a suitable sequence of switchings. I have conducted several interesting investigations into properties of the graph of realizations for certain important classes of degree sequences $\mathbf{d}$ (notably regular sequences, in which all terms are equal), and also for important subgraphs of the graph of realizations (notably the connected realizations). I have particularly studied the graph of realizations in relation to those properties of graphs which can be described by some integer (the integer-valued graph parameters). I have proved some fundamental interpolation theorems for various important parameters $\pi$ (such as the chromatic number of a graph, the clique number of graph, the matching number of graph, the decycling number of graph, etc.), showing that if a degree sequence $\mathbf{d}$ has two realizations $A$ and $B$ with $\pi(A) = m$ and $\pi(B) = n$, then for any integer $r$ between $m$ and $n$ there is a realization $X$ of $\mathbf{d}$ with $\pi(X) = r$. Moreover, I proved some interpolation theorems in which $A$ and $B$ are restricted to be of some particular type of graphs, and $X$ is of the same type. The references at the end of this report lists papers by some 34 graph theorists spanning the last 60 years, among them some of the most famous in the field. Many of the papers have been published in the last 20 years. Thus, the area of Interpolation Theorems for Graph Parameters and Extremal Problems in Graph Theory have attracted a good number of the world's top graph theorists, and it is an active area of current research interest.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by **Chartrand** during a conference held at **Kalamazoo** in 1980. He posed the following question: If a graph $G$ contains spanning trees having $m$ and $n$ end-vertices, with $m < n$, does $G$ contain a spanning tree with $k$ end-vertices for every integer $k$ with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph.

I introduce a simple but useful concept of graph transformation called **adding edge transformation** and this leads to the definition of **$(\Delta, n)$-graph**. A $(\mathbf{\Delta}, \mathbf{n})$-graph is a graph in which the vertex set is the set of all graphs of order $n$ and of maximum degree $\Delta$; two vertices are adjacent if one can be obtained from the other by adding or deleting an edge. I have used this transformation

as a tool in proving several extremal results on graph parameters.

The definitions of **graph transformation graph invariant** and **graph parameter** are given in Section 2.1. A review of several interpolation results on graph parameters are given in Section 2.2.

**Extremal problems** in graph theory is a huge area in graph theory. It is introduced in Section 2.3 by starting with question and elementary examples in this direction. The distinction between the extremal problem and the optimization problem are also given. Graph theorists have agreed that the extremal problem in graph theory was originated after **Turán** (1910-1976) answered a question concerning the maximum number of edges in a $K_{r+1}$-free graph of order $n$.

Chapter 3 contains all results that I have done in this direction.

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# Chapter 1

# INTRODUCTION

This chapter is to give a brief history and introduction to the readers in order to learn something about how graph theory was developed and understand how we use graphs to model real world problems. We will see that many real world situations can be conveniently described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For examples, the points could represent people, with lines joining pairs of friends; or the points might be communication centers, with lines representing communication links.

## 1.1 Graphs and Graph Models

We know that various pure mathematical concepts, such as groups, rings, fields, vector spaces, topology, measured theory, etc. were derived by abstracting special properties of some familiar objects like real number system. Similarly, the generic concept of graphs also arose by abstracting some properties common to the diagrams representing various physical system like the molecular structure of chemical compound, the interconnections in an electrical network, a railway network or road network or interrelations among a set of people. What is common to these and several similar systems is that each of them consists of a number parts or components interlinked or interrelated in specific ways. Thus **graph** arose as diagrams representing specific relations among a set of objects. However, various different diagrams may be drawn to represent the same relation among the set of objects.

**Königsberg** is a city which was the capital of East Prussia but now is known as **Kaliningrad** in Russia. The city is built around the River **Pregel** where it joins another river. An island named **Kniephof** is in the middle of where the two rivers join. There are seven bridges that join the different parts of the city on both sides of the rivers and the island. People tried to find a way to walk all seven bridges without crossing a bridge twice, but no one could find a way to do it. The problem came to the attention of a Swiss mathematician named **Leonhard Euler** (pronounced "oiler"). In 1735, Euler presented the solution to the problem before the Russian Academy. He explained why crossing all

seven bridges without crossing a bridge twice was impossible. While solving this problem, he developed a new mathematics field called **graph theory**, which later served as the basis for another mathematical field called **topology**. Euler simplified the bridge problem by representing each land mass as a point and each bridge as a line. He reasoned that anyone standing on land would have to have a way to get on and off. Thus each land mass would need an even number of bridges. But in Königsberg, each land mass had an odd number of bridges. This was why all seven bridges could not be crossed without crossing one more than once. The problem could have been solved if ONE bridge was removed or added. Which bridge would you remove? Where could you add a bridge?

That is the history of the original of a new branch of Mathematics called **Graph Theory**. As we have agreed that Graph Theory was originated from the city of Königsberg, it is coincident that later there were two great mathematicians named **König** and **Berge** who have significantly contributed in this area. **Denes König** was born in Budapest, Hungary on 1884. He has been influenced by Minkowski's lectures on the four color problem. These lectures contributed to his growing interest in Graph Theory. König's work on the factorization of bipartite graphs related closely to the marriage problem of Hall. König died in Budapest on 1944. **Claude Berge** was born in Paris, France in 1926. He wrote five books. Each book, originally written in French, was a classic, and was quickly translated into English and half a dozen other languages. His work emphasized on perfect graphs and hypergraphs. He passed away on 30 June 2002 at the age of 76.

**König + Berge = Königsberg**

From almost 300-year history, graph theory has played a crucial role considerable success in problem solving and has developed into an elegant and rapidly growing branch of mathematics. In particular, it has been grown for the past forty years by the number of books and the numerous research papers that have been published in this area. A major reason for this rapidly growth is its applicability to a wide range of problems arising in modern science, engineering, computer science, modern business and many more. Examples include the following:

- Determining of production schedules and workplace configurations (plant layout) to maximize efficient production.
- The design and operation of complex networks such as: transportation networks for the movement of commodities; communication networks for the transmission of information; data communications networks capable of supporting large scale on-line applications; powerful multiprocessor systems for solving complex problems such as radar signal processing; and many more.
- Identification of complicated organic chemicals.
- The design and operation of a large open pit mine, including the determi-

nation of an extraction sequence which maximizes the total net profit.

- Determining of schedules such as: class timetables; bus/train timetables; rounds in sport competition.
- Design of experiments to accurately test: the quality of products; the effectiveness of particular drugs to cure a disease; public opinion to a particular issue; and many more. Usually the idea is to eliminate bias.
- The efficient allocation of containers for the storage of chemicals.
- The efficient design of Very Large Integrated Circuits.
- Determining an optimal set of tours for a delivery vehicle.
- The optimal allocation of security passwords.

A **graph** may be considered to be a collection of points called **vertices** connected by lines called **edges**. This very simple structure makes a graph suitable for developing structural models for a variety of situations. Indeed any system which can be viewed as a collection of objects some of which are related in some way or other can be modelled as a graph in which the vertices represent the objects and the edges represent the relationship between the objects. In many applications one encounters such systems. Formally, a **graph** $G$ consists of a finite nonempty set $V$ of objects called **vertices** (the singular is **vertex**) and a set $E$ of 2-element subsets of $V$ called **edges**. The set $V$ and $E$ are the **vertex set** and **edge set** of $G$, respectively. So a graph $G$ is an order pair of two sets $V$ and $E$. For this reason, some write $G = (V, E)$. At times, it is useful to write $V(G)$ and $E(G)$ rather than $V$ and $E$ to emphasize that the vertex and edge sets of a particular graph $G$.

The following examples show how we use graphs to model real systems.

- A chemical structure is a collection of atoms some of which are boned.
- A communication network is a collection of centers some of which can communicate directly.
- A traffic network is a collection of roads, some of which are intersect. Such a network can be modelled by a graph with vertices representing intersections and the edges representing direct traffic links. The edges can be assigned a direction to indicate the direction of traffic flow.
- A printed circuit board is a collection of components, some of which are connected by a conductor path.
- An assembly production line is a collection of workstations organized in some order.
- A multiprocessor is a collection of processors and memory modules linked through an interconnection network.

The above are just a few examples of the application of graphs to model real systems. In each case, the graph provides a structural model of the system being studied, displaying clearly the interactions and relationships a model

provides very often the study of the underlying graph theory provides tools for the analysis of the system becomes one of finding: An arrangement of the system which satisfies certain requirements, and which is optimal according to some criterion such as cost, output or performance. The basic problems which arise in graph theory applications can be classified as:

**1. The Existence Problem**

Is there at least one arrangement of a particular type?

**2. The Counting Problem**

How many arrangements of a particular type are there?

**3. The Optimization Problem**

Concerned with choosing from among all possible arrangements of a certain type, that which is best according to some criterion.
We illustrate these problems with an example.

**Example 1.1.1** An Allocation Problem
Consider the problem of allocating mobile frequencies to transmitting stations. Suppose there are $m$ possible channels (frequencies) available for use by $n$ transmitting stations. Stations that are located "close" to each other cannot use the same channel without causing interference. Thus given two stations it may or may not be the case that they use the same channel. A number of problems arise.

**Existence problem**

It is possible to allocate a channel to each station in such a way that any two stations which need to have different channels get different channels? If so, how does one find such an allocation?

**Enumeration problem**

How many "proper" allocations are there?

**Optimization problem**

What is the minimum number of channels that will suffice to satisfy the allocation conditions. How can one model this situation? A graph model can be developed as follows: Represent each station by a vertex. Join two vertices by an edge if and only if the corresponding stations cannot use the same channel. Note that here the relationship between the vertices is "conflict" with respect to the same frequency. As an actual example, consider the case when there are 5 stations and the information is represented the following **"conflict matrix"**.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | – | 1 | 0 | 0 | 1 |
| 2 | 1 | – | 1 | 1 | 0 |
| 3 | 0 | 1 | – | 1 | 1 |
| 4 | 0 | 1 | 1 | – | 1 |
| 5 | 1 | 0 | 1 | 1 | – |

Note that a "1" represents a conflict (i.e. the stations cannot be assigned the same channel), and a "0" represents no conflict. The corresponding graph is drawn in Figure 1.

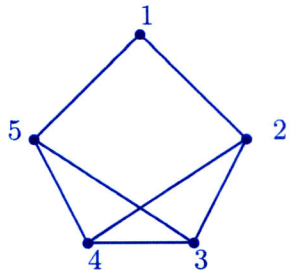



**Figure 1**

Representing the conflict matrix by a graph does not of course, resolve any of the problems mentioned above. However, the visual aspects of the model does provide us with some insight. For instance, the presence of triangles in the graph implies that we need at least three different channels to satisfy the restrictions imposed by nearby stations. Can we make the allocations if we have just three channels? How do we resolve the existence problem when we have $m$ channels? An allocation is possible only if we can partition the vertices of our graph into $m$ sets in such a way that no two vertices in the same set are joined by an edge. This leads us to the concept of coloring. A graph is **$k$-colorable** if its vertices can be colored, using no more than $k$ colors, in such a way that no two "adjacent" vertices are colored alike. For example, the graphs $G_i$, $i = 1, 2, 3, 4$ displayed in Figure 2 are $i$-colorable.

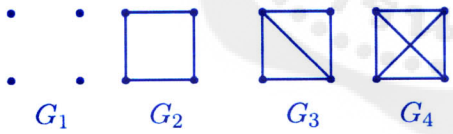



**Figure 2**

Note that each graph is $k$-colorable for any $k \geq 4$. Also note that there is no stipulation that all $k$-colors are used. A **$k$-coloring** of a graph is just a partitioning of its vertices into $k$ disjoint sets so that two vertices in the same set are colored stations get different channels if and only if the corresponding graph is $m$-colorable (each color corresponds to a channel). In a specific application, we might be interested in determining the minimum value of $m$ for which such a coloring exists. This minimum is called **the chromatic number** of the graph.

The above example demonstrates how a given problem can be formulated and solved using graph theory concepts. By studying colorings in graphs one can devise algorithms for testing whether or not a given graph is $m$-colorable.

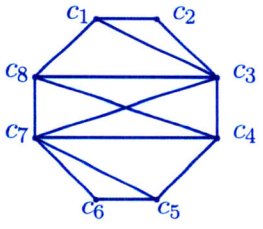
There are many applications, particularly in the area of scheduling, where the problem that arises is one of partitioning the vertex set of an associated graph so that adjacent vertices belong to different sets of the partition.

**Example 1.1.2** Construction of Timetables Consider the problem of constructing the weekly schedule of lectures at a university. We have a list $L$ of lectures to be given in the various subjects. Certain of these lectures must not be given at the same time as certain others, since there are students who wish to attend both. Therefore we have also a list $C$ of pairs of lectures which must not clash. If we label the list of lectures $L = \{1, 2, \ldots, n\}$, then $C$ could be represented by an $n \times n$ conflict matrix with the $(i,\ j)$-element 1 if lectures $i$ and $j$ clash; 0 otherwise. Consequently, we can model this situation with a graph in which the vertices represent the lectures and the edges represent the conflicts. There will be a set of time slots $T = \{1, 2, \ldots, m\}$ during the week, usually 7 or 8 per day, when lectures may be given. If we take these as our colors, then the problem of constructing a weekly corresponds to finding an $m$-coloring of the corresponding graph. Lectures corresponding to vertices allocated a certain color will be scheduled in the period corresponding to that color. Thus the problem is exactly the same as that in the preceding example.

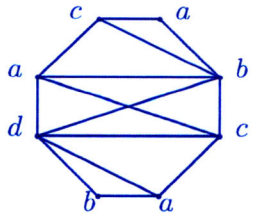
**Remark 1.1.3** In an actual application, the problem would be a little more complicated with additional restrictions. For instance, the individuals giving the lectures need to be taken into account as well as the maximum number of lectures that can be scheduled in the same period (restricted by the available rooms).

**Example 1.1.4** A Storage Problem
A chemical company manufactures $n$ chemicals $C_1, C_2, \ldots, C_n$. These chemicals are stored in containers until required by the customers. Certain pairs of these chemicals are incompatible and would cause dangerous reactions if they were stored together. Thus we have a list of pairs of chemicals which cannot be stored together. Again this information can be represented in a $n \times n$ conflict matrix in which the $(i,\ j)$-entry is 1 if chemicals $C_i$, and $C_j$ are incompatible; and 0 otherwise. This in our graph model the vertices represent the chemicals and the edges the conflicts.

The problems faced by the chemical company is to determine the minimum number of containers necessary for the safe storage of the chemicals. Thus we need to determine the chromatic number of the corresponding graphs.
As an example, consider the case when $n = 8$ and the graph is:



Since the four vertices $c_3, c_4, c_7, c_8$ are pairwise joined, we need at least 4 colors to color the graph. A 4-coloring with colors $a, b, c, d$ is:




Thus the chromatic number, and hence the minimum number of containers is 4.

**Example 1.1.5** Very Large Integrated Circuit Design

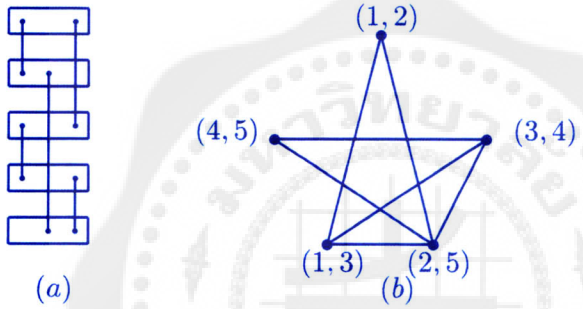



**Figure 3**

Consider a simplified circuit; consisting of a row of gates $g_1, g_2, \ldots, g_n$ and specified connections between various pairs of these gates. The connections are accomplished through conductor paths or tracks with "overlapping connections" on different tracks. Consider the simple case exhibited in Figure 3(a). Here we have 5 gates and 3 tracks. Track 1 has connections $(g_1, g_2)$ and $(g_3, g_4)$; Track 2 has connections $(g_2, g_5)$ and track 3 has connections $(g_1, g_3)$ and $(g_4, g_5)$. We can form a graph in which the connections are the vertices and the edges correspond to "overlaps"; this is called the interval graph and is displayed in Figure 3(b). In the above we have made the connections on three tracks.

$$(g_1,\ g_2),\ (g_3,\ g_4),\ (g_2,\ g_5),\ (g_1,\ g_3) \text{ and } (g_4,\ g_5).$$

Is this the minimum number of tracks? Now the graphs clearly exhibits the overlap relations; an edge between two vertices means that the corresponding connections must be on different tracks. Thus the problem of determining the minimum number of tracks is equivalent to determining the number of the associated graph. In this case the graph is 3-colorable but cannot be colored with fewer colors. So 3 tracks is indeed minimum.

**Example 1.1.6** Scheduling a Tournament

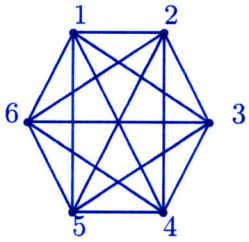



**Figure 4**

Consider a competition in which there are $n$ teams $T_1, T_2, \ldots, T_n$ each pair of which meets exactly once in a round robin tournament. The matches are played once per week with each team playing at most one match per week. The problem is to schedule the weekly matches, using as few weeks as possible, for the round. We can represent this situation by a graph in which the vertices correspond to the teams and the edges to the matches. Consider the simple case of $n = 6$; the corresponding graph is displayed in Figure 4 with team $T_i$ labelled $i$.

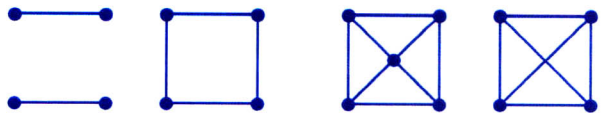
As no team can play more than one match per week, a minimum of 5 weeks is required to schedule the tournament. One possible schedule is:

| Week | Schedule |
|---|---|
| 1 | $(1,2),(3,5),(4,6)$ |
| 2 | $(1,3),(2,6),(5,4)$ |
| 3 | $(1,4),(2,3),(6,5)$ |
| 4 | $(1,5),(2,4),(3,6)$ |
| 5 | $(1,6),(2,5),(4,3)$ |

Observe that the graph corresponding to each weekly set of matches consists of 3 edges with no two sharing a common vertex. Thus, in effect, we have partitioned the edge set of the graph into 5 sets, each set consisting of three edges with distinct ends. Now if we color the edges in the set corresponding to Week $i$ by Color $i$, we have a coloring of the edges of the graph, using 5 colors, such that no edges sharing a common vertex are colored alike. This leads us to the concept of edge-colorings.

A graph is **$k$-edge-colorable** if its edges can be colored, using no more than

$k$ colors, in such a way that no two "adjacent" edges are colored alike. For example, the graphs $G_i$, $i = 1, 2, 3, 4$ displayed in Figure 5 are $i$-edge-colorable.



$G_1$ $G_2$ $G_3$ $G_4$

**Figure 5**

So a $k$-edge-coloring of a graph $G$ is a partitioning of its edges into $k$ sets (some possibly empty) so that no two edges in the same set are adjacent. The minimum $k$ for which a $k$-edge-coloring exists is called **the edge-chromatic number** of $G$. Like vertex colorings, edge colorings have many applications.

**Example 1.1.7** A Timetabling Problem

Consider the following class timetabling problem. There are $m$ lecturers $L_1, L_2, \ldots, L_m$ and $n$ courses $C_1, C_2, \ldots, C_n$. Given that lecturer $L_i$ is required to teach class $C_j$, $P_{ij}$ periods per week, the problem is to schedule a complete timetable using as few periods as possible. We can represent the above situation as graph in which the vertices correspond to the lecturers and courses. Vertices $L_i$ and $C_j$ are joined by $P_{ij}$ edges. For example, when the teaching requirements are given by the following matrix:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_1$ | 3 | 1 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 0 |
| $L_2$ | 0 | 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $L_3$ | 0 | 0 | 0 | 1 | 1 | 1 | 3 | 0 | 0 | 0 |
| $L_4$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 3 | 0 | 0 |
| $L_5$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 | 0 |
| $L_6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 4 |

Since, in any one period, each lecturer can teach at most one class, an each class can be taught by at most one lecturer, the problem of scheduling classes using as few periods as possible is equivalent the edge-chromatic number of the corresponding graph.

For our example, since the largest number of edges incident to any vertex is 8 (lecturer 1), at least 8 periods are required. Is this possible?

## 1.2 Notation, Terminology and Basic Results

We have already introduced graphs and graph models in Section 1.1. We shall provide more notation and terminology in this section.

A **graph** $G$ consists of a set $V(G)$ of vertices, a set $E(G)$ of edges, and an incidence relation which associates with each edge of $G$ an unordered pair of (not necessarily distinct) vertices of $G$ called its ends. If $e$ is an edge of $G$ with ends $u$ and $v$, then $e$ is said to join $u$ and $v$. An edge with identical ends is called a **loop**. We can represent a graph by a diagram in which the vertices are points and edges are line segments. Thus, in general, there is no unique way of drawing a graph. Two vertices which are joined by an edge are said to be **adjacent**. If more than one edge joins the same pair of vertices, we say the graph has **multiple edges**. A graph is **simple** if it has no loops or multiple edges. A simple graph in which each pair of distinct vertices is joined by an edge is called a **complete graph**. A complete graph on $n$ vertices is denoted by $K_n$. Note that $K_n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

Let $G = (V, E)$ be a simple graph and $X \subseteq V$. We define a graph $G - X$ to be the graph obtained from $G$ by deleting all vertices in $X$ and all edges incident with them. If $X = \{v\}$, we simply write $G - v$ for $G - \{v\}$. If $F \subseteq E$, we can analogously define a graph $G - F$ as the graph obtained from $G$ by removing all edges in $F$. Also we simply write $G - e$ for $G - \{e\}$. Furthermore, if $e \notin E$, we write $G + e$ for the graph obtained from $G$ and adding the new edge $e$.

There are considerable variations in graph theory notation and terminology used in the literature, we present, in this section, basic notation and terminology that will be used. For most part, our graph theoretic notation and terminology can be found in the textbooks of **Bondy and Murty** [4] and **West** [40].

Since we deal only with finite graphs, we set up the following notation and terminology for a typical graph $G$. For a graph $G = (V, E)$, we may write $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Denoting by $|S|$ the cardinality of a set $S$ we define $n = |V|$ the **order** of $G$ and $m = |E|$ the **size** of $G$. If an edge $e$ corresponds to the vertex pair $\{u, v\}$, to simplify writing, we will write $e = uv$ and we say that the edge $e$ **joins** the vertices $u$ and $v$. For each $v \in V$, a vertex $u \in V$ is called a **neighbor** of $v$ if $vu \in E$. The **neighborhood** of $v \in V$ in $G$, denoted by $N_G(v)$, is defined by

$$N_G(v) = \{x \in V(G) : xv \in E(G)\}.$$

If $S \subseteq V$, we denote $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_S(v) = N_G(v) \cap S$. The idea of **sameness** or **identical** of the graphs is formalized by the following definition.

Two simple graphs $G = (V, E)$ and $H = (U, F)$ are **isomorphic** if and only if there is a bijection $\phi : V \to U$ such that for every pair $u, v$ in $V$ we have $uv \in E$ if and only if $\phi(u)\phi(v) \in F$. The function $\phi$ is called an **isomorphism** from $G$ onto $H$ and $\phi^{-1}$ is also an isomorphism from $H$ onto $G$. We use the notation $G \cong H$ to denote that $G$ is isomorphic to $H$. It is clear that the relation " $\cong$ " is an equivalence relation on the set of all simple graphs. Consequently, it partitions the set of all graphs into equivalence classes each of which consists of all graphs having the same **algebraic properties**. The meaning of what we call algebraic property can be defined in the following definition.

Let $\mathbb{G}$ be the class of simple graphs and let $X$ be any set. A function $\pi : \mathbb{G} \to X$ is called an **invariant** if $\pi(G) = \pi(H)$ for all isomorphic graphs $G$ and $H$. When $X = \mathbb{Z}$, a graph invariant is called a **graph parameter**. If $X$ is the set of all integer sequences, then $\pi(G) = \mathbf{d}(G)$, where $\mathbf{d}(G)$ is the degree sequence of $G$, is an example of invariants. If $X$ is the category of all groups, then the function $\pi(G) = \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the group of automorphisms of $G$, is a group invariant of graphs. An example of an integer invariant of graphs defined algebraically is the function $\pi(G) = \det A(G)$ where $A(G)$ is the adjacency matrix of $G$.

It would be worthwhile to observe that **Mathematical structures** usually come together with structure preserving maps, for examples, groups and isomorphisms, vector spaces and invertible linear transformations, topological spaces and open maps or continuous maps, Hilbert spaces and linear continuous operators, etc. Structures of graphs are obviously derived by graph isomorphisms.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (U, F)$ with $U \subseteq V$ and $F \subseteq E$. Since $H$ is a graph, the end-vertices of every edge of $F$ will also be in $H$. When $U = V$, we say $H$ is a **spanning subgraph** of $G$. When $F$ consists of all edges of $G$ joining pairs of vertices of $U$, we say $H$ is an **induced subgraph** of $G$ and write $H = G[U]$.

A graph $G = (V, E)$ is said to **$r$-partite** (where $r$ is a positive integer) if its vertex set can be partitioned as $V_1 \cup V_2 \cup \ldots \cup V_r$ such that $G[V_i]$ contains no edge for all $i = 1, 2, \ldots, r$. A 2-partite graph is usually referred as a **bipartite graph.** A **complete bipartite graph** is a simple bipartite graph with partition $(X, Y)$ in which each vertex of $X$ is joined to each vertex of $Y$. $K_{m,\,n}$ denotes a complete bipartite graph with $|X| = m$ and $|Y| = n$. The graph $K_{1,\,n}$ is called a **star**.

**Remark 1.2.1** A bipartite graph cannot have loops. Of course a bipartite graph need not be simple as it may have multiple edges. Observe that if a graph $G$ is 2-colorable, then it has a bipartition $(X, Y)$ with the vertices of $X$ colored in one color and the vertices of $Y$ colored in the other color. Conversely, If $G$ is a bipartite graph then it is 2-colorable. Thus we have the characterization that a graph $G$ is bipartite if and only if it is 2-colorable.

For any positive integer $n$ and $V = \{v_1, v_2, \ldots, v_n\}$, up to isomorphism, we will denote by $K_n$ the simple graph with vertex set $V$ and every pair of distinct vertices is joined by an edge, and call it **the complete graph of order** $n$. The **complement** $\overline{G} = (V, \overline{E})$ of a simple graph $G = (V, E)$ of order $n$ has the same vertex set as $G$ and its edge set is the set complement $\overline{E} = \{e \in E(K_n) : e \notin E\}$ that is $uv$ is an edge in $\overline{G}$ if and only if $uv$ is not an edge in $G$. It is clear that the complement $\overline{K}_n$ of $K_n$ is the empty graph, i.e., the graph on the vertex set $V$ containing no edge. A graph $G$ is said to be **self-complementary** if $G \cong \overline{G}$.

A property $P$ is called a **parametric property** for graphs if whenever a graph $G$ has property $P$, then every graph isomorphic to $G$ also has property $P$. In what follows, by a property $P$ of a graph we mean a parametric property. A graph with property $P$ will be referred as a $P$-graph. A subgraph $H$ of a graph $G$ is said to be **maximal** with respect to a property $P$ (or $P$-maximal for short) if $H$ is a $P$-graph and there is no subgraph $K$ of $G$ which properly contains $H$ and has property $P$. For example, $K_{m,n}$ is a maximal bipartite graph.

A subgraph $H$ of a graph $G$ is said to be **minimal** with respect to a property $P$ (or $P$-minimal for short) if $H$ is a $P$-graph and there is no proper subgraph $K$ of $H$ which is also a $P$-graph. For example, a tree on $n$ vertices is a minimal connected graph.

We now define a number of concept arising from the adjacency and incidence relations in a graph, leading to the concept of a connected graph.

A sequence $v_1e_1v_2e_2v_3 \cdots v_{k-1}e_{k-1}v_k$ with elements alternately from the vertex set $V$ and the edge set $E$ of a graph $G$, such that $e_i = v_iv_{i+1}$ for $1 \leq i \leq k-1$ is called a $v_1 - v_k$ **walk** $W$ in $G$; $v_1$ and $v_k$ are called the **initial** and **terminal** vertices of the walk $W$, and $k$, the number of vertices in $W$, is called the order of $W$. The walk is said to be **open** if $v_1$ and $v_k$ are distinct vertices, and closed if $v_1 = v_k$. A **trail (edge-trail)** in a graph is a walk in which no edge is repeated. A **tour** is a closed trail. A **path** in a graph is an open walk in which no vertex (and therefore no edge) is repeated. A closed walk in which no vertex (and edge) is repeated is called a **cycle.** A path of order $n$ is called an $n$-path and is denoted by $P_n$. A cycle of order $n$ is called an $n$-cycle and is denoted by $C_n$. Two distinct vertices $u$ and $v$ of a graph $G$ are said to be **connected** if there is a $u - v$ walk in $G$. By convention, a vertex is connected to itself. A graph is said to be connected if for every two of its vertices are connected; otherwise it is **disconnected.**

Let $k$ be a positive integer. A graph $G$ is said to be **$k$-connected** if for any subset $X$ of cardinality $k-1$ of $V(G)$, $G - X$ is connected. A maximal connected subgraph of a graph $G$ is called a **component** of $G$.

A graph is **acyclic** if it has no cycle as a subgraph. A **forest** is an acyclic

graph. A connected acyclic graph is called a **tree.** A connected acyclic spanning subgraph of $G$ is called a **spanning tree** of $G$.

The **degree** of a vertex $v$ of a graph $G$ is the number of edges of $G$ which are incident with $v$. In symbol

$$d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|.$$

The **minimum degree** and the **maximum degree** of a graph $G$ are usually denoted by the special symbols $\delta(G)$ and $\Delta(G)$ respectively.

A simple graph is said to be **$r$-regular** if all of its vertices have degree $r$. A 3-regular graph is usually called a **cubic graph**. A vertex with degree zero is called an **isolated vertex**. An $r$-regular spanning subgraph of a simple graph $G$ is called an $r$-factor of $G$. A vertex of odd degree is called an **odd vertex** and a vertex of even degree is called an **even vertex.** The following theorem is known as **The First Theorem of Graph Theory.**

**Theorem 1.2.2** *If $G$ is a graph of size $m$, then*

$$\sum_{v \in V(G)} d_G(v) = 2m.$$

**Corollary 1.2.3** *Every graph has an even number of odd vertices.*

An interesting question in graph theory is:

To what extent do the degrees of the vertices of a graph determine the structure of a graph? We now consider this question.

**Theorem 1.2.4** *A sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is a degree sequence of some graph if and only if $\sum_{i=1}^{n} d_i$ is even.*

**Proof.** We must prove two facts: (i) If $(d_1, d_2, \ldots d_n)$ is a degree sequence of some graph, then $\sum_{i=1}^{n} d_i$ is even ("sufficiency"). (ii) If $(d_1, d_2, \ldots d_n)$ is a sequence of non-negative integers such that $\sum_{i=1}^{n} d_i$ is even, then there is a graph whose degree sequence is $(d_1, d_2, \ldots d_n)$ ("Necessity"). Fact (i) follows immediately from Theorem 1. To prove (ii) we construct a graph $G$ with degree sequence $(d_1, d_2, \ldots, d_n)$ as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since $\sum_{i=1}^{n}(d_i)$ is even, the number of $d_i$ which are odd is even. Pair $i$ for those $d_i$ and join each pair by an edge. Now add $\frac{d_i - 1}{2}$ loops at each vertex $v_i$. The resulting graph has degree sequence $(d_1, d_2, \ldots, d_n)$.

**Example 1.2.5** It is easy to construct a graph with the degree sequence $(4, 1, 3, 6, 5, 1)$. But a graph with this degree sequence is not simple.

What happens, if we insist that the graph be simple? A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is **graphic** if there is a simple graph with degree sequence $\mathbf{d}$.

**Example 1.2.6** Are the following sequences graphic?
(i) $(4,1,3,6,5,1)$,
(ii) $(6,6,5,5,4,3,1)$.

**Solution** (i) No, since the maximum degree in a graph on 6 vertices is 5.

(ii) Suppose the sequence is graphic and let $G$ be a graph with such a sequence. Let $v_1$ and $v_2$ denote the two vertices of $G$ of degree 6. Since $G$ has 7 vertices, $v_1$ and $v_2$ are adjacent. Further, $v_1$ and $v_2$ must be adjacent to every other vertex of $G$ implying that $\delta(G) \geq 2$. Hence sequence is not graphic.

If we concentrate in a simple graph, a sequence of non-increasing non-negative integers $(d_1, d_2, \ldots, d_n)$ is graphic, it is necessary that $d_1 \leq n-1$ and $\sum_{i=1}^{n} d_i$ is even. Those conditions, in general, are not sufficient. We will show that for any positive integers $r$ and $n$, there exists an $r$-regular graph of order $n$ if and only if $nr \equiv 0 (\mathrm{mod}\ 2)$ and $r \leq n-1$. For general sequence of non-negative integers **Havel** [21] and **Hakimi** [12] independently obtained a recursive necessary and sufficient conditions for the sequence, in terms of laying of a largest integer in the sequence.

**Theorem 1.2.7** *Let* $\mathbf{d} = r^n = (r, r, \ldots, r)$ *be a constant sequence of length* $n$. *Then* $\mathbf{d}$ *is a degree sequence of an* $r$*-regular graph of order* $n$ *if and only if* $r \leq n-1$ *and* $rn$ *is even.*

**Proof.** If $\mathbf{d} = r^n$ is a degree sequence of an $r$-regular graph $G$ of order $n$, then, by Theorem 1, $rn$ is even and $G$ is simple, the degree $r$ cannot be greater than $n-1$. Conversely, suppose $rn$ is even and $r \leq n-1$, the following two cases will be considered.

**Case 1** $r$ is even. Consider a regular polygon with $n$ vertices. Join by an edge neighboring vertices which are two apart, ..., vertices which are $\frac{r}{2}$ apart. Since $r < n$, it follows that $\frac{r}{2} < \frac{n}{2}$, and thus by construction each vertex has degree $r$ and there are no multiple edges.

**Case 2** $r$ is odd. Thus $n$ is even. The $(r-1)$-regular graph of order $n$ can be constructed as in Case 1. Since $r < n$, $\frac{r-1}{2} \leq \frac{n-2}{2} = \frac{n}{2} - 1$, and thus the endpoints of the longest diagonals of the polygon are not joined by an edge. Thus by joining diametrical vertices of the polygon by an edge, each vertex will have degree $r$, and hence an $r$-regular graph with $n$ vertices is obtained.

The following result is important in that it provides us with a mechanism for determining whether or not a given set of non-negative integers is graphic.

**Theorem 1.2.8** *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers, and denote the sequence*

$$\mathbf{d}' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

*Then* $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

**Proof.** Suppose $\mathbf{d}'$ is graphic. Then, by definition, there exists a simple graph $G'$ with $V(G') = \{v_2, v_3, \ldots, v_n\}$ such that

$$d_{G'}(v_i) = \begin{cases} d_i - 1 & \text{if } 2 \leq d_i \leq d_1 + 1, \\ d_i & \text{if } d_1 + 2 \leq i \leq n. \end{cases}$$

A new graph $G$ can be constructed by adjoining a new vertex, $v_1$ say, to $G'$ and joining $v_1$ to $v_2, v_3, \ldots, v_{d_1+1}$. Clearly $G$ is simple and has a degree sequence $\mathbf{d}$. Hence $\mathbf{d}$ is also graphic. Conversely, let $\mathbf{d}$ be a graphic sequence. Thus a graph $G$ exists with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_i) = d_i$ for $1 \leq i \leq n$. We consider two cases.

**Case 1** $v_1$ is adjacent to the vertices $v_2, v_3, \ldots, v_{d_1+1}$. Then the graph $G - v_1$ has degree sequence $\mathbf{d}'$ and hence $\mathbf{d}'$ is graphic.

**Case 2** $v_1$ is not adjacent to all of the vertices $v_2, v_3, \ldots, v_{d_1+1}$. Then there must be vertices $v_j$ and $v_k$ with $j < k$ such that $v_1$ is adjacent to $v_k$ but not to $v_j$. We may take $d_j > d_k$ since if $d_j = d_k$ we can interchange $v_j$ and $v_k$. Hence there must be a vertex, $v_m$ say, which is adjacent to $v_i$ but not to $v_k$. Let $G'$ be the graph obtained from $G$ by deleting the edges $v_1v_k$ and $v_mv_j$, and adding the edges $v_1v_j$, and $v_kv_m$.

Clearly $G'$ has degree sequence $\mathbf{d}$. However, in $G'$ the sum of the degrees of the vertices adjacent with $v_1$ is larger than in $G$. Consequently, if we continue this procedure we end up with a graph with degree sequence $\mathbf{d}$ in which $v_1$ satisfies the hypothesis of Case (1). Hence $\mathbf{d}$ is graphic.

**Remark 1.2.9** The above proof suggests an algorithm for testing whether or not a given sequence is graphic. We illustrate this in the following example.

**Example 1.2.10** Determine whether or not the following sequences are graphic. If so, construct a graph with the appropriate degree sequence.

(i) $\mathbf{d} = (4, 4, 3, 2, 1, 0)$.

(ii) $\mathbf{d} = (5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0)$.

**Solution** (i) $(4, 4, 3, 2, 1, 0)$ is graphic iff $(3, 2, 1, 0, 0)$ is graphic. $(3, 2, 1, 0, 0)$ is clearly not graphic.

(ii) $(5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0)$ is graphic iff $(2, 2, 2, 2, 1, 2, 2, 1, 1, 1, 0)$ is graphic. Rearranging $(2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 0)$ is graphic iff $(1, 1, 2, 2, 2, 1, 1, 1, 1, 0)$ is graphic. Rearranging $(2, 2, 2, 1, 1, 1, 1, 1, 1, 0)$ is graphic iff $(1, 1, 1, 1, 1, 1, 1, 1, 0)$ is graphic.

Observe that there is a unique graph with degree sequence $(1, 1, 1, 1, 1, 1, 1, 1, 0)$ but there may be more graphs with the original degree sequence $(5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0)$.

**Theorem 1.2.11** *If $G$ is a bipartite graph, then it contains no cycle of odd order.*

**Proof.** Let $C = v_1, v_2, \ldots, v_k, v_1$ be a cycle of $G$. Since $G$ is bipartite it must be 2-colorable. So the vertices of $C$ can be colored using two colors, in such a way that alternate vertices are colored differently. Clearly in any 2-coloring of $C$ the vertices $v_1, v_3, v_5, \ldots, v_{k-1}$ are assigned one color whilst the vertices $v_2, v_4, \ldots, v_k$ are assigned a different color. The only possibility is for $k$ to be an even integer, and hence $C$ is a cycle of even order.

**Remark 1.2.12** It is also true that a graph with no cycles of odd length is bipartite. So a graph is bipartite if and only if it has no cycle of odd length.

If two vertices $u$ and $v$ are connected in a graph $G$, the **distance** $d(u, v)$ between $u$ and $v$ is the length of the shortest $(u, v)$-path in $G$. When there is no path connecting $u$ and $v$ in $G$ we define $d(u, v)$ to be infinite. The **diameter** of a graph $G$, denoted $\text{diam}(G)$, is defined as $\text{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}$. If $\text{diam}(G) = k$, then any two vertices of $G$ are connected by a path of length at most $k$ and for at least one pair of vertices the path length is exactly $k$. In network design the diameter of a graph is an important parameter as it corresponds to the maximum number of links over which a message between two centers in the network must travel. Thus it is a useful measure of efficiency and reliability in networks in which the time delay or signal degradation is approximately proportional to the length of the path being used; this is the case, for example, in communication networks.

In many applications, the graph theoretic problem that arise concerns traversing the graph in certain ways. A walk that traverses every edge of $G$ exactly once is called an **Euler walk** of $G$. An **Euler tour** of $G$ is a closed Euler walk. A graph $G$ is **Eulerian** if it contains an Euler tour. Eulerian graphs have many applications. Euler established an important characterization of Eulerian graphs. We begin with the following useful result.

**Theorem 1.2.13** *Let $G$ be a graph with $|E(G)| > 0$, having no vertices of odd degree. Then $G$ contains a closed walk in which no edge is repeated.*

**Proof.** Let $v_0$ be a vertex of $G$ having positive degree. A closed walk beginning at $v_0$ in which no edge is repeated is constructed as follows:

**Step 1**: Choose an edge incident to $v_0$. Move along this edge to its other end, $v_1$ say.

**Step 2**: On arriving at a vertex, select any edge incident with that vertex which has not been transversal and move along this edge to its other end.

**Step 3**: Stop when Step 2 cannot be further implemented.

On arriving at a vertex $v$ different from $v_0$ we can always continue along a new edge, since the degree of $v$ is even and only an odd number of edges incident to $v$ have been used. The walk can therefore finish only at $v_0$, which happens sooner or later. Thus a closed walk in which no edge is repeated is obtained.

**Remark 1.2.14** 1. The closed walk $W$ constructed by the above algorithm will contain every edge incident to $v_0$.

2. If every edge of the walk $W$ constructed above were deleted from the graph $G$, the resulting graph $G'$ will contain no vertices of odd degree. Does this suggest anything?

We are now ready to prove Euler's result.

**Theorem 1.2.15** *A connected graph with at least one edge is Eulerian if and only if it has no vertices of odd degree.*

**Proof.** Let $G$ be Eulerian, and let $W$ be an Euler tour of $G$ with origin $v_0$. Let $v$ be any vertex of $G$ different from $v_0$. Clearly $W$ must enter $v$ as many times as it departs $v$, implying that $d_G(v)$ is even. Therefore $d_G(v)$ is even for every vertex $v \neq v_0$. Since a graph must have an even number of vertices of odd degree, $d_G(v_0)$ must also be even. Thus $G$ has no vertices of odd degree.

Conversely, suppose that $G$ is connected with no vertices of odd degree. Thus $G$ contains a closed walk, $W_0$ say, in which no edge is repeated. Let $v_0$ be the origin of $W_0$. Consider the subgraph $G'$ obtained from $G$ by deleting the edges of the walk $W_0$. $G'$ has no vertices of odd degree. If $G'$ has no edges, then $W_0$ must be an Euler tour, in which case $G$ is Eulerian. If, on the other hand, $G'$ has edges then because $G$ is connected there must at least one vertex, $v_1$ say, of $W_0$ which has an edge of $G'$ incident to it. Therefore $G'$ contains a closed walk $W_1$ with origin $v_1$ in which no edge is repeated. We construct a new walk $W_2$ as follows:
Starting at $v_0$ we move along $W_0$ until we reach $v_i$. At this point we leave $W_0$ and move along $W_1$ and return to $v_1$. We complete the walk by moving along the remaining part of $W_0$. $W_2$ is a closed walk with origin $v_0$ in which no edge is repeated. Note that $W_2$ need not be unique and it has more edges than $W_0$. This process of enlarging closed walks can be continued; ultimately an Euler tour is constructed. This completes the proof.

We now obtain the following corollary.

**Corollary 1.2.16** *A connected graph with at least one edge has an Euler walk if and only if it has at most two vertices of odd degree.*

A path that contains every vertex of $G$ is called a **Hamilton path** of $G$. A cycle that contains every vertex of $G$ is called a **Hamilton cycle**. A graph is **hamiltonian** if it contains a Hamilton cycle.

The following result of **Dirac** (1952) gives a sufficient condition for a graph to be hamiltonian.

**Theorem 1.2.17** *If $G$ is a simple graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then $G$ is hamiltonian.*

**Proof.** Suppose the assertion is false and let $G$ be a maximal non-hamiltonian simple graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Since $K_n$ is hamiltonian, $G$ cannot be complete. Let $u$ and $v$ be two non-adjacent vertices of $G$. By choice of $G$, $G + uv$ is Hamiltonian. Furthermore, since a Hamilton cycle of $G + uv$ must contain the edge $uv$, there is a $(u, v)$-Hamilton path $P = v_1, v_2, \ldots, v_n$, where $v_1 = u$ and $v_n = v$. Let $S = \{v_i : uv_{i+1} \in E(G)\}$, and $T = \{v_i : v_i v \in E(G)\}$. If $v_i \in S \cap T$, then $G$ would have a Hamilton cycle:

$$v_1, v_2, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1} v_1.$$

Therefore $S \cap T = \emptyset$. Further, since $v_n \notin S \cap T$, $|S \cup T| < n$. Now $d_G(u) + d_G(v) = |S| + |T| < n$. This contradicts the fact that $\delta(G) \geq \frac{n}{2}$, completing our proof.

**Remark 1.2.18** The cycle $C_n$ does not satisfy any of the sufficient conditions given above for $n \geq 5$.

An important concept in graph theory is that of a matching. A **matching** in a graph is any subset of edges so that no two members of the subset share a common vertex. Matchings arise naturally in many applications in combinatorics and optimization. These applications include: optimal assignments, scheduling, and allocation problems. Further, matchings also arise as subproblems in many applications such as the Chinese Postman Problem.

As an example, consider the Personnel Assignment Problem where we needed to determine an optimal assignment of persons to jobs. We can associate with this problem a bipartite graph in which the vertices in one of the bipartitioning sets represent the workers and those in the other set represent the jobs. In the case of $n$ jobs and $n$ workers, the solution is a 1-regular spanning subgraph. Such a subgraph is called a perfect matching. So the problem is to find, in a given weighted bipartite graph a perfect matching of optimal weight (maximum if the weights represent effectiveness; minimum if they represent costs). A subset $M$ of $E$ is called a matching in $G$ if its elements are links and no two are adjacent in $G$; the two ends of an edge in $M$ are said to be matched under $M$.

A vertex $v$ is said to be **$M$-saturated** if some edge of the matching $M$ is incident with $v$; otherwise $v$ is **$M$-unsaturated**. If every vertex of $G$ is $M$-saturated, then the matching is perfect. $M$ is a maximum matching if $G$ has no matching $M'$ with $|M'| > |M|$. Every perfect matching is, of course, maximum. Let $M$ be a matching in $G$. An $M$-alternating path in $G$ is a path whose edges are alternately in $E - M$ and $M$. An $M$-augmenting path is an $M$-alternating path whose origin and terminus are $M$-unsaturated.

The following fundamental result of **Berge** (1957) provides a good characterization of a maximum matching and we will state his theorem without proof.

**Theorem 1.2.19** *(Berge's Theorem) A matching $M$ in $G$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.*

So to determine a maximum matching in a graph $G$ we need an efficient method for searching for an $M$-augmenting path; this is accomplished through a tree growing routine. Note that since a maximum matching is finite and each augmentation increases the cardinality of the current matching by one, the above procedure will terminate. An important result on matchings in bipartite graphs is:

**Theorem 1.2.20** *(Hall's Theorem) (1935) Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ contains a matching that saturates every vertex of $X$ if and only if $|N(S)| \geq |S|$ for every $S \subseteq X$.*

**Proof.** Suppose that $G$ has a matching $M$ which saturates every vertex in $X$, and let $S \subseteq X$. Clearly $|N(S)| \geq |S|$. Conversely, suppose $G$ satisfies the condition of the theorem but has no matching saturating all the vertices in $X$. Let $M$ be a maximum matching in $G$ and let $u$ be an $M'$-unsaturated vertex in $X$. Let $Z$ denote the set of all vertices connected to $u$ by $M'$-alternating paths. Berge's theorem implies that $u$ is the only $M'$-unsaturated vertex in $Z$. Set $S = Z \cap X$ and $T = Z \cap Y$. The vertices of $S - \{u\}$ are matched under $M'$ with the vertices in $T$. Hence $|T| = |S| - 1$ and $T \subseteq N(S)$. In fact, $N(S) = T$, since every vertex in $N(S)$ is connected to $u$ by an $M'$-alternating path. Consequently $|N(S)| = |S| - 1 < |S|$, we get a contradiction. This completes the proof.

As a corollary to the preceding Theorem we have:

**Corollary 1.2.21** *(Marriage Theorem) If $G$ is a $k$-regular bipartite graph with $k > 0$, then $G$ has a perfect matching.*

The following example gives an application of matchings.

**Example 1.2.22** Consider the following coding problem. There are $n$ files $F_1, F_2, \ldots, F_n$, each having a four-digit codeword associated with it; the digits available are: $0, 1, 2, \ldots, t$. The problem is to determine whether or not it is possible to associate with each entry in the file just one of the digits of its codeword so that the entry can be uniquely recovered from this single digit.

Formulation: We can represent the above situation with a bipartite graph with bipartition $(X, Y)$, where

$$X = \text{the set of four} - \text{digit codewords}$$

$$Y = \{0, 1, 2, \ldots, t\}$$

$$(|x| = n, |Y| = t + 1).$$

$x \in X$ is joined to $y \in Y$ if $y$ appears as a digit in $x$.

**Problem** Does the graph $G$ contain a maximum matching that saturates the vertices of $X$? Hall's theorem says yes, provided $|N(S)| \geq |S|$ for all $S \subseteq V$.

The above example can be viewed combinatorially in terms of sets. Let $\mathbb{F} = \{S_1, S_2, \ldots, S_n\}$ be a family of not necessarily distinct sets. An $n$-tuple $T = (a_1, a_2, \ldots, a_n)$ with $a_i \in S_i, 1 \leq i \leq n$, is called a system of distinct representatives (SDR) for $\mathbb{F}$ if all the $a_i$'s are distinct. A question of interest is when does a given $\mathbb{F}$ have an SDR? The following result can be obtained.

**Theorem 1.2.23** *The family $\mathbb{F} = \{S_1, S_2, \ldots, S_n\}$ has a SDR if and only if for all $k, i \leq k \leq n$, any $k$ $S_i$'s together contain at least $k$ distinct elements.*

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it, and a **vertex cover** of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called its **vertex covering number** of $G$ and is denoted by $\beta_0(G)$. Clearly for any matching $M$ and covering $K$ of $G$, $|M| \leq |K|$. In particular, if $M^*$ is a maximum matching and $K^*$ a minimum covering, then $|M^*| \leq |K^*|$.

A relationship between matching and covering of a bipartite graph was obtained by **König** in the following theorem.

**Theorem 1.2.24** *(König) In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

**Proof.** Let $G$ be a bipartite graph with bipartition $(X, Y)$ and let $M^*$ be a maximum matching of $G$. Let $U$ be the set of unsaturated vertices of $X$. Let $Z$ be the set of vertices of $G$ connected to vertices of $U$ by $M^*$-alternating paths. Consider $S = Z \cap X$ and $T = Z \cap Y$. Clearly every vertex of $T$ is $M^*$-saturated and $N_G(S) = T$. Now let $K = (X - S) \cup T$. Claim that every edge of $G$ has at least one end in $K$. For, if this is not the case, there would be an edge with one end in $S$ and one in $Y - T$, contradicting the fact that $N(S) = T$. Hence $K$ is a covering of $G$. Now if $K$ is not a minimum vertex cover then for a minimum vertex cover $K^*$ we have $|K| = |M^*| \leq |K^*| < |K|$, a contradiction. This completes the proof.

Hall's theorem gives a necessary and sufficient conditions for the existence of a maximum matching in a bipartite graph. We now establish the important result of **Tutte** which gives necessary and sufficient conditions for a graph to have a perfect matching. Obviously if $G$ has a perfect matching then $|V(G)|$ must be even. However, $2C_5$ has an even order but it contains no perfect matching. There are also connected graphs of even order containing no perfect matching.

A necessary and sufficient condition for a graph to have a perfect matching

was obtained by Tutte [37]. A component of a graph is **odd** or **even** according as it has an odd or even number of vertices. We denote by $O(G)$ the number of odd components of $G$. The following theorem is due to Tutte [37].

**Theorem 1.2.25** *$G$ has a perfect matching if and only if*

$$O(G-S) \leq |S| \text{ for all } S \subseteq V.$$

Berge [3] generalized Tutte's result and it makes easier for application.

**Theorem 1.2.26** *The number of edges in a maximum matching of a graph $G$ is $\frac{1}{2}(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)}\{O(G-S) - |S|\}$.*

A simple application of Tutte's theorem yields an old result of **Petersen** (1891).

**Theorem 1.2.27** *Every 3-regular graph without cut edges has a perfect matching.*

**Proof.** Let $G$ be a 3-regular graph without cut edges, and let $S \subseteq V$. Denote the odd components of $G - S$ by $G_1, G_2, \ldots, G_n$. Let $m_i$ be the the number of edges in $G$ with one end in $G_i$ and the other in $S$. We have

$$m_i = \sum_{v \in V(G_i)} d_G(v) - 2|E(G_i)|, \ 1 \leq i \leq n.$$

Since $G$ has no cut edges, $m_i > 1$ and $m_i$ must be odd. Hence $m_i \geq 3$, for $1 \leq i \leq n$ and so

$$O(G-S) = n \leq \frac{1}{3}\sum_{i=1}^{n} m_i \leq \frac{1}{3}\sum_{v \in S} d_G(v) = |S|.$$

Hence, by Tutte's theorem $G$ has a perfect matching.

**Corollary 1.2.28** *Let $G$ be a 3-regular graph without any cut edge. Then $E(G)$ can be written as a disjoint union of 1-factor and 2-factor.*

**Corollary 1.2.29** *Let $G$ be a 2-connected 3-regular graph. Then $E(G)$ can be written as a disjoint union of 1-factor and 2-factor.*

One objective in studying graph theory is to develop algorithms which can be implemented on a computer and so we need to be able to specify a graph in a computer program. Indeed, the performance of a graph algorithm is highly dependent upon the manner in which the graph is stored and manipulated. We do not discuss data structure in detail here but give just two simple ways in which a graph can be specified in terms of matrices. In particular, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ we define the **incident matrix** of a graph $G$ as the $n \times m$ matrix $M(G) = [m_{ij}]$, where $m_{ij}$ is the number of times (0, 1 or 2) that $v_i$ and $e_j$ are incident. The **adjacency matrix** of $G$ is the $n \times n$ matrix $A = [a_{ij}]$ is the number of edges joining $v_i$ and $v_j$.

**Remark 1.2.30** The adjacency matrix of a graph is usually smaller than the incident matrix, and so it is in this form that graphs are commonly stored in computers. We could also specify a graph in a computer program by a list of edges or a list of vertices adjacent to each vertex.

# Chapter 2

# INTERPOLATION THEOREMS

This chapter is to give motivation and review some known results on interpolation theorems for graph parameters. We review the basic concept of graph transformations in Section 2.1. A list of graph parameters which will be used through out this chapter and next chapter is given in Section 2.2. In Section 2.3, we review results on interpolation graph parameters. The results on extremal problems on graph theory are presented in Section 2.4.

## 2.1 Graph transformations

Let $\mathbb{J}$ be a class of graphs or subgraphs of a graph. If there is a fixed type of transformation of a member of $\mathbb{J}$ into another, then this will provide a definition of a **graph transformation on $\mathbb{J}$**. Equivalently, a graph transformation may be defined as a relation $\rho$ from $\mathbb{J}$ to $\mathbb{J}$. In other words $\rho \subseteq \mathbb{J} \times \mathbb{J}$. Let $\rho$ be a graph transformation on $\mathbb{J}$. We can define **the $\rho$-graph** having $\mathbb{J}$ as its vertex set and there is an edge between two vertices $G$ and $H$ if and only if $(G, H) \in \rho$. If $\rho$ is symmetric, we have an undirected graph and if not a directed graph. Thus $\rho$-graph arises from graphs in at least two different ways:

1. from a class of graphs, or

2. from a class of subgraphs of a fixed graph.

Harary introduced a graph transformation and called it a **fundamental exchange** or an **edge exchange** as follows: Let $G$ be a graph of order $n \geq 3$. The **tree graph,** $\mathbf{T}(G)$, of $G$ is defined by specifying $V(\mathbf{T}(G))$ to be the set of all spanning trees of $G$, and two vertices $T_1, T_2 \in V(\mathbf{T}(G))$ are adjacent in $\mathbf{T}(G)$ if and only if $T_1$ and $T_2$ have exactly $n-2$ edges in common. This is an example of an undirected $\rho$-graph. It was proved by Harary and others that the tree graph $\mathbf{T}(G)$ is connected. Details can be found in [13, 14, 15, 16, 17, 18, 19, 20].

Let $G$ be a graph of order $n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$.

A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a **graphic degree sequence** if it is a degree sequence of some graph $G$. In this case, $G$ is called a **realization** of $\mathbf{d}$.

Let $G$ be a graph. For the distinct vertices $a$, $b$, $c$, and $d$ in $G$ such that $ab$ and $cd$ are edges in $G$, but $ac$ and $bd$ are not edges in $G$. Define $G^{\sigma(a,b;c,d)}$, simply written $G^{\sigma}$, to be the graph obtained from $G$ by deleting the edges $ab$ and $cd$ and adding the edges $ac$ and $bd$. The operation $\sigma(a,b;c,d)$ is called a **switching operation**. Note that $G$ and $G^{\sigma}$ have the same degree sequence.

Let $\mathbf{d}$ be a graphic degree sequence. Let $\mathbb{R}(\mathbf{d})$ and $\mathbb{CR}(\mathbf{d})$ be the sets of realizations and connected realizations of $\mathbf{d}$, respectively. Since $\sigma$ is a degree preserving transformation, it is reasonable to define $\Sigma(\mathbf{d})$ as a relation on $\mathbb{R}(\mathbf{d})$ as $(G,H) \in \Sigma(\mathbf{d})$ if $G \ncong H$ and there is a switching $\sigma$ on $G$ such that $H = G^{\sigma}$. It is easy to see that $\sigma(a,b;c,d)$ is well defined on $G$ if and only if $\sigma(a,c;b,d)$ is well defined on $G^{\sigma(a,b;c,d)}$. Thus the $\Sigma(\mathbf{d})$-graph is simple. The $\Sigma(\mathbf{d})$-graph was proved to be connected by Taylor [35]. One year later, it was also proved by Taylor [36] that the subgraph of $\Sigma(\mathbf{d})$-graph induced by $\mathbb{CR}(\mathbf{d})$ is connected.

For positive integers $m$ and $m$ with $0 \leq m \leq \binom{n}{2}$, let $\mathbb{G}(m,n)$ and $\mathbb{CG}(m,n)$ be the sets of all graphs and connected graphs, respectively, of order $n$ and size $m$. Let $G \in \mathbb{G}(m,n)$ with $e \in E(G)$ and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G-e+f)$. A transformation $t(e,f)$ is called an **edge jump**. Now let $\mathbb{T}(m,n)$ be a relation on $\mathbb{G}(m,n)$ defined by $(G,H) \in \mathbb{T}(m,n)$ if $G \ncong H$ and $H$ can be obtained from $G$ by a jumping transformation. Since $\mathbb{T}(m,n)$ is symmetric, it follows that $\mathbb{T}(m,n)$-graph is simple.

We proved recently in [33] that the $\mathbb{T}(m,n)$-graph and the subgraph of the $\mathbb{T}(m,n)$-graph induced by $\mathbb{CG}(m,n)$ are connected.

## 2.2 The interpolation graph parameters

Let $\mathbb{G}$ be the class of all graphs. A function $\pi : \mathbb{G} \to \mathbb{Z}$ is called a **graph parameter** if $\pi(G) = \pi(H)$ for all isomorphic graphs $G$ and $H$. A graph parameter $\pi$ is called an **interpolation graph parameter over** $\mathbb{J} \subseteq \mathbb{G}$ if there exist integers $x$ and $y$ such that

$$\{\pi(G) : G \in \mathbb{J}\} = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

If $\pi$ is an interpolation graph parameter over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J}) := \min\{\pi(G) : G \in \mathbb{J}\}$ and $\max(\pi, \mathbb{J}) := \max\{\pi(G) : G \in \mathbb{J}\}$.

In 1963, Erdős and Gallai proved that any regular graph on $n$ vertices has

chromatic number $k \leq \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communication, Erdős wrote to Pullman, "probably such a graph exists for every $k \leq \frac{3n}{5}$, except possibly for trivial exceptional cases." Caccetta and Pullman [6] confirmed and strengthened Erdős' conjecture by showing that if $k > 1$, then for every $n \geq \frac{5k}{3}$, there exists a connected, regular, $k$-chromatic graph on $n$ vertices. This is an example of interpolation graph parameter $\chi$ over the class of all connected regular graphs of order $n$.

The main idea of proving the interpolation property of various kinds of graph parameters is by using the connectivity of the tree graph $\mathbf{T}(G)$ of a connected graph $G$ and proving that for a graph parameter $\pi$, $\pi(T_1)$ and $\pi(T_2)$ differ by at most one for any $T_1$ and $T_2$ that are adjacent in $\mathbf{T}(G)$.

### 2.2.1 The $\Sigma(\mathbf{d})$-graphs

We will review in this subsection the interpolation properties of various kinds of graph parameters over $\mathbb{R}(\mathbf{d})$ and $\mathbb{CR}(\mathbf{d})$. We first state the following general result.

**Theorem 2.2.1** *Let $\mathbb{J} \subseteq \mathbb{R}(\mathbf{d})$ and the subgraph of $\Sigma(\mathbf{d})$-graph induced by $\mathbb{J}$ is connected. Let $\pi$ be a graph parameter. For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|\pi(G) - \pi(G^\sigma)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{J}$.*

The following results can be obtained as consequences of Taylor [35, 36].

**Theorem 2.2.2** *Let $\pi$ be a graph parameter. For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|\pi(G) - \pi(G^\sigma)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{R}(\mathbf{d})$.*

**Theorem 2.2.3** *Let $\pi$ be a graph parameter. For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|\pi(G) - \pi(G^\sigma)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{CR}(\mathbf{d})$.*

We will now review interpolation results on various graph parameters over $\mathbb{R}(\mathbf{d})$ and $\mathbb{CR}(\mathbf{d})$. Since our work started with the graph parameter $\chi$, we first state the definition of $\chi$.

Graph coloring takes its name from the map-coloring application. We assign labels to vertices when the numerical value of labels is unimportant, we call them **colors** to indicate that they may be elements of any set.

A ***k*-coloring** of a graph $G = (V, E)$ is a partition of its vertex set $V$ as $V_1 \cup V_2 \cup \ldots \cup V_k$ such that no two vertices in $V_i (1 \leq i \leq k)$ are adjacent. The $V_i$'s are called the **color classes.** A function $f : V \to \{1, 2, \ldots, k\}$ such that $f(v) = i$ for each $v \in V_i (1 \leq i \leq k)$ is called a **color function.** If $G$ has a $k$-coloring, it is said to be ***k*-colorable** and the minimum integer $k$ for

which $G$ is $k$-colorable is called the **chromatic number** of $G$ and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that $G$ is **$k$-chromatic.** We proved in [25] the following result.

**Theorem 2.2.4** *Let $G$ be a graph and $\sigma$ be a switching on $G$. Then $|\chi(G) - \chi(G^\sigma)| \leq 1$.*

A maximal complete subgraph of a graph $G$ is called a **clique** of $G$. The maximum order of clique of $G$ is called the **clique number** of $G$ and is denoted by $\omega(G)$. In general, there is no formula for the chromatic number of a graph. In fact, determining the chromatic number of even a relatively small graph is often a challenging problem. However, a lower bound for the chromatic number of a graph $G$ can be given in term of the clique number of $G$. That is for any graph $G$, $\chi(G) \geq \omega(G)$. We proved in [26] the following result on $\omega$.

**Theorem 2.2.5** *Let $G$ be a graph and $\sigma$ be a switching on $G$. Then $|\omega(G) - \omega(G^\sigma)| \leq 1$.*

**Acyclic graph** is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree. Let $G$ be a graph and $F \subseteq V(G)$, $F$ is called an **induced forest** of $G$, if the induced subgraph $G[F]$ of $G$ contains no cycle. For a graph $G$, we define, $f(G)$ as:

$$f(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

We proved in [27] the following results on $f$.

**Theorem 2.2.6** *If $S$ is any subset of vertices of $G$ such that $G[S]$ is a forest, and $\sigma$ is any switching on $G$, then $G^\sigma[S]$ contains at most one cycle.*

**Corollary 2.2.7** *If $\sigma$ is any switching on $G$, then $|f(G) - f(G^\sigma)| \leq 1$.*

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph $G$ is difficult even for some simply defined graphs. For a graph $G$, this minimum number is known as the **decycling number** of $G$, and is denoted by $\phi(G)$. It is easy to see that for a graph $G$ of order $n$, $\phi(G) + f(G) = n$. Thus we obtain the following corollary.

**Corollary 2.2.8** *If $\sigma$ is any switching on $G$, then $|\phi(G) - \phi(G^\sigma)| \leq 1$.*

A subset $U$ of the vertex set $V$ of a graph $G = (V, E)$ is said to be an **independent set** of $G$ if the induced subgraph $G[U]$ of $G$ is an empty graph. An independent set of $G$ with maximum number of vertices is called a **maximum independent set** of $G$. The number of vertices of a maximum independent set of $G$ is called the **independence number** of $G$ and is denoted by $\alpha_0(G)$. It

is clear that $\alpha_0(G)$ is a graph parameter and $\alpha_0(G) = \omega(\overline{G})$, for any graph $G$. Observe that for a graph $G$ and a switching $\sigma = \sigma(a, b; c, d)$ on $G$, $\overline{G^\sigma} = \overline{G}^{\sigma'}$, where $\sigma' = \sigma(a, c; b, d)$. Thus we obtain the following corollary.

**Corollary 2.2.9** *If $\sigma$ is any switching on $G$, then $|\alpha_0(G) - \alpha_0(G^\sigma)| \leq 1$.*

A subset $M$ of the edge set $E$ of a graph $G = (V, E)$ is an **independent edge set** or **matching** in $G$ if no two distinct edges in $M$ have a common vertex. A matching $M$ is **maximum** in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality of a maximum matching of $G$ is denoted by $\alpha_1(G)$ and is called the **matching number** of $G$.

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A **vertex cover** of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the **vertex covering number** of $G$ and is denoted by $\beta_0(G)$.

There is an analogous covering concept for edges. An edge of a graph $G = (V, E)$ is said to cover the two vertices incident with it. An **edge cover** of a graph $G$ is a set of edges covering all the vertices of $G$. The minimum cardinality of an edge cover of $G$ is called the **edge covering number** of $G$ and is denoted by $\beta_1(G)$.

**Theorem 2.2.10** *If $G$ is a graph with $\alpha_1(G) = \alpha_1$ and $\sigma$ is a switching on $G$, then $\alpha_1(G^\sigma) \geq \alpha_1 - 1$.*

**Corollary 2.2.11** *If $\sigma$ is a switching on $G$, then $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$.*

Let $n$ be a positive integer. The **star** of order $n + 1$ is the complete bipartite graph $K_{n,1}$. The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest number of stars. This is equivalent to our next graph parameter.

A **dominating set** of a graph $G = (V, E)$ is a subset $D$ of $V$ such that each vertex of $V - D$ is adjacent to at least one vertex of $D$. The **domination number** $\gamma(G)$ is the cardinality of a minimal dominating set with the least number of elements.

**Theorem 2.2.12** *If $G$ is a graph with $\gamma(G) = \gamma$ and $\sigma$ is a switching on $G$, then $\gamma(G^\sigma) \leq \gamma + 1$.*

**Corollary 2.2.13** *If $\sigma$ is a switching on $G$, then $|\gamma(G) - \gamma(G^\sigma)| \leq 1$.*

Gallai [11] and Norman-Rabin [22] proved the following results concerning relationship between $\alpha_0$, $\beta_0$, and $\alpha_1$, $\beta_1$, respectively.

**Theorem 2.2.14** *For any graph $G$ of order $n$, $\alpha_0 + \beta_0 = n$.*

**Theorem 2.2.15** *For any graph $G$ of order $n$ and $\delta \geq 1, \alpha_1 + \beta_1 = n$.*

The following corollaries are easily obtained.

**Corollary 2.2.16** *If $\sigma$ is a switching on $G$, then $|\beta_0(G) - \beta_0(G^\sigma)| \leq 1$.*

**Corollary 2.2.17** *If $\sigma$ is a switching on $G$, then $|\beta_1(G) - \beta_1(G^\sigma)| \leq 1$.*

Combining the results in this section we can conclude the following results.

**Theorem 2.2.18** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence. Then $\chi$, $\omega$, $f$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$, and $\gamma$ are interpolation graph parameters over $\mathbb{R}(\mathbf{d})$ and $\mathbb{CR}(\mathbf{d})$.*

**Theorem 2.2.19** *Let $\pi \in \{\chi, \omega, f, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(\pi)$ and $b := b(\pi)$ such that $\mathbf{d}$ has a realization $G$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

**Theorem 2.2.20** *Let $\pi \in \{\chi, \omega, f, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(\pi)$ and $b := b(\pi)$ such that $\mathbf{d}$ has a connected realization $G$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

### 2.2.2 The $\mathbb{T}(m, n)$-graphs

Analogous interpolation results on the graph parameters over $\mathbb{G}(m, n)$ and $\mathbb{CG}(m, n)$ will be reviewed in this subsection. We proved in [33] the following results.

**Theorem 2.2.21** *Let $G, H \in \mathbb{G}(m, n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that $H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G$.*

**Corollary 2.2.22** *The graph $\mathbb{G}(m, n)$ is connected.*

**Theorem 2.2.23** *Let $G, H \in \mathbb{CG}(m, n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$, $\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1}) \cdots \sigma(e_1, f_1)G \in \mathbb{CG}(m, n)$ and $H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G$.*

**Corollary 2.2.24** *The graph $\mathbb{CG}(m, n)$ is connected.*

**Theorem 2.2.25** *For a graph $G \in \mathbb{G}(m,n)$ and an edge jump $\sigma(e,f)$, if $|\pi(G) - \pi(\sigma(e,f)G)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{G}(m,n)$.*

**Theorem 2.2.26** *For a graph $G \in \mathbb{CG}(m,n)$ and an edge jump $\sigma(e,f)$, if $|\pi(G) - \pi(\sigma(e,f)G)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{CG}(m,n)$.*

**Theorem 2.2.27** *Let $\mathbb{J} \subseteq \mathbb{G}(m,n)$ and the subgraph of the graph $\mathbb{G}(m,n)$ induced by $\mathbb{J}$ be connected. For a graph $G \in \mathbb{J}$ and an edge jump $\sigma(e,f)$, if $|\pi(G) - \pi(\sigma(e,f)G)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{J}$.*

We will now present interpolation results on various graph parameters with over $\mathbb{G}(m,n)$ and $\mathbb{CG}(m,n)$.

**Theorem 2.2.28** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\chi(G) - \chi(\sigma(e,f)G)| \leq 1$.*

**Theorem 2.2.29** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\omega(G) - \omega(\sigma(e,f)G)| \leq 1$.*

**Theorem 2.2.30** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|f(G) - f(\sigma(e,f)G)| \leq 1$.*

Since $f(G) + \phi(G) = n$ for any graph $G$ of order $n$, we have the following corollary.

**Corollary 2.2.31** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\phi(G) - \phi(\sigma(e,f)G)| \leq 1$.*

**Corollary 2.2.32** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\alpha_0(G) - \alpha_0(\sigma(e,f)G)| \leq 1$.*

**Theorem 2.2.33** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\alpha_1(G) - \alpha_1(\sigma(e,f)G)| \leq 1$.*

**Theorem 2.2.34** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\gamma(G) - \gamma(\sigma(e,f)G)| \leq 1$.*

By Theorem 2.2.14 and Theorem 2.2.15 we obtain the following corollary.

**Corollary 2.2.35** *Let $G \in \mathbb{G}(m,n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\beta_0(G) - \beta_0(\sigma(e,f)G)| \leq 1$. and $|\beta_1(G) - \beta_1(\sigma(e,f)G)| \leq 1$.*

The following theorems are obtained as consequences of previous results.

**Theorem 2.2.36** *$\chi$, $\omega$, $f$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$ and $\gamma$ are interpolation graph parameters over $\mathbb{G}(m,n)$.*

**Theorem 2.2.37** *$\chi$, $\omega$, $f$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$ and $\gamma$ are interpolation graph parameters over $\mathbb{CG}(m,n)$.*

**Theorem 2.2.38** *Let $\pi \in \{\chi, \omega, f, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then there exist integers $a := \min(\pi; m, n)$ and $b := \max(\pi; m, n)$ such that there exists $G \in \mathbb{G}(m,n)$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

**Theorem 2.2.39** *Let $\pi \in \{\chi, \omega, f, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then there exist integers $a := \mathrm{Min}(\pi; m, n)$ and $b := \mathrm{Max}(\pi; m, n)$ such that there exists $G \in \mathbb{CG}(m,n)$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

## 2.3 Extremal problems

In this section we discuss the second part of interpolation theorems for graph parameters. We first recall what we mean by an extremal problem in graph theory. An *extremal problem* asks for minimum and maximum values of a function $\pi$ over a class of objects.

**Remark 2.3.1** Proving that $A$ is the minimum of $\pi(G)$ for graphs in a class $\mathbb{J}$ requires showing two things:

1. $\pi(G) \geq A$ for all $G \in \mathbb{J}$.
2. $\pi(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of $\pi$.
Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

It will be seen that we have produced several results in the extremal graph theory for graph parameters $\chi, \omega, f, \phi$, and $\alpha_1$ in the class of all $r$-regular graphs of order $n$ and some other related classes.

**Theorem 2.3.2** *An $r$-regular graph $G$ of order $n > r+1$ has chromatic number $k \leq \frac{3n}{5}$, with equality holds if and only if the complementary graph $\overline{G}$ of $G$ is a union of 5-cycles.*

They used the concept of the **partition number** of a graph $G$. A **vertex partition** of a graph $G$ is a partition $P$ of $V(G)$ such that for each $V_i \in P$, the induced subgraph $G[V_i]$ in $G$ is a complete graph. The **partition number** of $G$, denoted by $p(G)$, and is defined by

$$p(G) := \min\{|P| \ : P \text{ is a covering of } G\}.$$

It is easy to see that for any graph $G$, $p(G) = \chi(\bar{G})$. We extended their result by improving the upper bound in [24].

Let $\pi$ be an interpolation graph parameter with respect to $\mathbb{R}(\mathbf{d})$. It is clear that $\{\pi(G) : G \in \mathbb{R}(\mathbf{d})\}$ is uniquely determined by $\min(\pi, \mathbf{d})$ and $\max(\pi, \mathbf{d})$. We will discuss in this section those extremal values for various kinds of graph parameters. For simplicity, we will consider the class of regular graphs and some other related classes of graphs.

For the graph parameter $\chi$, we obtained in [25] the following results:

**Theorem 2.3.3** *If $r \geq 2$ and $n \geq 2r$, then*

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 2.3.4** *If $r \geq 2$, then*
$\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r + 1$, *and*
$\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = (r + 2)/2$.

Proofs of Theorem 2.3.3 and Theorem 2.3.4 were based on graph constructions.

**Theorem 2.3.5** *For any $r \geq 4$ and odd integer $s$ such that $3 \leq s \leq r$, let $q$ and $t$ be integers satisfying $r + s = sq + t, 0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases}$$

**Theorem 2.3.6** *For any even integer $r \geq 6$ and any even number $s$ such that $4 \leq s \leq r$, let $q$ and $t$ be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases}$$

Proofs of Theorem 2.3.5 and Theorem 2.3.6 were based on the concept of the covering number of graphs and graph constructions depending on the parity of $s$.

**Theorem 2.3.7** *Let $r \geq 2$. Then*

(1) $\max(\chi, r^{2r}) = r$,

(2) $$\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases}$$

(3) $\max(\chi, r^n) = r + 1$ *for* $n \geq 2r + 2$.

Brooks' theorem [5] and graph constructions were applied to prove this theorem.

**Theorem 2.3.8** *For any $r$ ad $s$ such that $3 \leq s \leq r-1$, we have*
(1) $\max(\chi, r^{r+s}) \geq (r+s)/2$ *if $r+s$ is even, and*
(2) $\max(\chi, r^{r+s}) \geq (r+s-1)/2$ *if $r+s$ is odd.*

Purely graph constructions were used in the proof of this theorem. The exact values of $\max(\chi, r^n)$ are not easy to obtain when $r+3 \leq n \leq 2r-1$. Theorem 2.3.2 was proved by Erdős and Gallai [9] to provide an upper bound in the class of connected proper regular graphs of order $n$.

It should be noted that the bound given by Theorem 2.3.2 does not depend on $r$ and this bound may be very far from the actual values of the chromatic numbers. Also the exact value of $\max(\chi, r^{r+3})$ was obtained by Theorem 2.3.2. We gave in [25] the following definition.

Let $j$ be a positive integer. An **$F(j)$-graph** is a $(j-1)$-regular graph $G$ of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$. It is easy to see that $F(3)$-graph is a unique graph $C_5$ and $f(3) = 5$. We will see later that $F(j)$-graphs, $j \geq 5$, are not unique. We found $F(j)$-graphs for all odd integers $j$ as we state in the following theorem.

**Theorem 2.3.9** *For odd integer $j$ with $j \geq 3$, we have $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 (\mathrm{mod}\ 4)$ and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 (\mathrm{mod}\ 4)$.*

**Theorem 2.3.10** *Any $r$-regular graph of order $n$ with $n - r = j$ odd and $j \geq 3$, has chromatic number at most $\frac{f(j)+1}{2f(j)} \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

Note that our results in Theorem 2.3.9 and Theorem 2.3.10 can be considered as a generalization of Theorem 2.3.2 as $F(3)$-graph is $C_5$ and $f(3) = 5$.

**Problem 1.** Find an $F(j)$-graph for even integer $j \geq 4$.

**Problem 2.** Find the value of $\max(\chi, r^{r+j})$, for even integer $j$ and $4 \leq j \leq r-2$.

For the graph parameter $\omega$ we completely answered in [26] the second part of interpolation theorem with respect to the class of $r$-regular graphs of order $n$ as stated in the following results.

**Theorem 2.3.11** *Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Then $\omega(G) \geq \frac{n}{n-\Delta}$.*

Since the complete graph $K_n$ is a clique, $\min(\omega, r^{r+1}) = \max(\omega, r^{r+1}) = r+1$. Given positive integers $n$ and $k$ with $k \leq n$, there exists a connected graph $G$ of order $n$ with $\omega(G) = k$. However, if $G$ is regular, we have the following remarkable result.

**Theorem 2.3.12** *Let* $\mathbf{d} = r^n$ *be a graphic degree sequence with* $r+2 \leq n \leq 2r+1$. *Then* $\max(\omega, r^n) = \lfloor \frac{n}{2} \rfloor$.

**Theorem 2.3.13** *For any* $r \geq 6$ *and odd integer* $s$ *such that* $5 \leq s < r$, *let* $q$ *and* $t$ *be integers satisfying* $r+s = sq+t, 0 \leq t < s$. *Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & if\ t = 0, \\ q+1 & if\ 1 \leq t \leq s-2, \\ q+2 & if\ t = s-1. \end{cases}$$

**Theorem 2.3.14** *For any even integer* $r \geq 6$ *and any even number* $s$ *such that* $4 \leq s \leq r$, *let* $q$ *and* $t$ *be integers satisfying* $r+s = sq+t$, $0 \leq t < s$. *Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & if\ t = 0, \\ q+1 & if\ t \geq 2. \end{cases}$$

Observe that results of Theorem 2.3.13 and Theorem 2.3.14 respectively coincide with Theorem 2.3.5 and Theorem 2.3.6 above.

For graph parameter $f$, we found in [28] a lower bound of $\min(f, \mathbf{d})$ by using **the probabilistic method**. In particular we proved that if $G$ is a graph with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots d_n \geq 1$, then

$$f(G) \geq 2 \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

As an application, we found all minimum values of the order of maximum induced forest in a $(\Delta, n)$-graph.

First of all we would like to introduce basic tools from discrete probability theory which shall be used for calculation of some bounds of graph parameters.

Let $(\Omega, p)$ be a finite **probability space,** where $\Omega$ is a finite set and a probability function $p$ that maps from $\Omega$ into the interval $[0, 1]$ with

$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

A **random variable** $X$ on $\Omega$ is a mapping $X : \Omega \to \mathbb{R}$. We define a probability space on the image set $X(\Omega)$ as

$$p(X = x) := \sum_{X(\omega) = x} p(\omega).$$

The **expectation** $E(X)$ of $X$ is

$$E(X) = \sum_{\omega \in \Omega} p(\omega) X(\omega).$$

Now suppose that $X$ and $Y$ are two random variables on $\Omega$, then the sum $X+Y$ is again a random variable, sharing a property $E(X+Y) = E(X) + E(Y)$. Clearly, this can be extended to any finite linear combination of random variables (the linearity of expectation). Note that it is not necessary that the random variables are "independent."

Thus we have the following theorems.

**Theorem 2.3.15** *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \ldots d_n \geq 1.$$

*Then*

$$f(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

**Proof.** Let $G$ be an arbitrary graph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Denote the degree of $v_i$ by $d_i$. We choose with equal probability $\frac{1}{n!}$ a random permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ of $V$ and construct the following set $F_\pi$. We put $\pi_i$ into $F_\pi$ if and only if $\pi_i$ is adjacent to at most one $\pi_j, (j < i)$. It is clear that $F_\pi$ is an induced forest of $G$. Let $X(\pi) = |F_\pi|$ be the corresponding random variable. We have $X = \sum_{i=1}^{n} X_i$, where $X_i$ is the indicator random variable of the vertex $v_i$, in other words $X_i = 1$ if $v_i \in F_\pi$ and $X_i = 0$ if $v_i \notin F_\pi$. In order to calculate $E(X_i)$, we first calculate the number of permutations $\pi$ which contain the vertex $v_i$. Since $v_i$ has $d_i$ neighbors, we partition $n!$ permutations on $V$ into $\binom{n}{d_i+1}$ classes according to $d_i + 1$ positions that $v_i$ and its neighbors appeared. It is clear that each class contains $(d_i+1)!(n-d_i-1)!$ permutations and $2d_i!(n - d_i - 1)!$ of which have the property that $v_i \in F_\pi$. Thus

$$E(X_i) = \sum_{\pi} p(\pi)X_i(\pi) = \frac{2}{d_i + 1}.$$

Therefore

$$E(X) = \sum_{i=1}^{n} E(X_i) = 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

**Corollary 2.3.16** *If the average degree of $G$ is at most $d$, then $f(G) \geq \frac{2n}{d+1}$.*

We now have the following corollary as a result of Caro [7].

**Corollary 2.3.17** *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \ldots d_n \geq 0.$$

*Then*

$$\alpha_0(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1},$$

*where $\alpha_0(G)$ is the independence number of $G$.*

**Proof.** The proof follows easily from the fact that a forest is bipartite.

**Corollary 2.3.18** *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \ldots d_n \geq 0.$$

*Then*

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i},$$

*where $\omega(G)$ is the clique number of $G$.*

**Proof.** The proof follows from the fact that $\omega(G) = \alpha_0(\overline{G})$.

Let $G$ be a graph. The problem of determining the decycling number $\phi(G)$ of $G$ is equivalent to finding the greatest order of an induced forest $f(G)$ of the graph $G$ and the sum of the two numbers equals the order of the graph. Observe that for a minimum decycling set $S$ of a graph $G$, if $v \in S$, then there exists a connected component $C$ of $G-S$ such that $v$ is adjacent to at least two vertices of $C$. Thus $\Delta(G[S]) \leq \Delta(G) - 2$. With this observation, we find that if $G$ is an $r$-regular graph and $S$ is a minimum decycling set of $G$, the graph $G[S]$ may not be an $(r-2)$-regular graph. This causes a difficulty in finding $\max(\phi, r^n)$ if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the following class of graphs. Let $\Delta$ be a nonnegative integer and $n$ be a positive integer such that $n \geq \Delta + 1$. Let $\mathbb{G}(\Delta, n)$ be the class of all graphs of order $n$ and of maximum degree $\Delta$. **The $(\Delta, n)$-graph** is a graph having $\mathbb{G}(\Delta, n)$ as its vertex set where two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge. This gives another graph transformation.

**Lemma 2.3.19** *The $(\Delta, n)$-graph is connected.*

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of non-negative integers. Define $\overline{\mathbf{d}}$ a degree sequence $(\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, where $\overline{d_i} = n - d_i - 1$, for $i = 1, 2, \ldots, n$. It is clear that $\mathbf{d}$ is graphic if and only if $\overline{\mathbf{d}}$ is.

**Theorem 2.3.20** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then*

1. $\min(f, \mathbf{d}) = 2$ *if and only if* $d_1 = d_2 = d_3 = \cdots = d_n$ *and* $n = d_1 + 1$ *and*
2. *if $\mathbf{d}$ does not have a complete graph as its realization, then* $\min(f, \mathbf{d}) = 3$ *if and only if $\overline{\mathbf{d}}$ has a union of stars as its realization.*

**Theorem 2.3.21** *Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then*
(1) $\min(f, \mathbb{G}(\Delta, n)) = 2q$, *if* $t = 0$,
(2) $\min(f, \mathbb{G}(\Delta, n)) = 2q + 1$, *if* $t = 1$, *and*
(3) $\min(f, \mathbb{G}(\Delta, n)) = 2q + 2$, *if* $2 \leq t \leq \Delta$.

We obtained in [27] the values of $\max(f, r^n)$, for all $r$ and $n$ as stated in the following theorems.

**Theorem 2.3.22**

$$\max(f, r^n) = \begin{cases} n-r+1 & if\ r+1 \le n \le 2r-1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & if\ n \ge 2r. \end{cases}$$

The values of $\min(f, r^n)$, for all $r$ and $n$, were obtained in [31] in terms of the graph parameter $\phi$ as stated in the following theorems. Note that $\min(f, r^n) = \max(\phi, r^n)$.

**Theorem 2.3.23** *For $r \ge 3$, and $n = r + j$, $1 \le j \le r+1$*
(1) $\min(f, r^n) = 2$, *if and only if* $n = r+1$,
(2) $\min(f, r^n) = 3$, *if and only if* $n = r+2$,
(3) $\min(f, r^n) = 4$, *for all even integers* $n$, $r+3 \le n$,
(4) $\min(f, r^n) = 4$, *for all odd integers* $n$, $r+3 \le n$ *and* $n \ge f(j)$,
(5) $\min(f, r^n) = 5$, *for all odd integers* $n$, $r+3 \le n$ *and* $n < f(j)$,
*where* $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 (\mathrm{mod}\ 4)$, *and*
$f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 (\mathrm{mod}\ 4)$.

**Theorem 2.3.24** *For $n \ge 2r+2$ and $r \ge 3$, write $n = (r+1)q + t, q \ge 2$ and $0 \le t \le r$. Then*
(1) $\min(f, r^n) = 2q$ *if* $t = 0$,
(2) $\min(f, r^n) = 2q+1$ *if* $t = 1$,
(3) $\min(f, r^n) = 2q+2$ *if* $2 \le t \le r-1$,
(4) $\min(f, r^n) = 2q+3$ *if* $t = r$.

In [30], we determined the values of $\min(\alpha_1, r^n)$ and $\max(\alpha_1, r^n)$ for all $r$ and $n$. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, from now on we will consider $r \ge 2$ and $n \ge r+1$.

We first investigate the value of $\max(\alpha_1, r^n)$ by the following theorem.

**Theorem 2.3.25** *For $r \ge 2, n \ge r+1$ and $nr \equiv 0 (\mathrm{mod}\ 2)$, there exists an $r$-regular hamiltonian graph of order $n$. In particular,* $\max(\alpha_1, r^n) = \lfloor \frac{n}{2} \rfloor$.

We will use the generalized result of Tutte and the result of Wallis [38] to obtain all values of $\min(\alpha_1, r^n)$.

It is well known that $K_{2n}$ and $K_{n,n}$ have 1-factorizations for all positive integers $n$. The question of which graphs contain 1-factors is one that has attracted considerable attention. For a comprehensive review we refer to the survey of Akiyama and Kano [1] and of Wallis [39]. We obtained the following results.

**Theorem 2.3.26** *The number of edges in a maximum matching of a graph $G$ is $\frac{1}{2}(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)}\{O(G-S) - |S|\}$.*

Let $F(r,d)$ be the minimum order of an $r$-regular graph $G$ with $\alpha_1(G) = \frac{1}{2}(|V(G)| - d)$. It is clear that $|V(G)| \equiv d(\mathrm{mod}\ 2)$. Wallis [38] found $F(r,2)$ for all $r \geq 3$. In other words, he proved the following theorem.

**Theorem 2.3.27** *Let $G$ be an $r$-regular graph with no 1-factor and no odd component. Then*

$$|V(G)| \geq \begin{cases} 3r+7 & \text{if } r \text{ is odd, } r \geq 3, \\ 3r+4 & \text{if } r \text{ is even, } r \geq 6, \\ 22 & \text{if } r = 4. \end{cases}$$

*Furthermore, no such graphs exist for $r = 1$ or 2.*

Suppose that $G$ is an $r$-regular graph with $\alpha_1(G) = \frac{1}{2}(|V(G)| - d)$. There exists a $k$-subset $K$ of $V(G)$ such that $O(G-K) = k+d$. If $k = 0$, then $r$ is even, $G$ contains $d$ odd components, and each component of $G$ has order at least $r+1$. Suppose $k \geq 1$ and $G-K$ has an odd component with $p$ vertices, where $p$ is less than or equal to $r$. The number of edges within the component is at most $\frac{1}{2}p(p-1)$. This means that the sum of degree of these $p$ vertices in $G-K$ is at most $p(p-1)$. But $G$ is an $r$-regular graph, so the sum of these $p$ vertices in $G$ is $pr$. The number of edges joining to the component $K$ must be at least $pr - p(p-1)$. For a fixed $r$ and for integer $p$ satisfying $1 \leq p \leq r$ the function $f(p) = pr - p(p-1)$, $1 \leq p \leq r$ has minimum value $f(1) = f(r) = r$. So any odd component with $r$ or less vertices is joined to $K$ by $r$ or more edges. Suppose that there are $O_+$ odd components of $G-K$ with more than $r$ vertices and $O_-$ odd components with less than or $r$ vertices. Thus

$$O_+ + O_- = k + d \tag{1}$$
$$O_+ + rO_- \leq kr. \tag{2}$$

From these 2 relations, we have $O_+ \geq \lceil \frac{rd}{r-1} \rceil = d + \lceil \frac{d}{r-1} \rceil$ and $k \geq \lceil \frac{d}{r-1} \rceil$. Thus we have the following results.

**Theorem 2.3.28** *Let $r$ be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$.*

**Corollary 2.3.29** *Let $r$ be an even integer, $r \geq 2$. If $n = (r+1)d + e, 0 \leq e \leq r$, then $\min(\alpha_1, r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$.*

Suppose that $r$ is odd and $r \geq 3$. Let $G$ be an $r$-regular graph of order $n$ such that $\alpha_1(G) = \frac{1}{2}(n-d)$. Then $d$ must be even. Put $d = 2q$. There exists a nonempty subset $K$ of $V(G)$ of cardinality $k$ such that $O(G-K) = k+2q$. By (1) and (2), we have

$$n \geq k + (r+2)O_+ \geq \lceil \tfrac{2q}{r-1} \rceil + (r+2)(2q + \lceil \tfrac{2q}{r-1} \rceil) = \lceil \tfrac{2q}{r-1} \rceil (r+3) + 2q(r+2).$$

Wallis [38] defined $G(x,y)$ to be a graph with $x+y$ vertices, $x$ being of degree $x+y-3$ and $y$ of degree $x+y-2$. $G(x,y)$ exists if and only if $y$ is even and $y \geq 2$. It is noted that for any graph $G(x,y)$, it has $y$ vertices of degree $r$

and $x$ vertices of degree $r-1$. Let $x_i, y_i, i = 1, 2, \ldots, m$ be integers such that $G(x_i, y_i)$ exists for all $i = 1, 2, \ldots, m$. We construct the graph

$$G(x_1, y_1) * G(x_2, y_2) * \ldots * G(x_m, y_m)$$

from disjoint copies of the graphs by inserting a new vertex, say $u$, and joining $u$ to all vertices of $G(x_i, y_i)$ which have the smallest degree, for $i = 1, 2, \ldots, m$. Using this notation, we see that for an odd integer $r \geq 3$ and $q = 1, 2, \ldots, \frac{r-1}{2}$, for any odd positive integers $a_i, i = 1, 2, \ldots, 1 + 2q$ whose sum is $r$,

$$G_q = G(a_1, r + 2 - a_1) * G(a_2, r + 2 - a_2) * \ldots * G(a_{1+2q}, r + 2 - a_{1+2q})$$

is an $r$-regular graph on $(r+2)(1+2q)+1$ vertices with $\alpha_1(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$. We have the following results.

**Theorem 2.3.30** *For an odd integer $r \geq 3$, then*
1. *$F(r, 2q) = (r + 2)(1 + 2q) + 1$, for $q = 1, 2, \ldots, \frac{r-1}{2}$,*
2. *if $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, then $F(r, 2q) = sF(r, r-1) + F(r, 2t)$, where $F(r, 0) = 0$.*

**Corollary 2.3.31** *Let $r$ be an odd integer, $r \geq 3$. If $F(r, 2q) \leq n < F(r, 2(q+1))$, then $\min(\alpha_1, r^n) = \frac{1}{2}(n - 2q)$.*

## 2.4 Induced subgraphs

It was observed by Eggleton and Holton [10] that the graph $\mathbb{R}(\mathbf{d})$ is connected. It is natural to investigate connected subgraphs of $\mathbb{R}(\mathbf{d})$. Let $P$ be a property which a graph may possess. Denote by $\mathbb{R}(\mathbf{d};\ P)$ the subgraph of the graph $\mathbb{R}(\mathbf{d})$ induced by those vertices which correspond to graphs with property $P$. Property $P$ for which $\mathbb{R}(\mathbf{d};\ P)$ is connected is called **complete**. If a property $P$ is complete we may find all the graphs of a given degree sequence with the property by switching constrained to graphs with the property. Colbourn [8] showed that the property of being a tree is complete and Syslo [34] extended this to the property of being unicyclic. Taylor [35] generalized these results by showing that the property of being connected is complete. The property of being 2-connected was shown to be complete also by Taylor [36]. By using results in Section 2 and the results of Taylor, we immediately obtain the following interpolation theorems.

**Theorem 2.4.1** *Let $\pi \in \{\chi, f, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(\pi)$ and $b := b(\pi)$ such that $\mathbf{d}$ has a connected realization $G$ with $f(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

**Theorem 2.4.2** *Let $\pi \in \{\chi, f, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(\pi)$ and $b := b(\pi)$ such that $\mathbf{d}$ has a 2-connected realization $G$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

We have known that

$$\chi(\mathbf{d}) := [a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\},$$

where $a = \min(\chi, \mathbf{d})$ and $b = \max(\chi, \mathbf{d})$.

Brooks [5] observed that every graph $G$ may be colored by $\Delta(G) + 1$ colors and he characterized the graphs for which $\Delta(G)$ colors are not enough.

**Theorem 2.4.3** *Any graph $G$ satisfies $\chi(G) \leq 1+\Delta(G)$, with equality holds if and only if either of the following holds:*
*(1) some component of $G$ is the complete graph $K_{\Delta+1}$, where $\Delta = \Delta(G)$;*
*(2) some component of $G$ is an odd cycle, and $\Delta(G) = 2$.*

Note that Brooks' theorem implies $b = \max(\chi, \mathbf{d}) \leq 1 + \Delta$, where $\Delta = \max\{d_1, d_2, \ldots, d_n\}$. Thus for a regular degree sequence $r^n$, we have $b = \max(\chi, r^n) \leq 1+r$. We found in [25] the corresponding value of $b$ for all values of $r$ and $n$ except the cases when $r$ and $n$ are even and $r + 4 \leq n \leq 2r - 2$. In general the Brooks' bound may be very far from the actual value.

The following results concerning connected subgraphs of $R(\mathbf{d})$ were obtained in [29]. For each $c \in \chi(\mathbf{d})$, let $\mathbb{R}(\mathbf{d});\ \chi = c)$ denote the subgraph of $\mathbb{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $c$. Similarly for any $c \in \chi(\mathbf{d})$, let $\mathbb{R}(\mathbf{d});\ \chi \leq c)$ denote the subgraph of $\mathbb{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $p \leq c$. We consider the problem of determining the structure of induced subgraph $\mathbb{R}(\mathbf{d});\ \chi = c)$ and $\mathbb{R}(\mathbf{d});\ \chi \leq c)$. In general, what is the structure of $\mathbb{R}(\mathbf{d});\ \chi = c)$ and of $\mathbb{R}(\mathbf{d});\ \chi \leq c)$? In particular, are these graphs connected? If $\mathbb{R}(\mathbf{d});\ \chi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with chromatic number $c$ by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with chromatic number $c$. Similarly if $\mathbb{R}(\mathbf{d});\ \chi \leq c)$ is connected.

Note that $\chi(0^n) = \{1\}$, $\chi(1^n) = \{2\}$ ($n$ is even), $\chi(2^4) = \{2\}$ $\chi(2^n) = \{2, 3\}$ if $n$ is even and $n \geq 6$, and $\chi(2^n) = \{3\}$ if $n$ is odd and $n \geq 3$. For $\chi(3^n)$, we have $\chi(3^4) = \{4\}$, $\chi(3^6) = \{2, 3\}$ and $\chi(3^n) = \{2, 3, 4\}$ if $n$ is even and $n \geq 8$.

Let $G$ be a graph with $\chi(G) = k$ and $\sigma$ be a switching on $G$. $\sigma$ is called a *$k$-safe switching* if $\chi(G^\sigma) = k$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a sequence of $k$-safe switchings if for each $i, i = 1, 2, \ldots, t$, $\chi(G^{\sigma_1\sigma_2\cdots\sigma_i}) = k$.

**Theorem 2.4.4** *If $r \geq 3$ and $\max(\chi, r^n) = r+1$, then the graphs $\mathbb{R}(r^n;\ \chi = r + 1)$ and $\mathbb{R}(r^n; p \leq r)$ are connected.*

**Proof.** Note that if $r \geq 3$, then $\max(\chi, r^n) = r+1$ if and only if $n = r+1$ or $n \geq 2r+2$. Moreover, if $G$ is a realization of $r^n$ and $\chi(G) = r+1$ if and only if $G$ has $K_{r+1}$ as a component. The theorem is true for $n = r+1$. For $n \geq 2r+2$, let $G_1$ and $G_2$ be any two realizations of $r^n$ such that $\chi(G_1) = \chi(G_2) = r+1$. Thus $G_1 = K_{r+1} \cup H_1$ and $G_2 = K_{r+1} \cup H_2$. Since $H_1$ and $H_2$ are $r$-regular graphs of order $n-r-1$, $H_2$ can be obtained from $H_1$ by a finite number of switchings. Thus $G_2$ can as well be obtained by those switchings. Furthermore, it is easy to observe that those switchings are $(r+1)$-safe switchings.
Let $G$ be a realization of $r^n$ such that $\chi(G) \leq r$. If $G$ is disconnected, then $G$ does not contain $K_{r+1}$ as its component. Thus there exists a suitable sequence of switchings which transforms $G$ to a connected realization of $r^n$ such that each resulting graph obtained in this transformation will have chromatic number less than or equal to $r$. By using the result by Taylor [35], the proof is complete.

Let $C_m$ be the cycle of order $m$. Thus $C_m$ exists for all integer $m \geq 3$, we call $C_m$ **odd cycle** or **even cycle** according to $m$ is odd or even. A realization of $2^n$ can be written as $\cup_{i=1}^{t} C_{n_i}$, where $\sum_{i=1}^{t} n_i = n$. It is well known that if $G = \cup_{i=1}^{t} C_{n_i}$, then $\chi(G) = 2$ if and only if for all $i$, $n_i$ is even. It is clear that any two even cycles $C_r$ and $C_s$, there is a 2-safe switching which transforms these two cycles to $C_{r+s}$. Thus $\mathbb{R}(2^n;\ \chi = 2)$ is connected. The corresponding result can be obtained for the graph $\mathbb{R}(2^n;\ \chi = 3)$ with only one exception.

**Theorem 2.4.5** *If $3 \in \chi(2^n)$, then the graph $\mathbb{R}(2^n;\ \chi = 3)$ is connected if and only if $n \neq 10$.*

**Proof.** Observe that if $n$ is odd and $n \geq 3$, then $\chi(2^n) = \{3\}$. Thus the graph $\mathbb{R}(2^n;\ \chi = 3)$ is connected. If $n = 10$, we can not transform $2C_5$ to $2C_3 \cup C_4$ without passing $C_{10}$. Thus the graph $\mathbb{R}(2^{10};\ \chi = 3)$ is not connected. For $n \geq 12$, let $\cup_{i=1}^{t} C_{n_i}$ be a realization of $2^n$ and suppose that $n_1$ is odd. Then there is a sequence of 3-safe switchings which transforms $\cup_{i=1}^{t} C_{n_i}$ to $C_{n_1} \cup C_{n-n_1}$. Since $n \geq 12$, $C_5 \cup C_{n-5}$ can be transformed to $C_5 \cup C_3 \cup C_{n-8}$ and to $C_3 \cup C_{n-3}$. If $m$ is odd and $m \geq 7$, then $C_m \cup C_{n-m}$ can be transformed to $C_3 \cup C_{m-3} \cup C_{n-m}$ then to $C_3 \cup C_{n-3}$. The proof is complete.

**Theorem 2.4.6** *Let $G$ and $H$ be bipartite graphs with bipartition $(X, Y)$. If $d_G(v) = d_H(v)$ for all $v \in X \cup Y$, then there is a sequence of 2-safe switchings that transforms $G$ into $H$.*

The following theorem is a consequence of Theorem 2.4.6.

**Theorem 2.4.7** *If $2 \in \chi(r^n)$, then the graph $\mathbb{R}(r^n;\ \chi = 2)$ is connected.*

**Theorem 2.4.8** *If $c \in \chi(3^n)$, then the graph $\mathbb{R}(3^n;\ \chi = c)$ is connected.*

**Proof.** We have already proved when $c = 2, 4$. Note that $G \in \mathbb{R}(3^n)$ with $\chi(G) = 3$ if and only if $G$ does not contain $K_4$ as its component and $G$ has an odd cycle as its subgraph. It is easy to check that the theorem is true if $n \leq 8$. Let $G$ be a cubic graph of order $n \geq 10$ with $\chi(G) = 3$. We first suppose that $G$ has a triangle $T$ with $V(T) = \{x, y, z\}$ and $E(T) = \{xy, yz, xz\}$. If $x$ and $y$ have a common neighbor $v \in G - T$, then there exists an edge $ab$ in $G - T$ independent to the edge $xv$. Thus there is a 3-safe switching $\sigma$ such that $G^\sigma$ contains a triangle $T$ where any two vertices of $T$ have no common neighbor in $G - T$. Now suppose that $G$ contains an odd cycle $C$ of smallest order $k \geq 5$. Let $x, y$ be two adjacent vertices in the cycle. Thus $x$ and $y$ have no common neighbor in $G$. Let $z$ be the neighbor of $y$ in $G - C$. Then there exists a 3-safe switching $\sigma$ such that $G^\sigma$ contains a triangle $T = \{x, y, z\}$. Thus for a graph $G \in \mathbb{R}(3^n)$ and $\chi(G) = 3$, there is a sequence of 3-safe switchings which transforms $G$ to $G'$ such that $G'$ contains a triangle $T' = \{x, y, z\}$ and $G' - T'$ is a graph having exactly 3 vertices of degree 2. Since $G$ is an arbitrary graph in $\mathbb{R}(3^n)$, the proof is complete.

**Conjecture.** If $c \in \chi(r^n)$, then the graph $\mathbb{R}(r^n;\ \chi = c)$ is connected.

For the graph parameter $\phi$ we investigated the values of $\phi(G)$ where $G$ runs over the class of cubic graphs in [32]. As an application we were able to answer the problem asked by Bau and Beineke [2].

For a graphic degree sequence $\mathbf{d}$, let $\phi(\mathbf{d}) = \{\phi(G) : G \in \mathbb{R}(\mathbf{d})\}$. Thus there exist integers $a$ and $b$ such that $\phi(\mathbf{d}) = \{k \in \mathbb{Z} : a \leq k \leq b\}$. For each $c \in \phi(\mathbf{d})$, let $\mathbb{R}(\mathbf{d};\ \phi = c)$ denote the subgraph of the graph $\mathbb{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number $c$. We consider the problem of determining the structure of induced subgraph $\mathbb{R}(\mathbf{d};\ \phi = c)$. In general, what is the structure of $\mathbb{R}(\mathbf{d};\ \phi = c)$? In particular, are these graphs connected? If $\mathbb{R}(\mathbf{d};\ \phi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number $c$ by beginning with one such realization and applying a suitable sequence of switchings to produce only graphs with decycling number $c$.

Bau and Beineke posed the following problems: Which cubic graphs $G$ of order $2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

We answered the problem by finding all cubic graphs $G$ of order $2n$ with $\phi(G) = \lceil \frac{n+1}{2} \rceil$ in [32]. Furthermore we proved that the induced subgraph $\mathbb{R}(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil)$ is connected. We proved in the same paper that if $\mathbb{R}(3^{2n})$ is the class of all cubic graphs of order $2n$, then $\min\{\phi(G) : G \in \mathbb{R}(3^{2n})\} = \lceil \frac{n+1}{2} \rceil$. Thus to answer the problem is equivalent to finding all cubic graphs of order $2n$ in $\mathbb{R}(3^{2n})$ having minimum cardinality of decycling set.

**Conjecture.** If $c \in \phi(r^n)$, then the graph $\mathbb{R}(r^n;\ \phi = c)$ is connected.

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# List of Notation

| Symbol | Meaning | Page |
|---|---|---|
| $V, V(G)$ | vertex set | 9 |
| $E, E(G)$ | edge set | 9 |
| $G = (V, E)$ | graph | 9 |
| $K_n$ | the complete graph | 16 |
| $G - v, G - X$ | deleting vertices from $G$ | 16 |
| $G + e, G + Y$ | adding edges to $G$ | 16 |
| $G - e, G - Y$ | deleting edges from $G$ | 16 |
| $G \cong H$ | isomorphic relation | 17 |
| $G[U]$ | induced subgraph | 17 |
| $K_{m,n}$ | complete bipartite graph | 17 |
| $\overline{G}$ | complementary graph | 18 |
| $P$-graph | graphs with property $P$ | 18 |
| $P_n$ | path of order $n$ | 18 |
| $C_n$ | cycle of order $n$ | 18 |
| $d_G(v)$ | degree of $v$ in $G$ | 19 |
| $\delta(G)$ | the minimum degree of $G$ | 19 |
| $\Delta(G)$ | the maximum degree of $G$ | 19 |
| $d(u, v)$ | distance between $u$ and $v$ | 22 |
| $O(G)$ | number of odd components of $G$ | 27 |
| $M(G)$ | incident matrix of $G$ | 27 |
| $\mathbf{T(G)}$ | set of all spanning trees of $G$ | 29 |
| $\min(\pi, \mathbb{J})$ | $\min\{\pi(G) : G \in \mathbb{J}\}$ | 30 |
| $\max(\pi, \mathbb{J})$ | $\max\{\pi(G) : G \in \mathbb{J}\}$ | 30 |
| $\mathbb{R}(\mathbf{d})$ | the graph of realizations | 30 |
| $\sigma,\ \sigma(a, b; c, d)$ | switching operation | 30 |
| $\chi(G)$ | the chromatic number of $G$ | 32 |
| $\omega(G)$ | the clique number of $G$ | 32 |
| $f(G)$ | the maximum induced forest of $G$ | 32 |
| $\phi(G)$ | the decycling number of $G$ | 32 |
| $\alpha_0(G)$ | the independence number of $G$ | 32 |

| Symbol | Meaning | Page |
|---|---|---|
| $\beta_0(G)$ | the vertex covering number of $G$ | 33 |
| $\alpha_1(G)$ | the matching number of $G$ | 33 |
| $\beta_1(G)$ | the edge covering number of $G$ | 33 |
| $\gamma(G)$ | the domination number of $G$ | 33 |
| $p(G)$ | the partition number of $G$ | 36 |
| $\min(\pi, \mathbf{d})$ | $\min\{\pi(G) : G \in \mathbb{R}(\mathbf{d})\}$ | 37 |
| $\max(\pi, \mathbf{d})$ | $\max\{\pi(G) : G \in \mathbb{R}(\mathbf{d})\}$ | 37 |
| $N_G(v)$ | neighborhood of $v$ in $G$ | 38 |
| $(\Omega, p)$ | finite probability space | 39 |
| $E(X)$ | expectation of $X$ | 39 |
| $\mathbb{G}(\Delta, n)$ | $(\Delta, n)$-graph | 41 |

# Chapter 3

# MAIN RESULTS

The purpose of this chapter is to present our work which has been carried out with financial support by the **Thailand Research Fund** under the Grant Number **BRG4880020** from July 2005 to July 2008. The project title is:

**"EXTREMAL PROBLEM ON GRAPH PARAMETERS".**

We have published 10 papers, 8 of which in graph theory context and the other 2 papers in combinatorial designs as shown below. Details of each paper can be found in the appendices.

**A1.** N. Punnim, **Switchings, Realizations, and Interpolation Theorems for Graph Parameters, International J. of Mathematics and Mathematical Sciences, 2005 (13)**, 2095-2117.

**A2.** N. Punnim, **Decycling connected regular graphs, Austral. J. of Combinatorics, 35** (2006), 155-169.

**A3.** A. Chantasartrassmee and N. Punnim, **Constrained switchings in cubic graphs, Ars. Combinatoria., 81** (2006), 65-79.

**A4.** N. Punnim, **The decycling number of regular graphs, Thai J. of Mathematics, 4 (1)** (2006) 145-161.

**A5.** N. Punnim, V. Seanpholphat and S. Thaithae **Almost Hamiltonian Cubic Graphs, International Journal of Computer Science and Network Security, 7 (1)** (2007) 83-86.

**A6.** N. Punnim, **The decycling number of cubic planar graphs, Lecture Notes in Computer Science (LNCS), 4381** (2007) 149-161.

**A7.** N. Punnim, **Interpolation theorems in jump graphs, Austral. J. of Combinatorics, 39** (2007), 103-114.

**A8.** N. Punnim and D. G. Sarvate, **A 3-$(v, 4, 1)$ packing construction, East-West J. of Mathematics, 7 (2)** (2007), 199-202.

**A9.** N. Punnim and D. G. Sarvate, **A construction for group divisible designs with two groups,** to appear in **CONGRESSUS NUMERANTIUM, (2008).**

**A10.** N. Punnim and S. Thaithae, **The Hamiltonian number of cubic graphs,** to appear in **Lecture Notes in Computer Science (LNCS), (2008).**

# SWITCHINGS, REALIZATIONS, AND INTERPOLATION THEOREMS FOR GRAPH PARAMETERS

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Interpolation theorems on several graph parameters obtained in the past few years will be reviewed in this paper. Some simplified proofs are provided. Open problems in this direction are reviewed.

## 1. Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [4]. Let $G = (V,E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph $G$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set $S$ and therefore we define $n = |V|$ the *order* of $G$ and $m = |E|$ the *size* of $G$. To simplify writing, we write $e = uv$ for the edge $e$ that joins the vertex $u$ to the vertex $v$. The *degree* of a vertex $v$ of a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph $G$ is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by $S$ in $G$. For a graph $G$ and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ and $X \subseteq E(\overline{G})$, we denote by $G + X$ the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs $G$ and $H$, we define $G \cup H$ (their union) by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs, since the operation "$\cup$" is associative. For a positive integer $p$ and a graph $G$, $pG$ is denoted for the union of $p$ copies of $G$. A graph $G$ is said to be *$r$-regular* if all of its vertices have degree $r$. A 3-regular graph is called a *cubic graph*.

Let $G$ be a graph of order $n$ and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is a *graphic degree sequence* if it is a degree sequence of some graph $G$. In this case, $G$ is called a *realization* of $\mathbf{d}$.

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An algorithm for determining whether or not a given sequence of nonnegative integers is graphic was independently obtained by Havel [22] and Hakimi [14]. We state their results in the following theorem.

THEOREM 1.1. *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a nonincreasing sequence of nonnegative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'. \tag{1.1}$$

*Then* $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

Let $G$ be a graph and let $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}. \tag{1.2}$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation*. It is easy to see that the graph obtained from $G$ by a switching has the same degree sequence as $G$. The following theorem has been shown by Havel [22] and Hakimi [14].

THEOREM 1.2. *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* $G_1$ *and* $G_2$ *are any two realizations of* $\mathbf{d}$*, then* $G_2$ *can be obtained from* $G_1$ *by a finite sequence of switchings.*

As a consequence of Theorem 1.2, Eggleton and Holton [10] defined in 1979 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

THEOREM 1.3. *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

The following theorem was shown by Taylor [38] in 1980.

THEOREM 1.4. *For a graphic degree sequence* $\mathbf{d}$*, let* $\mathscr{C}\mathcal{R}(\mathbf{d})$ *be the set of all connected realizations of* $\mathbf{d}$*. Then the induced subgraph* $\mathscr{C}\mathcal{R}(\mathbf{d})$ *of* $\mathcal{R}(\mathbf{d})$ *is connected.*

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If $f$ is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, $f$ is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers $x$ and $y$ such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \le k \le y\}. \tag{1.3}$$

If $f$ is an interpolation graph parameter with respect to $\mathbb{J}$, $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathcal{R}(\mathbf{d})$, we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$, respectively, and in the case where $\mathbb{J} = \mathscr{C}\mathcal{R}(\mathbf{d})$, we write $\mathrm{Min}(f, \mathbf{d})$ and $\mathrm{Max}(f, \mathbf{d})$ for $\min(f, \mathscr{C}\mathcal{R}(\mathbf{d}))$ and $\max(f, \mathscr{C}\mathcal{R}(\mathbf{d}))$, respectively.

## 2. Interpolation theorems

Studying interpolation theorems for graph parameters may be divided into two parts: the first part deals with the question that given a graph parameter $f$ and a subset $\mathbb{J}$ of $\mathbb{G}$, does

$f$ interpolate over $\mathbb{J}$? If $f$ interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f,\mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f,\mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$; thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f,\mathbf{d})$ and $\max(f,\mathbf{d})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph $G$ contains spanning trees having $m$ and $n$ end-vertices, with $m < n$, does $G$ contain a spanning tree with $k$ end-vertices for every integer $k$ with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. Details can be found in [15, 16, 17, 18, 19, 20, 21]. Another interpolation theorem for the graph parameter $\chi + \overline{\chi}$ was obtained by Fink [12].

Let $\mathbf{d}$ be a graphic degree sequence. It was shown that the graph $\mathcal{R}(\mathbf{d})$ is connected and its subgraph induced by the set $\mathscr{C}\mathcal{R}(\mathbf{d})$ is also connected.

THEOREM 2.1. *For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|f(G) - f(G^\sigma)| \le 1$, then $f$ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$.*

THEOREM 2.2. *For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|f(G) - f(G^\sigma)| \le 1$, then $f$ is an interpolation graph parameter with respect to $\mathscr{C}\mathcal{R}(\mathbf{d})$.*

THEOREM 2.3. *Let $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$ and the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathbb{J}$ be connected. For a graph $G$ of degree sequence $\mathbf{d}$ and a switching $\sigma$ if $|f(G) - f(G^\sigma)| \le 1$, then $f$ is an interpolation graph parameter with respect to $\mathbb{J}$.*

We will now review interpolation results on various graph parameters with respect to $\mathcal{R}(\mathbf{d})$. Since our work started with the graph parameter $\chi$, we first state the definition of $\chi$.

Graph coloring takes its name from the map-coloring application. We assign labels to vertices when the numerical value of labels is unimportant, we call them *colors* to indicate that they may be elements of any set.

A *$k$-coloring* of a graph $G = (V,E)$ is a partition of its vertex set $V$ as $V_1 \cup V_2 \cup \cdots \cup V_k$ such that no two vertices in $V_i$ $(1 \le i \le k)$ are adjacent. The $V_i$'s are called the *color classes*. A function $f : V \to \{1,2,\ldots,k\}$ such that $f(v) = i$ for each $v \in V_i$ $(1 \le i \le k)$ is called a *color function*. If $G$ has a $k$-coloring, it is said to be *$k$-colorable* and the minimum integer $k$ for which $G$ is $k$-colorable is called the *chromatic number* of $G$ and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that $G$ is *$k$-chromatic*.

*Remark 2.4.* In a proper coloring, each color class contains no edge, so $G$ is $k$-colorable if and only if $G$ is a $k$-partite graph. Thus a graph is 2-colorable if and only if it is bipartite. Thus a graph containing an odd cycle must be at least 3-colorable.

We proved in [25] the following results.

THEOREM 2.5. *Let $G$ be a graph and let $\sigma$ be a switching on $G$. Then $|\chi(G) - \chi(G^\sigma)| \le 1$.*

*Proof.* Without loss of generality, we can assume that $\chi(G) < \chi(G^\sigma)$. Since $G^\sigma$ is the graph obtained from $G$ by a switching, there exist $a,b,c,d \in V(G)$ such that $ab, cd \in E(G)$,

$ac, bd \notin E(G)$, $ac, bd \in E(G^\sigma)$, and $ab, cd \notin E(G^\sigma)$. Let $\pi : V(G) \to \{1,2,\ldots,\chi(G)\}$ be a color function of $G$ such that $\pi(a) = i$, $\pi(b) = j$, $\pi(c) = p$, and $\pi(d) = q$. Then we will define $\pi' : V(G^\sigma) \to \{1,2,\ldots,\chi(G),\chi(G)+1\}$ by $\pi'(c) = \pi'(d) = \chi(G)+1$ and $\pi'(v) = \pi(v)$ for all $v \in V(G^\sigma) - \{c,d\}$. Thus the graph $G^\sigma$ is $(\chi(G)+1)$-colorable. This completes the proof. □

A maximal complete subgraph of a graph $G$ is called a *clique* of $G$. The maximum order of clique of $G$ is called the *clique number* of $G$ and is denoted by $\omega(G)$.

In general, there is no formula for the chromatic number of a graph. In fact, determining the chromatic number of even a relatively small graph is often a challenging problem. However, lower bounds for the chromatic number of a graph $G$ can be given in terms of the clique number of $G$. That is for any graph $G$, $\chi(G) \geq \omega(G)$.

We proved in [26]the following results on interpolation theorems on $\omega$.

THEOREM 2.6. *Let $G$ be a graph and let $\sigma$ be a switching on $G$. Then $|\omega(G) - \omega(G^\sigma)| \leq 1$.*

*Proof.* Let $\sigma(a,b;c,d) = \sigma$ be a switching on $G$, such that $ab, cd \in E(G)$, $ac, bd \notin E(G)$. Then $G^\sigma$ is the graph obtained from $G$ by deleting edges $ab$, $cd$ and adding edges $ac$, $bd$. Since vertices $a$ and $c$ cannot lie in the same complete subgraph of $G$, $\omega(G^\sigma) \geq \omega(G) - 1$. By symmetry of switching, we may assume that $\omega(G) \geq \omega(G^\sigma)$. Thus $\omega(G^\sigma)$ is either $\omega(G) - 1$ or $\omega(G)$. In both cases, we have $|\omega(G) - \omega(G^\sigma)| \leq 1$. □

*Acyclic graph* is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree. Since a tree is connected, every two vertices in a tree are connected by a unique path.

Let $G$ be a graph and $F \subseteq V(G)$, $F$ is called an *induced forest* of $G$, if the induced subgraph $G[F]$ of $G$ contains no cycle. For a graph $G$, we define $I(G)$ as

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}. \tag{2.1}$$

We proved in [28] the following results on interpolation theorems on the graph parameter $I$.

THEOREM 2.7. *If $S$ is any subset of vertices of $G$ such that $G[S]$ is a forest, and $\sigma$ is any switching on $G$, then $G^\sigma[S]$ contains at most one cycle.*

COROLLARY 2.8. *If $\sigma$ is any switching on $G$, then $|I(G) - I(G^\sigma)| \leq 1$.*

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph $G$ is difficult even for some simply defined graphs. For a graph $G$, this minimum number is known as the *decycling number* of $G$, and is denoted by $\phi(G)$. It is easy to see that for a graph $G$ of order $n$, $\phi(G) + I(G) = n$. Thus the interpolation result for $\phi$ in $\mathcal{R}(\mathbf{d})$ is easily obtained.

A subset $U$ of the vertex set $V$ of a graph $G = (V,E)$ is said to be an *independent set* of $G$ if the induced subgraph $G[U]$ of $G$ is an empty graph. An independent set of $G$ with maximum number of vertices is called a *maximum independent set* of $G$. The number of vertices of a maximum independent set of $G$ is called the *independence number* of $G$ and is denoted by $\alpha_0(G)$.

It is clear that $\alpha_0(G)$ is a graph parameter and $\alpha_0(G) = \omega(\overline{G})$, for any graph $G$. Observe that for a graph $G$ and a switching $\sigma$ on $G$, $\overline{G}^{\sigma} = \overline{G^{\sigma}}$. Thus the interpolation result for $\alpha_0$ in $\mathcal{R}(\mathbf{d})$ follows directly from the graph parameter $\omega$.

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A *vertex cover* of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the *vertex covering number* of $G$ and is denoted by $\beta_0(G)$.

Within a set of people, some pairs are compatible as roommates; under what condition can we pair them all up? Many applications of graphs involve such pairings. Problem of job assignment with qualified applicants is also an example of matching.

A subset $M$ of the edge set $E$ of a graph $G = (V, E)$ is an *independent edge set* or *matching* in $G$ if no two distinct edges in $M$ have a common vertex. A matching $M$ is *maximum* in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality of a maximum matching of $G$ is denoted by $\alpha_1(G)$ and is called the *matching number* of $G$.

There is an analogous covering concept for edges.

An edge of a graph $G = (V, E)$ is said to cover the two vertices incident with it. An *edge cover* of a graph $G$ is a set of edges covering all the vertices of $G$. The minimum cardinality of an edge cover of $G$ is called the *edge covering number* of $G$ and is denoted by $\beta_1(G)$.

THEOREM 2.9 [31]. *If $G$ is a graph with $\alpha_1(G) = \alpha_1$ and $\sigma$ is a switching on $G$, then $\alpha_1(G^{\sigma}) \geq \alpha_1 - 1$.*

*Proof.* Let $M$ be an independent set of edges of $E$ with $|M| = \alpha_1(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on $G$. If $\{ab, cd\} \cap M = \emptyset$, then $|M| = |M^{\sigma}|$. If $\{ab, cd\} \subseteq M$, then $|M| = |M^{\sigma}|$. Finally, if $M$ contains exactly one edge from the set $\{ab, cd\}$, then $|M^{\sigma}| = |M| - 1$. Therefore, $\alpha_1(G^{\sigma}) \geq \alpha_1 - 1$. □

COROLLARY 2.10 [31]. *If $\sigma$ is a switching on $G$, then $|\alpha_1(G) - \alpha_1(G^{\sigma})| \leq 1$.*

Let $n$ be a positive integer. The *star* of order $n+1$ is the complete bipartite graph $K_{n,1}$. The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest number of stars. This is equivalent to our next graph parameter.

A *dominating set* of a graph $G = (V, E)$ is a subset $D$ of $V$ such that each vertex of $V - D$ is adjacent to at least one vertex of $D$. The *domination number* $\gamma(G)$ is the cardinality of a minimal dominating set with the least number of elements.

THEOREM 2.11 [31]. *If $G$ is a graph with $\gamma(G) = \gamma$ and $\sigma$ is a switching on $G$, then $\gamma(G^{\sigma}) \leq \gamma + 1$.*

*Proof.* Let $D$ be a dominating set of vertices of $V$ with $|D| = \gamma(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on $G$. If $\{a, b, c, d\} \cap D = \emptyset$, then $|D| = |D^{\sigma}|$. If $\{a, b, c, d\} \subseteq D$, then $|D| = |D^{\sigma}|$. If $a \in D$ and $b, c, d \in V - D$, then $D \cup \{b\}$ is a dominating set of $G^{\sigma}$. If $a, b \in D$ and $c, d \in V - D$, then $D$ is a dominating set of $G^{\sigma}$. If $a, c \in D$, then $D \cup \{b\}$ is a dominating set of $G^{\sigma}$. Finally, if $a, b, c \in D$ and $d \in V - D$, then $D \cup \{d\}$ is a dominating set of $G^{\sigma}$. Thus $\gamma(G^{\sigma}) \leq \gamma(G) + 1$. □

COROLLARY 2.12 [31]. *If $\sigma$ is a switching on $G$, then $|\gamma(G) - \gamma(G^{\sigma})| \leq 1$.*

*Proof.* Since a switching is symmetric, we may assume that $\gamma(G) \leq \gamma(G^\sigma)$. By Theorem 2.11, $\gamma(G^\sigma)$ is either $\gamma(G)+1$ or $\gamma(G)$. In both cases, we have $|\gamma(G) - \gamma(G^\sigma)| \leq 1$. □

Gallai [13] and Norman-Rabin [23] proved the following results concerning relationship between $\alpha_0$ and $\beta_0$, and $\alpha_1$, and $\beta_1$, respectively.

THEOREM 2.13 [13]. *For any graph $G$ of order $n$, $\alpha_0 + \beta_0 = n$.*

THEOREM 2.14 [23]. *For any graph $G$ of order $n$ and $\delta \geq 1$, $\alpha_1 + \beta_1 = n$.*

The interpolation property of the graph parameters $\beta_0$ and $\beta_1$ follows from the last two theorems. Thus combining the results in this section, we can conclude the following results.

THEOREM 2.15. *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be a graphic degree sequence. Then $\chi$, $\omega$, $I$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$, and $\gamma$ are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$.*

THEOREM 2.16. *Let $f \in \{\chi, \omega, I, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(f)$ and $b := b(f)$ such that $\mathbf{d}$ has a realization $G$ with $f(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

## 3. Extremal graph parameters

In this section, we discuss the second part of interpolation theorems for graph parameters. We first recall what we mean by an extremal problem in graph theory. An *extremal problem* asks for minimum and maximum values of a function over a class of objects.

*Remark 3.1.* Proving that $A$ is the minimum of $f(G)$ for graphs in a class $\mathbb{J}$ requires showing two things:

(1) $f(G) \geq A$ for all $G \in \mathbb{J}$;

(2) $f(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of $f$.

Changing "$\geq$" to "$\leq$" yields the criteria for a maximum.

It will be seen that we have produced several results in the extremal graph theory for graph parameters $\chi$, $\omega$, $I$, $\phi$, and $\alpha_1$ in the class of all $r$-regular graphs of order $n$ and some other related classes.

We start with the graph parameter $\chi$. In the graph-theoretic colloquium at Smolenice in 1963, Dirac conjectured that the chromatic number of a proper regular subgraph of a complete $n$-gon is at most $3n/5$. Erdős and Gallai answered this conjecture immediately and presented their result during the conference. Their article was entitled "Solution to a problem of Dirac," and it was appeared in the proceedings of the symposium, Smolenice, in 1964. In fact, the result was more than of what Dirac asked. They have, in addition, found all regular graphs reaching the upper bound as stated in the following theorem.

THEOREM 3.2. *An $r$-regular graph $G$ of order $n > r+1$ has chromatic number $k \leq 3n/5$, with equality holds if and only if the complementary graph $\overline{G}$ of $G$ is a union of 5 cycles.*

In their proof, they used the concept of the *covering number* of a graph $G$. A *covering* of a graph $G$ is a partition $P$ of $V(G)$ such that for each $V_i \in P$, the induced subgraph $G[V_i]$ in $G$ is a complete graph. The covering number of $G$, denoted by $c(G)$, is defined by

$$c(G) := \min\{|P| \mid P \text{ is a covering of G}\}. \tag{3.1}$$

It is easy to see that for any graph $G$, $c(G) = \chi(\bar{G})$. We extended their result by improving the upper bound in [24].

Let $f$ be an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$. It is clear that $\{f(G) : G \in \mathcal{R}(\mathbf{d})\}$ is uniquely determined by $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$. We will discuss in this section those extremal values for various kinds of graph parameters. For simplicity reason, we will consider the class of regular graphs and some other related classes of graphs.

For the graph parameter $\chi$, we obtained in [25] the following results.

THEOREM 3.3. *If $r \geq 2$ and $n \geq 2r$, then*

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases} \tag{3.2}$$

THEOREM 3.4. *If $r \geq 2$, then*

$$\begin{aligned} \min(\chi, r^{r+1}) &= \max(\chi, r^{r+1}) = r+1, \\ \min(\chi, r^{r+2}) &= \max(\chi, r^{r+2}) = \frac{r+2}{2}. \end{aligned} \tag{3.3}$$

The proofs of Theorems 3.3 and 3.4 were based on graph constructions.

THEOREM 3.5. *For any $r \geq 4$ and odd integer $s$ such that $3 \leq s \leq r$, let $q$ and $t$ be integers satisfying $r + s = sq + t, 0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases} \tag{3.4}$$

THEOREM 3.6. *For any even integer $r \geq 6$ and any even number $s$ such that $4 \leq s \leq r$, let $q$ and $t$ be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases} \tag{3.5}$$

The proofs of Theorems 3.5 and 3.6 were based on the concept of the covering number of graphs and graph constructions depending on the parity of $s$.

THEOREM 3.7. *Let $r \geq 2$. Then*

(1)

$$\max(\chi, r^{2r}) = r, \tag{3.6}$$

(2)

$$\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases} \tag{3.7}$$

(3)

$$\max(\chi, r^n) = r+1 \quad \text{for } n \geq 2r+2. \tag{3.8}$$

Brooks' theorem [5] and graph constructions were applied to prove this theorem.

THEOREM 3.8. *For any $r$ and $s$ such that $3 \leq s \leq r-1$,*

(1)

$$\max(\chi, r^{r+s}) \geq \frac{r+s}{2} \quad \text{if } r+s \text{ is even}, \tag{3.9}$$

(2)

$$\max(\chi, r^{r+s}) \geq \frac{r+s-1}{2} \quad \text{if } r+s \text{ is odd}. \tag{3.10}$$

Purely graph constructions were used in the proof of this theorem. The exact values of $\max(\chi, r^n)$ are not easy to obtain when $r+3 \leq n \leq 2r-1$. Theorem 3.2 was proved by Erdős and Gallai [11] to provide an upper bound in the class of connected proper regular graphs of order $n$.

It should be noted that the bound given by Theorem 3.2 does not depend on $r$ and this bound may be very far from the actual values of the chromatic numbers. Also the exact value of $\max(\chi, r^{r+3})$ was obtained by Theorem 3.2. We gave in [24] the following definition.

Let $j$ be a positive integer. An *$F(j)$-graph* is a $(j-1)$-regular graph $G$ of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$.

It is easy to see that $F(3)$-graph is a unique graph $C_5$ and $f(3) = 5$. We will see later that $F(j)$-graphs, $j \geq 5$, are not unique.

We found $F(j)$-graphs for all odd integers $j$ as we state in the following theorem.

THEOREM 3.9 [24]. *For odd integer $j$ with $j \geq 3$, $f(j) = (5/2)(j-1)$ if $j \equiv 3 (\mathrm{mod}\, 4)$ and $f(j) = 1 + (5/2)(j-1)$ if $j \equiv 1 (\mathrm{mod}\, 4)$.*

THEOREM 3.10 [24]. *Any $r$-regular graph of order $n$ with $n - r = j$ odd and $j \geq 3$ has chromatic number at most $((f(j)+1)/2f(j)) \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

Note that our results in Theorems 3.9 and 3.10 can be considered as a generalization of Theorem 3.2 as $F(3)$-graph is $C_5$ and $f(3) = 5$.

*Problem 3.11.* Find an $F(j)$-graph for even integer $j \geq 4$.

*Problem 3.12.* Find the value of $\max(\chi, r^{r+j})$ for even integer $j$ and $4 \leq j \leq r-2$.

For the graph parameter $\omega$, we completely answered in [26] the second part of interpolation theorem with respect to the class of $r$-regular graphs of order $n$ as stated in the following results.

THEOREM 3.13. *Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Then $\omega(G) \geq n/(n-\Delta)$.*

Since the complete graph $K_n$ is a clique, $\min(\omega, r^{r+1}) = \max(\omega, r^{r+1}) = r+1$. Given positive integers $n$ and $k$ with $k \leq n$, there exists a connected graph $G$ of order $n$ with $\omega(G) = k$. However, if $G$ is regular, we have the following remarkable result.

THEOREM 3.14. *Let $\mathbf{d} = r^n$ be a graphic degree sequence with $r+2 \leq n \leq 2r+1$. Then $\max(\omega, r^n) = \lfloor n/2 \rfloor$.*

THEOREM 3.15. *For any $r \geq 6$ and odd integer $s$ such that $5 \leq s < r$, let $q$ and $t$ be integers satisfying $r+s = sq+t$, $0 \leq t < s$. Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases} \tag{3.11}$$

THEOREM 3.16. *For any even integer $r \geq 6$ and any even number $s$ such that $4 \leq s \leq r$, let $q$ and $t$ be integers satisfying $r+s = sq+t$, $0 \leq t < s$. Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases} \tag{3.12}$$

Observe that results of Theorems 3.15 and 3.16, respectively, coincide with Theorems 3.5 and 3.6 above.

For graph parameter $I$, we found in [27] a lower bound of $\min(I, \mathbf{d})$ by using *the probabilistic method*. In particular, we proved that if $G$ is a graph with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots d_n \geq 1$, then

$$I(G) \geq 2 \sum_{i=1}^{n} \frac{1}{d_i + 1}. \tag{3.13}$$

As an application, we found all minimum values of the order of maximum induced forest in a $(\Delta, n)$-graph.

First of all we would like to introduce basic tools from discrete probability theory which will be used for calculation of some bounds of graph parameters.

Let $(\Omega, p)$ be a finite *probability space*, where $\Omega$ is a finite set and $p$ is a probability function that maps from $\Omega$ into the interval $[0,1]$ with

$$\sum_{\omega \in \Omega} p(\omega) = 1. \tag{3.14}$$

A *random variable* $X$ on $\Omega$ is a mapping $X : \Omega \to \mathbb{R}$. We define a probability space on the image set $X(\Omega)$ as

$$p(X = x) := \sum_{X(\omega)=x} p(\omega). \tag{3.15}$$

The *expectation* $E(X)$ of $X$ is

$$E(X) = \sum_{\omega\in\Omega} p(\omega)X(\omega). \tag{3.16}$$

Now suppose $X$ and $Y$ are two random variables on $\Omega$, then the sum $X + Y$ is again a random variable, sharing a property $E(X + Y) = E(X) + E(Y)$.

Clearly, this can be extended to any finite linear combination of random variables (the linearity of expectation). Note that it is not necessary that the random variables are "independent."

Thus we have the following theorems.

THEOREM 3.17 [27]. *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 1. \tag{3.17}$$

*Then*

$$I(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}. \tag{3.18}$$

*Proof.* Let $G$ be an arbitrary graph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Denote the degree of $v_i$ by $d_i$. We choose with equal probability $1/n!$ a random permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ of $V$ and construct the following set $F_\pi$. We put $\pi_i$ into $F_\pi$ if and only if $\pi_i$ is adjacent to at most one $\pi_j$ $(j < i)$. It is clear that $F_\pi$ is an induced forest of $G$. Let $X(\pi) = |F_\pi|$ be the corresponding random variable. We have $X = \sum_{i=1}^{n} X_i$, where $X_i$ is the indicator random variable of the vertex $v_i$, in other words $X_i = 1$ if $v_i \in F_\pi$ and $X_i = 0$ if $v_i \notin F_\pi$. In order to calculate $E(X_i)$, we first calculate the number of permutations $\pi$ which contain the vertex $v_i$. Since $v_i$ has $d_i$ neighbors, we partition $n!$ permutations on $V$ into $\binom{n}{d_i+1}$ classes according to $d_i + 1$ positions that $v_i$ and its neighbors appeared. It is clear that each class contains $(d_i + 1)!(n - d_i - 1)!$ permutations and $2d_i!(n - d_i - 1)!$ of which have the property that $v_i \in F_\pi$. Thus

$$E(X_i) = \sum_{\pi} p(\pi)X_i(\pi) = \frac{2}{d_i + 1}. \tag{3.19}$$

Therefore,

$$E(X) = \sum_{i=1}^{n} E(X_i) = 2\sum_{i=1}^{n} \frac{1}{d_i + 1}. \tag{3.20}$$

□

COROLLARY 3.18. *If the average degree of $G$ is at most $d$, then $I(G) \geq 2n/(d+1)$.*

We now have the following corollary as a result of Caro [7].

COROLLARY 3.19. *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 0. \tag{3.21}$$

*Then*

$$\alpha_0(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}, \tag{3.22}$$

*where $\alpha_0(G)$ is the independence number of $G$.*

*Proof.* The proof follows easily from the fact that a forest is bipartite. □

COROLLARY 3.20. *Let $G$ be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \ldots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 0. \tag{3.23}$$

*Then*

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i}, \tag{3.24}$$

*where $\omega(G)$ is the clique number of $G$.*

*Proof.* The proof follows from the fact that $\omega(G) = \alpha_0(\overline{G})$. □

Let $G$ be a graph. The problem of determining the decycling number $\phi(G)$ of $G$ is equivalent to finding the greatest order of an induced forest $I(G)$ of the graph $G$ and the sum of the two numbers equals the order of the graph. Observe that for a minimum decycling set $S$ of a graph $G$, if $v \in S$, then there exists a connected component $C$ of $G - S$ such that $v$ is adjacent to at least two vertices of $C$. Thus $\Delta(G[S]) \leq \Delta(G) - 2$. With this observation, we find that if $G$ is an $r$-regular graph and $S$ is a minimum decycling set of $G$, the graph $G[S]$ may not be an $(r-2)$-regular graph. This causes a difficulty in finding $\max(\phi, r^n)$ if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the following class of graphs. Let $\Delta$ be a nonnegative integer and let $n$ be a positive integer such that $n \geq \Delta + 1$. Let $\mathbb{G}(\Delta, n)$ be the class of all graphs of order $n$ and of maximum degree $\Delta$. *The $(\Delta, n)$-graph* is a graph having $\mathbb{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

LEMMA 3.21 [27]. *The $(\Delta, n)$-graph is connected.*

*Proof.* For any graph $G \in \mathbb{G}(\Delta, n)$, if $F = K_{1,\Delta} \cup (n - \Delta - 1)K_1$ and $G \neq F$, $G$ can be obtained from $F$ by a finite sequence of adding edges. The proof is complete. □

THEOREM 3.22 [27]. *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ *be a graphic degree sequence and* $d_1 + 1 \leq n \leq 2d_1 + 1$. *Then*

(1) $\min(I, \mathbf{d}) = 2$ *if and only if* $d_1 = d_2 = d_3 = \cdots = d_n$ *and* $n = d_1 + 1$,

(2) *if* $\mathbf{d}$ *does not have a complete graph as its realization, then* $\min(I, \mathbf{d}) = 3$ *if and only if* $\overline{\mathbf{d}}$ *has a union of stars as its realization.*

THEOREM 3.23 [27]. *Let* $n = (\Delta + 1)q + t$, $0 \leq t \leq \Delta$. *Then*

(1) $\min(I, \mathbb{G}(\Delta, n)) = 2q$ *if* $t = 0$,

(2) $\min(I, \mathbb{G}(\Delta, n)) = 2q + 1$ *if* $t = 1$,

(3) $\min(I, \mathbb{G}(\Delta, n)) = 2q + 2$ *if* $2 \leq t \leq \Delta$.

We obtained in [28] the values of $\max(I, r^n)$, for all $r$ and $n$ as stated in the following theorems.

THEOREM 3.24.

$$\max(I, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \left\lfloor \dfrac{nr - 2}{2(r-1)} \right\rfloor & \text{if } n \geq 2r. \end{cases} \tag{3.25}$$

The values of $\min(I, r^n)$, for all $r$ and $n$, were obtained in [33] in terms of the graph parameter $\phi$ as stated in the following theorems. Note that $\min(I, r^n) + \max(\phi, r^n) = n$.

THEOREM 3.25. *For* $r \geq 3$, *and* $n = r + j$, $1 \leq j \leq r + 1$,

(1) $\min(I, r^n) = 2$ *if and only if* $n = r + 1$,

(2) $\min(I, r^n) = 3$ *if and only if* $n = r + 2$,

(3) $\min(I, r^n) = 4$ *for all even integers* $n$, $r + 3 \leq n$,

(4) $\min(I, r^n) = 4$ *for all odd integers* $n$, $r + 3 \leq n$ *and* $n \geq f(j)$,

(5) $\min(I, r^n) = 5$ *for all odd integers* $n$, $r + 3 \leq n$ *and* $n < f(j)$,

*where*

$$\begin{aligned} f(j) &= \frac{5}{2}(j - 1) \quad \text{if } j \equiv 3 (\mathrm{mod}\, 4), \\ f(j) &= 1 + \frac{5}{2}(j - 1) \quad \text{if } j \equiv 1 (\mathrm{mod}\, 4). \end{aligned} \tag{3.26}$$

THEOREM 3.26. *For* $n \geq 2r + 2$ *and* $r \geq 3$, *write* $n = (r + 1)q + t$, $q \geq 2$ *and* $0 \leq t \leq r$. *Then*

(1) $\min(I, r^n) = 2q$ *if* $t = 0$,

(2) $\min(I, r^n) = 2q + 1$ *if* $t = 1$,

(3) $\min(I, r^n) = 2q + 2$ *if* $2 \leq t \leq r - 1$,

(4) $\min(I, r^n) = 2q + 3$ *if* $t = r$.

In [32], we investigated the values of $\min(\alpha_1, r^n)$ and $\max(\alpha_1, r^n)$ for all $r$ and $n$. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, from now on we will consider $r \geq 2$ and $n \geq r + 1$.

We first investigate the value of $\max(\alpha_1, r^n)$ by the following theorem.

THEOREM 3.27. *For* $r \geq 2$, $n \geq r + 1$, *and* $nr \equiv 0 (\mathrm{mod}\, 2)$, *there exists an* $r$*-regular Hamiltonian graph of order* $n$. *In particular,* $\max(\alpha_1, r^n) = \lfloor n/2 \rfloor$.

*Proof.* Since there is a well-known result by Dirac (cited from [4]) that an $r$-regular graph of order $n$ with $r \geq n/2$ is Hamiltonian, we need to consider only when $r < n/2$. It is easy to construct a Hamiltonian graph with 10 or fewer vertices. It is also easy to construct such a graph with $r = 2$ or 3. Let $X = \{x_0, x_1, \ldots, x_{t-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{t-1}\}$, where $t \geq 5$. For an integer $r$ with $1 \leq r \leq t$, take the edge set

$$E = \{x_i y_{i+j} : i = 0,1,2,\ldots,t-1,\ j = 0,1,2,\ldots,r-1\}, \tag{3.27}$$

where all subscripts are taken mod $t$. It is clear that the graph $B(r;X,Y) = (X \cup Y, E)$ is an $r$-regular bipartite graph and it is Hamiltonian if $r \geq 2$. Suppose that $n = 2t+1$ and $r$ is an even integer with $4 \leq r \leq t$, the graph $G = (X \cup Y \cup \{u\}, E')$, where $E' = [E(B(r-2;X,Y)) - \{x_i y_i : i = 0,1,2,\ldots,r/2\}] \cup \{x_i x_{i+1}, y_i y_{i+1} : i = 0,1,2,\ldots,t-1\} \cup \{ux_i, uy_i : i = 0,1,2,\ldots,r/2\}$, is an $r$-regular Hamiltonian graph. □

We will use the generalized result of Tutte and the result of Wallis [41] to obtain all values of $\min(\alpha_1, r^n)$.

A 1-*factor* of a graph $G$ is a 1-regular spanning subgraph of $G$. A 1-*factorization* of $G$ is a set of pairwise edge-disjoint 1-factors which together contain each edge of $G$.

It is well known that $K_{2n}$ and $K_{n,n}$ have 1-factorizations for all positive integers $n$. The question of which graphs contain 1-factors is one that has attracted considerable attention. For a comprehensive review, we refer to the survey of Akiyama and Kano [1] and of Wallis in [9].

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte [40]. A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $O(G)$ the number of odd components of $G$. The following theorem is due to Tutte [40]

THEOREM 3.28. *A graph $G$ has a perfect matching if and only if*

$$O(G-S) \leq |S| \quad \forall S \subseteq V. \tag{3.28}$$

Berge [3] generalized Tutte's result and it makes easier for application.

THEOREM 3.29 [3]. *The number of edges in a maximum matching of a graph $G$ is $(1/2)(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)}\{O(G-S) - |S|\}$.*

Let $F(r,d)$ be the minimum order of an $r$-regular graph $G$ with $\alpha_1(G) = (1/2)(|V(G)| - d)$. It is clear that $|V(G)| \equiv d \pmod 2$. Wallis [41] found $F(r,2)$ for all $r \geq 3$. In other words, he proved the following theorem.

THEOREM 3.30. *Let $G$ be an $r$-regular graph with no 1-factor and no odd component. Then*

$$|V(G)| \geq \begin{cases} 3r+7 & \text{if } r \text{ is odd},\ r \geq 3, \\ 3r+4 & \text{if } r \text{ is even},\ r \geq 6, \\ 22 & \text{if } r = 4. \end{cases} \tag{3.29}$$

*Furthermore, no such graphs exist for $r = 1$ or 2.*

Suppose that $G$ is an $r$-regular graph with $\alpha_1(G) = (1/2)(|V(G)| - d)$. There exists a $k$-subset $K$ of $V(G)$ such that $O(G-K) = k+d$. If $k = 0$, then $r$ is even, $G$ contains $d$ odd components, and each component of $G$ has order at least $r+1$. Suppose $k \geq 1$ and $G-K$ has an odd component with $p$ vertices, where $p$ is less than or equal to $r$. The number of edges within the component is at most $(1/2)p(p-1)$. This means that the sum of degree of these $p$ vertices in $G-K$ is at most $p(p-1)$. But $G$ is an $r$-regular graph, so the sum of these $p$ vertices in $G$ is $pr$. The number of edges joining to the component $K$ must be at least $pr - p(p-1)$. For a fixed $r$ and for integer $p$ satisfying $1 \leq p \leq r$, the function $f(p) = pr - p(p-1)$, $1 \leq p \leq r$, has minimum value $f(1) = f(r) = r$. So any odd component with $r$ or less vertices is joined to $K$ by $r$ or more edges.

Suppose that there are $O_+$ odd components of $G-K$ with more than $r$ vertices and $O_-$ odd components with less than or $r$ vertices. Thus

$$O_+ + O_- = k+d, \qquad O_+ + rO_- \leq kr. \tag{3.30}$$

From these two relations, we have $O_+ \geq \lceil rd/(r-1) \rceil = d + \lceil d/(r-1) \rceil$ and $k \geq \lceil d/(r-1) \rceil$. Thus we have the following results.

THEOREM 3.31 [32]. *Let $r$ be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$.*

COROLLARY 3.32 [32]. *Let $r$ be an even integer, $r \geq 2$. If $n = (r+1)d + e$, $0 \leq e \leq r$, then* $\min(\alpha_1, r^n) = dr/2 + \lfloor (1+e)/2 \rfloor$.

Suppose that $r$ is odd and $r \geq 3$. Let $G$ be an $r$-regular graph of order $n$ such that $\alpha_1(G) = (1/2)(n-d)$. Then $d$ must be even. Put $d = 2q$. There exists a nonempty subset $K$ of $V(G)$ of cardinality $k$ such that $O(G-K) = k+2q$. By (3.30), we have

$$n \geq k + (r+2)O_+ \geq \left\lceil \frac{2q}{r-1} \right\rceil + (r+2)\left(2q + \left\lceil \frac{2q}{r-1} \right\rceil\right) = \left\lceil \frac{2q}{r-1} \right\rceil (r+3) + 2q(r+2). \tag{3.31}$$

Wallis [41] defined $G(x,y)$ to be a graph with $x+y$ vertices, $x$ being of degree $x+y-3$ and $y$ of degree $x+y-2$. $G(x,y)$ exists if and only if $y$ is even and $y \geq 2$. It is noted that for any graph $G(x,y)$, it has $y$ vertices of degree $r$ and $x$ vertices of degree $r-1$. Let $x_i, y_i, i = 1,2,\ldots,m$, be integers such that $G(x_i, y_i)$ exists for all $i = 1,2,\ldots,m$. We construct the graph

$$G(x_1,y_1) * G(x_2,y_2) * \cdots * G(x_m,y_m) \tag{3.32}$$

from disjoint copies of the graphs by inserting a new vertex, say $u$, and joining $u$ to all vertices of $G(x_i, y_i)$ which have the smallest degree, for $i = 1,2,\ldots,m$.

Using this notation, we see that for an odd integer $r \geq 3$ and $q = 1,2,\ldots,(r-1)/2$, for any odd positive integers $a_i, i = 1,2,\ldots,1+2q$, whose sum is $r$,

$$G_q = G(a_1, r+2-a_1) * G(a_2, r+2-a_2) * \cdots * G(a_{1+2q}, r+2-a_{1+2q}) \tag{3.33}$$

is an $r$-regular graph on $(r+2)(1+2q)+1$ vertices with $\alpha_1(G_q) = (1/2)(|V(G_q)| - 2q)$. We have the following results.

THEOREM 3.33 [32]. *For an odd integer $r \geq 3$, then*

(1) $F(r,2q) = (r+2)(1+2q)+1$, *for* $q = 1,2,\ldots,(r-1)/2$,

(2) *if* $q = ((r-1)/2)s + t$, $0 \leq t < (r-1)/2$, *then* $F(r,2q) = sF(r,r-1) + F(r,2t)$, *where* $F(r,0) = 0$.

COROLLARY 3.34 [32]. *Let $r$ be an odd integer, $r \geq 3$. If $F(r,2q) \leq n < F(r,2(q+1))$, then* $\min(\alpha_1, r^n) = (1/2)(n-2q)$.

## 4. Induced subgraphs of $\mathcal{R}(\mathbf{d})$

It was observed by Eggleton and Holton [10] that the graph $\mathcal{R}(\mathbf{d})$ is connected. It is natural to investigate connected subgraphs of $\mathcal{R}(\mathbf{d})$.

Let $P$ be a property which a graph may possess. Denote by $\mathcal{R}(\mathbf{d};P)$ the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by those vertices which correspond to graphs with property $P$. Property $P$ for which $\mathcal{R}(\mathbf{d};P)$ is connected is called *complete*. If a property $P$ is complete, we may find all the graphs of a given degree sequence with the property by switching constrained to graphs with the property.

Colbourn [8] showed that the property of being a tree is complete and Syslo [37] extended this to the property of being unicyclic. Taylor [38] generalized these results by showing that the property of being connected is complete. The property of being 2-connected was shown to be complete also by Taylor [39].

By using results in Section 2 and the results of Taylor, we immediately obtain the following interpolation theorems.

THEOREM 4.1. *Let $f \in \{\chi, I, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence* $\mathbf{d}$, *there exist integers $a := a(f)$ and $b := b(f)$ such that* $\mathbf{d}$ *has a connected realization $G$ with $f(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

THEOREM 4.2. *Let $f \in \{\chi, I, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence* $\mathbf{d}$, *there exist integers $a := a(f)$ and $b := b(f)$ such that* $\mathbf{d}$ *has a* 2-connected realization $G$ with $f(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.

Let $f$ be an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$. Then $\{f(G) : G \in \mathcal{R}(\mathbf{d})\}$ is uniquely determined by $\min(f,\mathbf{d})$ and $\max(f,\mathbf{d})$. In particular, if $f = \chi$, we can define the *chromatic range*, $\chi(\mathbf{d})$, as the interval of integers as

$$\chi(\mathbf{d}) := [a,b] = \{c \in \mathbb{Z} : a \leq c \leq b\}, \tag{4.1}$$

where $a = \min(\chi,\mathbf{d})$ and $b = \max(\chi,\mathbf{d})$. The following results concerning connected subgraphs of $\mathcal{R}(\mathbf{d})$ were obtained in [35].

We consider the problem of determining the structure of induced subgraphs of the graph of realizations of a degree sequence $\mathbf{d}$ with prescribed chromatic number. We obtained some significant results when $\mathbf{d}$ is a regular degree sequence.

Brooks [4] observed that every graph $G$ may be colored by $\Delta(G)+1$ colors and he characterized the graphs for which $\Delta(G)$ colors are not enough.

THEOREM 4.3. *Any graph $G$ satisfies $\chi(G) \le 1 + \Delta(G)$, with equality holds if and only if either of the following holds:*

(1) *some component of $G$ is the complete graph $K_{\Delta+1}$, where $\Delta = \Delta(G)$;*

(2) *some component of $G$ is an odd cycle, and $\Delta(G) = 2$.*

Note that Brooks' theorem implies $b = \max(\chi, \mathbf{d}) \le 1 + \Delta$, where $\Delta = \max\{d_1, d_2, \ldots, d_n\}$. Thus for a regular degree sequence $r^n$, we have $b = \max(\chi, r^n) \le 1 + r$. We found in [25] the corresponding value of $b$ for all values of $r$ and $n$ except the cases when $r$ and $n$ are even and $r + 4 \le n \le 2r - 2$. In general, the Brooks bound may be very far from the actual value.

For each $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; \chi = c)$ denote the subgraph of $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $c$. Similarly for any $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; \chi \le c)$ denote the subgraph of $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $p \le c$. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d}; \chi = c)$ and $\mathcal{R}(\mathbf{d}; \chi \le c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d}; \chi = c)$ and of $\mathcal{R}(\mathbf{d}; \chi \le c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d}; \chi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with chromatic number $c$ by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with chromatic number $c$. The same applies if $\mathcal{R}(\mathbf{d}; \chi \le c)$ is connected.

Note that $\chi(0^n) = \{1\}$, $\chi(1^n) = \{2\}$ ($n$ is even), $\chi(2^4) = \{2\}$, $\chi(2^n) = \{2,3\}$ if $n$ is even and $n \ge 6$, and $\chi(2^n) = \{3\}$ if $n$ is odd and $n \ge 3$. For $\chi(3^n)$, we have $\chi(3^4) = \{4\}$, $\chi(3^6) = \{2,3\}$, and $\chi(3^n) = \{2,3,4\}$ if $n$ is even and $n \ge 8$.

Let $G$ be a graph with $\chi(G) = k$ and let $\sigma$ be a switching on $G$. $\sigma$ is called a *$k$-safe switching* if $\chi(G^\sigma) = k$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a sequence of $k$-safe switchings if for each $i$, $i = 1, 2, \ldots, t$, $\chi(G^{\sigma_1\sigma_2\cdots\sigma_i}) = k$.

THEOREM 4.4 [35]. *If $r \ge 3$ and $\max(\chi, r^n) = r + 1$, then the graphs $\mathcal{R}(r^n; \chi = r + 1)$ and $\mathcal{R}(r^n; p \le r)$ are connected.*

*Proof.* Note that if $r \ge 3$, then $\max(\chi, r^n) = r + 1$ if and only if $n = r + 1$ or $n \ge 2r + 2$. Moreover, if $G$ is a realization of $r^n$ and $\chi(G) = r + 1$ if and only if $G$ has $K_{r+1}$ as a component, the theorem is true for $n = r + 1$. For $n \ge 2r + 2$, let $G_1$ and $G_2$ be any two realizations of $r^n$ such that $\chi(G_1) = \chi(G_2) = r + 1$. Thus $G_1 = K_{r+1} \cup H_1$ and $G_2 = K_{r+1} \cup H_2$. Since $H_1$ and $H_2$ are $r$-regular graphs of order $n - r - 1$, $H_2$ can be obtained from $H_1$ by a finite number of switchings. Thus $G_2$ can as well be obtained by those switchings. Furthermore, it is easy to observe that those switchings are $(r + 1)$-safe switchings.

Let $G$ be a realization of $r^n$ such that $\chi(G) \le r$. If $G$ is disconnected, then $G$ does not contain $K_{r+1}$ as its component. Thus there exists a suitable sequence of switchings which transforms $G$ to a connected realization of $r^n$ such that each resulting graph obtained in this transformation will have chromatic number less than or equal to $r$. By using the result by Taylor [38], the proof is complete. □

Let $C_m$ be the cycle of order $m$. Thus $C_m$ exists for all integer $m \ge 3$, we call $C_m$ *odd cycle* or *even cycle* according to whether $m$ is odd or even. A realization of $2^n$ can be written as $\bigcup_{i=1}^{t} C_{n_i}$, where $\sum_{i=1}^{t} n_i = n$. It is well known that if $G = \bigcup_{i=1}^{t} C_{n_i}$, then $\chi(G) = 2$ if and

only if for all $i$, $n_i$ is even. It is clear that for any two even cycles $C_r$ and $C_s$, there is a 2-safe switching which transforms these two cycles to $C_{r+s}$. Thus $\mathcal{R}(2^n; \chi = 2)$ is connected. The corresponding result can be obtained for the graph $\mathcal{R}(2^n; 3)$ with only one exception.

THEOREM 4.5 [35]. *If $3 \in \chi(2^n)$, then the graph $\mathcal{R}(2^n; \chi = 3)$ is connected if and only if $n \neq 10$.*

*Proof.* Observe that if $n$ is odd and $n \geq 3$, then $\chi(2^n) = \{3\}$. Thus the graph $\mathcal{R}(2^n; \chi = 3)$ is connected. If $n = 10$, we cannot transform $2C_5$ to $2C_3 \cup C_4$ without passing $C_{10}$. Thus the graph $\mathcal{R}(2^10; \chi = 3)$ is not connected. For $n \geq 12$, let $\bigcup_{i=1}^{t} C_{n_i}$ be a realization of $2^n$ and suppose that $n_1$ is odd. Then there is a sequence of 3-safe switchings which transforms $\bigcup_{i=1}^{t} C_{n_i}$ to $C_{n_1} \cup C_{n-n_1}$. Since $n \geq 12$, $C_5 \cup C_{n-5}$ can be transformed to $C_5 \cup C_3 \cup C_{n-8}$ and to $C_3 \cup C_{n-3}$. If $m$ is odd and $m \geq 7$, then $C_m \cup C_{n-m}$ can be transformed to $C_3 \cup C_{m-3} \cup C_{n-m}$ and then to $C_3 \cup C_{n-3}$. The proof is complete. □

THEOREM 4.6 [42, page 53]. *Let $G$ and $H$ be bipartite graphs with bipartition $(X, Y)$. If $d_G(v) = d_H(v)$ for all $v \in X \cup Y$, then there is a sequence of 2-safe switchings that transforms $G$ into $H$.*

The following theorem is a consequence of Theorem 4.6.

THEOREM 4.7 [35]. *If $2 \in \chi(r^n)$, then the graph $\mathcal{R}(r^n; \chi = 2)$ is connected.*

THEOREM 4.8 [35]. *If $c \in \chi(3^n)$, then the graph $\mathcal{R}(3^n; \chi = c)$ is connected.*

*Proof.* We have already proved when $c = 2, 4$. Note that $G \in \mathcal{R}(3^n)$ with $\chi(G) = 3$ if and only if $G$ does not contain $K_4$ as its component and $G$ has an odd cycle as its subgraph. It is easy to check that the theorem is true if $n \leq 8$. Let $G$ be a cubic graph of order $n \geq 10$ with $\chi(G) = 3$. We first suppose that $G$ has a triangle $T$ with $V(T) = \{x, y, z\}$ and $E(T) = \{xy, yz, xz\}$. If $x$ and $y$ have a common neighbor $v \in G - T$, then there exists an edge $ab$ in $G - T$ independent of the edge $xv$. Thus there is a 3-safe switching $\sigma$ such that $G^\sigma$ contains a triangle $T$ where any two vertices of $T$ have no common neighbor in $G - T$.

Now suppose that $G$ contains an odd cycle $C$ of smallest order $k \geq 5$. Let $x$, $y$ be two adjacent vertices in the cycle. Thus $x$ and $y$ have no common neighbor in $G$. Let $z$ be the neighbor of $y$ in $G - C$. Then there exists a 3-safe switching $\sigma$ such that $G^\sigma$ contains a triangle $T = \{x, y, z\}$. Thus for a graph $G \in \mathcal{R}(3^n)$ and $\chi(G) = 3$, there is a sequence of 3-safe switchings which transforms $G$ to $G'$ such that $G'$ contains a triangle $T' = \{x, y, z\}$ and $G' - T'$ is a graph having exactly 3 vertices of degree 2. Since $G$ is an arbitrary graph in $\mathcal{R}(3^n)$, the proof is complete. □

CONJECTURE 4.9. *If $c \in \chi(r^n)$, then the graph $\mathcal{R}(r^n; \chi = c)$ is connected.*

For the graph parameter $\phi$, we investigated the values of $\phi(G)$ where $G$ runs over the class of cubic graphs in [34]. As an application, we were able to answer a problem asked by Bau and Beineke [2].

For a graphic degree sequence $\mathbf{d}$, let $\phi(\mathbf{d}) = \{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$. Thus there exist integers $a$ and $b$ such that $\phi(\mathbf{d}) = \{k \in \mathbb{Z} : a \leq k \leq b\}$. For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; \phi = c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number $c$. We consider the problem of determining the structure

of induced subgraph $\mathcal{R}(\mathbf{d};\ \phi = c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\ \phi = c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\ \phi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number $c$ by beginning with one such realization and applying a suitable sequence of switchings to produce only graphs with decycling number $c$.

Bau and Beineke posed the following problem: Which cubic graphs $G$ with $|\ G\ | = 2n$ satisfy $\phi(G) = \lceil (n+1)/2 \rceil$?

We answered the problem by finding all cubic graphs $G$ of order $2n$ with $\phi(G) = \lceil (n+1)/2 \rceil$. Furthermore, we proved that the induced subgraph $\mathcal{R}(3^{2n};\ \phi = \lceil (n+1)/2 \rceil)$ is connected. We proved in the same paper that if $\mathcal{R}(3^{2n})$ is the class of all cubic graphs of order $2n$, then $\min\{\phi(G) : G \in \mathcal{R}(3^{2n})\} = \lceil (n+1)/2 \rceil$. Thus to answer the problem is equivalent to finding all cubic graphs of order $2n$ in $\mathcal{R}(3^{2n})$ having minimum cardinality of decycling set.

CONJECTURE 4.10. *If $c \in \phi(r^n)$, then the graph $\mathcal{R}(r^n;\ \phi = c)$ is connected.*

## 5. New approach on graph parameters

We have introduced a new area of research on graph parameters. Let $f$ be a graph parameter and let $f(n,k,r)$ be the set of all connected $r$-regular graphs $G$ of order $n$ and $f(G) = k$. We consider the problem of determining the set of all integers $r$ for which $f(n,k,r) \neq \varnothing$.

Let $\chi(n,k,r)$ be the set of all connected, $r$-regular, $k$-chromatic graphs on $n$ vertices. By using the results in [25], we completely solved the problem for the graph parameter $\chi$, for all $n$ and $k$ with $n \geq 2k$ as stated in the following theorems.

THEOREM 5.1 [29]. *Let $n$, $k$, $r$ be integers such that $3 \leq k \leq r \leq n-2$. Let $D(n,k) = \max\{r : \chi(n,k,r) \neq \varnothing\}$. Suppose $n = kt + l$ with $t \geq 2$ and $0 \leq l \leq k-1$. Let $\epsilon(n,t) = 1$ or $0$ according to whether $nt$ is odd or even. Then*

(1) $D(kt,k) = (k-1)t$,
(2) $D(kt+k-1,k) = (k-1)t + k - 3 - \epsilon(n,t+1)$,
(3) $D(kt+l,k) = (k-1)t + l - 1 - \epsilon(n,t)$, $1 \leq l \leq k-2$.

Under the conditions of Theorem 5.1, we get the following theorem.

THEOREM 5.2 [29]. *$\chi(n,k,r) \neq \varnothing$ if $k \leq r \leq D(n,k)$.*

It is interesting to solve the problem by removing the condition $n \geq 2k$. It is also interesting to solve the problem for other graph parameters such as $\omega$, $\phi$, $\alpha_0$, $\alpha_1$, and $\gamma$.

## 6. Applications

In 1963, Erdős and Gallai [11] proved that any regular graph on $n$ vertices has chromatic number $k \leq 3n/5$ unless the graph is complete. Commenting on their result in a personal communication, Erdős wrote,"probably such a graph exists for every $k \leq 3n/5$, except possibly for trivial exceptional cases."

Caccetta and Pullman [6] confirmed and strengthened their conjecture by showing that if $k > 1$, then for every $n \geq 5k/3$, there exists a connected, regular, $k$-chromatic graph

on $n$ vertices. By using the interpolation results on graph parameter $\chi$ in [25], we are able to provide an alternate proof of Erdős' question.

In [24] we generalized the result of Erdős and Gallai by introducing the concept of $F(j)$-graphs and showed that their result is a special case when $j = 3$.

Let $n$ be an integer with $n \geq 4$, $r = n - 3$ and $n = 5p + i$, $0 \leq i \leq 4$. Then an explicit formula for $\max(\chi, r^n)$ can be obtained as follows:

$$\max(\chi, r^n) = \begin{cases} 3p & \text{if } i = 0, 1, \\ 3p+1 & \text{if } i = 2, 3, \\ 3p+2 & \text{if } i = 4. \end{cases} \tag{6.1}$$

For an integer $n \geq 4$, let $n = 5p + i$, $0 \leq i \leq 4$, and $n = 3q + t$, $0 \leq t \leq 2$.

If $n$ is even and $n = 2m$, then

$$\chi(m^n) = [2, m], \qquad \chi((n-3)^n) = \left[q + t, 3p + \left\lfloor \frac{i}{2} \right\rfloor\right]. \tag{6.2}$$

If $n$ is odd, $n \geq 7$, and $n = 2m + 1$, then

$$\chi(r^n) = [3, r], \qquad \chi((n-3)^n) = \left[q + t, 3p + \left\lfloor \frac{i}{2} \right\rfloor\right], \tag{6.3}$$

where $r$ is an even integer and either $r = m - 1$ or $r = m$.

It is easy to check that

$$\chi(m^n) \cap \chi((n-3)^n) \neq \varnothing, \qquad \chi(r^n) \cap \chi((n-3)^n) \neq \varnothing. \tag{6.4}$$

Thus we have the following interpolation theorem.

THEOREM 6.1 [30]. (1) *If $n$ is even and $n \geq 6$, then there exists a connected noncomplete regular graph $G$ of order $n$ with $\chi(G) = k$ if and only if $k$ is an integer satisfying $2 \leq k \leq 3n/5$.*

(2) *If $n$ is odd and $n \geq 7$, then there exists a connected noncomplete regular graph $G$ of order $n$ with $\chi(G) = k$ if and only if $k$ is an integer satisfying $3 \leq k \leq 3n/5$.*

## 7. Suggested problems

This final section is to give some problems related to interpolation graph parameters. We first consider the graph parameter $\phi$. We have already mentioned that $\phi$ is an interpolation graph parameter with respect to $\mathscr{GR}(\mathbf{d})$. The values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$ have been obtained in [36]. Let $G$ be a connected $r$-regular graph and let $S$ be a minimum

decycling set of $G$. Since for any $v \in S$ there is a connected component $C$ of $G - S$ such that $v$ is adjacent to at least two vertices of $C$, there exists $u \in G - S$ such that $vu = e \in E(G)$ and $G - e$ is connected. Thus for two disjoint connected $r$-regular graphs $G$ and $H$ with minimum decycling sets $S$ and $T$ of $G$ and $H$, respectively, there exist $u \in S$, $v \in G - S$, $x \in T$, $y \in H - T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$, and $G - e$, $H - f$ are connected. A connected $r$-regular graph $K = ((G - e) \cup (H - f)) + \{ux, vy\}$ satisfies

$$\phi(K) \leq \phi(G \cup H) = \phi(G) + \phi(H), \tag{7.1}$$

and the following theorem holds.

THEOREM 7.1. *Let $r \geq 3$ and $nr$ be even. Then*

$$\mathrm{Min}(\phi, r^n) = \begin{cases} r - 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \left\lceil \dfrac{nr - 2n + 2}{2(r-1)} \right\rceil & \text{if } n \geq 2r. \end{cases} \tag{7.2}$$

Thus the values of $\mathrm{Min}(\phi, r^n)$, for all $r$ and $n$, are already obtained. Moreover, $\mathrm{Min}(\phi, r^n) = \min(\phi, r^n)$.

It is clear that $\mathrm{Max}(\phi, r^n) = \max(\phi, r^n)$ for all $r$ and $n$ with $r + 1 \leq n \leq 2r + 1$.

Let $G$ be a $K_5$-free graph of order $n$, $\Delta(G) = 4$. Let $F$ be a maximal induced forest of $G$. We denote by $c(F)$ the number of cycles in $G - F$. A pair $(X, Y)$, where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $c((F - X) \cup Y) < c(F)$. In general, we can define an interchangeable pair of vertices for a graph $G$ with $\Delta(G) > 4$ as follows. Let $G$ be a graph of order $n$, $\Delta(G) = \Delta > 4$, and $G$ contains no $K_{\Delta+1}$ as its component. Let $F$ be a maximal induced forest of $G$. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G - F$. A pair $(X, Y)$, where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $k((F - X) \cup Y) < k(F)$.

By using the notion of interchangeable pair of vertices, we obtained in [36] the following results.

THEOREM 7.2. *Let $n$ be an even integer where $n \geq 12$. Then*

$$\mathrm{Max}(\phi, 3^n) = \begin{cases} \dfrac{3}{8}n + \dfrac{1}{4} & \text{if } n \equiv 2 (\mathrm{mod}\, 8), \\ \left\lfloor \dfrac{3}{8}n \right\rfloor & \text{otherwise.} \end{cases} \tag{7.3}$$

THEOREM 7.3. *Let $G$ be a connected $r$-regular graph of order $n \geq 2r + 2$. Then $\phi(G) \leq n(r - 2)/r$ for all $r \geq 4$.*

THEOREM 7.4. *Let $n = rq + t$. Then $\mathrm{Max}(\phi, r^n) = n - 2q$ if $t = 0$, $\mathrm{Max}(\phi, r^n) = n - 2q - 1$ if $t = 1, 2$, $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ if $2t > r$, and $\mathrm{Max}(\phi, r^n) \in \{n - 2q - 2, n - 2q - 1\}$ if $3 \leq t \leq r/2$.*

CONJECTURE 7.5. $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ *if* $3 \leq t \leq r/2$, *for all* $r \geq 6$.

It is interesting to determine the values of $\mathrm{Min}(f, r^n)$ and $\mathrm{Max}(f, r^n)$ for other graph parameter $f$.

It is also interesting to consider the following problems.

*Problem 7.6.* Find all values of positive integers $n \geq 10$ such that the graph $\mathcal{R}(2^n)$ contains a Hamiltonian cycle.

*Problem 7.7.* Find the longest length of induced path in the graph $\mathcal{R}(2^n)$.

*Problem 7.8.* Find the longest length of induced path in the subgraph of $\mathcal{R}(3^{2n})$ induced by the set of 2-connected realizations of $3^{2n}$.

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# Decycling connected regular graphs

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**Abstract**

For a graph $G$ and $S \subseteq V(G)$, if $G - S$ is acyclic, then $S$ is said to be a *decycling set* of $G$. The cardinality of a smallest decycling set of $G$ is called the *decycling number* of $G$ and is denoted by $\phi(G)$. We prove in this paper that if $G$ runs over the set of connected graphs with a fixed degree sequence $\mathbf{d}$, then the values $\phi(G)$ completely cover a line segment $[A, B]$ of positive integers. Let $\mathcal{CR}(\mathbf{d})$ be the class of all connected graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$A := \mathrm{Min}(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\}$$

and

$$B := \mathrm{Max}(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$, where $r$ is the vertex degree and $n$ is the order of the graph, we obtain some significant results on the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$.

## 1. Introduction

Let $G$ be a connected graph and $X \subseteq E(G)$. Then the minimum $|X|$ such that $G - X$ is acyclic is known as the dimension of the cycle space of $G$ and it is equal to $|E(G)| - |V(G)| + 1$. It is natural to investigate the corresponding problem in terms of vertices, and this was indeed considered by Kirchhoff [8] in his work on spanning trees.

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph $G$ is difficult even for some simply defined graphs. For

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a graph $G$, this minimum number is known as the *decycling number* of $G$, and is denoted by $\phi(G)$. The class of those graphs $G$ for which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if $G$ has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $K_{p,q} = p - 1$ if $p \leq q$, where $K_n$ denotes the complete graph of order $n$ and $K_{p,q}$ denotes the complete bipartite graph with partite sets of cardinality $p$ and $q$. The exact values of decycling numbers for many classes of graphs were obtained and cited in [2].

We proved recently in [10] that if $G$ runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[a, b]$ of nonnegative integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. Thus for an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$ where $r$ is the vertex degree and $n$ is the number of graph vertices, we obtained in [10] the exact values of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ in all situations. It is natural to extend this problem to the class of connected graphs with a degree sequence $\mathbf{d}$. As a direct consequence of Taylor [15] and our result in [10], we have that if $G$ runs over the set of connected graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[A, B]$ of nonnegative integers. Let $\mathcal{CR}(\mathbf{d})$ be the class of all connected graphs having degree sequence $\mathbf{d}$. Thus for an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$A := \mathrm{Min}(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\}$$

and

$$B := \mathrm{Max}(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{CR}(\mathbf{d})\},$$

arise naturally. We will find the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$.

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph $G$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set $S$ and therefore we define $n = |V|$ to be the *order* of $G$ and $m = |E|$ the *size* of $G$. To simplify writing, we write $e = uv$ for the edge $e$ that joins the vertex $u$ to the vertex $v$. A *path* of length $k$ in a graph $G$, denoted by $P_k$, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of $G$ such that for all $i = 1, 2, \ldots, k-1$, $u_i u_{i+1}$ are edges of $G$. A $u, v$-path is a path which has $u$ as its first vertex and $v$ as its last vertex in the path. The *degree* of a vertex $v$ of a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph $G$ is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by $S$ in $G$. For a graph $G$, if $X \subseteq E(G)$, we denote by

$G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$, if $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ with $X \subseteq E(\overline{G})$, we denote by $G + X$ the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs $G$ and $H$, we define $G \cup H$, their union, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs, since the operation "$\cup$" is associative. For a graph $G$ and $s \in V(G)$, the *neighborhood* of $s$ in $G$ is defined by

$$N(s) = \{v \in V(G) : sv \in E(G)\}.$$

If $S \subseteq V(G)$, then we define

$$N(S) = \bigcup_{s \in S} N(s).$$

If $F \subseteq V(G)$, we write $N_F(S)$ for $N(S) \cap F$. A graph $G$ is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called a *cubic graph.*

Let $G$ be a graph of order $n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph $G$ is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph $G$. In this case, $G$ is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [7] and Hakimi [6]. We state their results in the following theorem.

**Theorem 1.1** *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

*Then* $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.* □

Let $G$ be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from $G$ by a switching has the same degree sequence as $G$. The following theorem has been shown by Havel [7] and Hakimi [6].

**Theorem 1.2** *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* $G_1$ *and* $G_2$ *are any two realizations of* $\mathbf{d}$*, then* $G_2$ *can be obtained from* $G_1$ *by a finite sequence of switchings.* □

As a consequence of Theorem 1.2, Eggleton and Holton [4] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two

vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

**Theorem 1.3** *The graph $\mathcal{R}(\mathbf{d})$ is connected.* □

The following theorem was shown by Taylor [15] in 1980.

**Theorem 1.4** *For a graphic degree sequence $\mathbf{d}$, let $\mathcal{CR}(d)$ be the set of all connected realizations of $\mathbf{d}$. Then the induced subgraph $\mathcal{CR}(\mathbf{d})$ of $\mathcal{R}(\mathbf{d})$ is connected.* □

## 2. Interpolation theorem

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If $f$ is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, $f$ is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers $x$ and $y$ such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

We have shown in [11, 12, 13] that the chromatic number $\chi$, the clique number $\omega$, and the matching number $\alpha_1$ are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$. If $f$ is an interpolation graph parameter with respect to $\mathbb{J}$, $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathcal{R}(\mathbf{d})$ we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively and in the case where $\mathbb{J} = \mathcal{CR}(\mathbf{d})$ we write $\mathrm{Min}(f, \mathbf{d})$ and $\mathrm{Max}(f, \mathbf{d})$ for $\min(f, \mathcal{CR}(\mathbf{d}))$ and $\max(f, \mathcal{CR}(\mathbf{d}))$ respectively.

We proved in [10] the following results.

**Theorem 2.1** *If $\sigma$ is a switching on $G$, then $|\phi(G) - \phi(G^\sigma)| \leq 1$.* □

**Theorem 2.2** *For a given graphic degree sequence $\mathbf{d}$, there exist integers $a$ and $b$ such that there is a graph $G$ with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.* □

By Theorem 1.4 and Theorem 2.1, we have the following interpolation theorem with respect to $\mathcal{CR}(\mathbf{d})$.

**Theorem 2.3** *For a given graphic degree sequence $\mathbf{d}$, there exist integers $A$ and $B$ such that there is a connected graph $G$ with degree sequence $\mathbf{d}$ and $\phi(G) = c$ if and only if $c$ is an integer satisfying $A \leq c \leq B$.* □

Let $G$ be a graph and $D$ be a minimum decycling set of $G$. Then $G - D$ is an induced forest of $G$ of maximum order. Erdős et al. [5] first defined a counterpart graph parameter $I$ as follows. Let $G$ be a graph and $F \subseteq V(G)$. $F$ is called an *induced forest* of $G$ if $G[F]$ contains no cycle. An induced forest $F$ of $G$ is *maximal* if for every $v \in G - F$, $F \cup \{v\}$ is not an induced forest of $G$. Let $I(G)$ be defined as

$$I(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

Thus $I(G)$ is the maximum cardinality of induced forests of $G$. An induced forest $F$ of $G$ with $|F| = I(G)$ is called a *maximum induced forest* of $G$. It is clear that $\phi(G) + I(G) = |V(G)|$ for any graph $G$. Consequently, if $\mathbf{d}$ is a graphic degree sequence of length $n$, then $n = \min(\phi, \mathbf{d}) + \max(I, \mathbf{d}) = \max(\phi, \mathbf{d}) + \min(I, \mathbf{d}) = \mathrm{Min}(\phi, \mathbf{d}) + \mathrm{Max}(I, \mathbf{d}) = \mathrm{Max}(\phi, \mathbf{d}) + \mathrm{Min}(I, \mathbf{d})$. Since $\phi$ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$, $I$ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. A *linear forest* is a forest with each component is a path.

## 3. Cubic graphs

It is easy to observe that the values of $\mathrm{Min}(\phi, r^n)$ and $\mathrm{Max}(\phi, r^n)$ are easily obtained for all $r \in \{0, 1, 2\}$. The problems of finding $\mathrm{Min}(\phi, 3^n)$ and $\mathrm{Min}(\phi, 3^n)$ are more difficult. We will consider such problems in terms of the graph parameter $I$. A *cubic tree* is a tree in which its vertices consisting of degree 1 or 3. It is easy to see that if $T$ is a cubic tree of order $n$, then $n = 2k + 2$, where $k$ is the number of vertices of degree 3 of $T$. Let $\mathbb{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of $K_4$ with one subdivided edge (the graph $K_4'$ in **Fig. 3.1**) at every vertex of degree 1.

It is easy to see that $\mathrm{Min}(I, 3^4) = 2$, $\mathrm{Min}(I, 3^6) = 4$, $\mathrm{Min}(I, 3^8) = 5$ and $\mathrm{Min}(I, 3^{10}) = 6$.

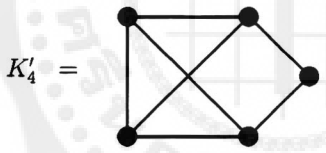


**Fig. 3.1**

A lower bound for the order of maximum induced forest in connected cubic graphs has been obtained by Liu and Zhao [9] as stated in the following theorem.

**Theorem 3.1** *Let $G$ be a connected cubic graph of order $n \geq 12$. Then $I(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathbb{T}$ and $I(G) \geq \frac{5}{8}n$ if $G \notin \mathbb{T}$.* □

It is clear that if $G \in \mathbb{T}$, then $G$ has order $8k + 10$, where $k$ is the number of vertices of degree 3 in the corresponding cubic tree. Thus $I(G) = \mathrm{Min}(I, 3^{8k+10}) = 5k + 6$. We now consider a cubic graph of order $8k + 8$. Let $C$ be a cubic graph of order $8k+8$. Then by Theorem 3.1, $I(C) \geq \frac{5}{8}(8k+8) = 5(k+1)$. A cubic graph $T$ obtained by taking cubic tree with $k$ vertices of degree 3, replacing $k - 1$ of the vertices by a triangle and attaching a copy of $K$ at every vertex of degree 1. Thus $T$ has order $8k+8$ and $I(T) = 5(k+1)$. Thus $\mathrm{Min}(I, 3^{8k+8}) = 5(k+1)$. The value of $\mathrm{Min}(I, 3^n)$, $n = 8k + 4, 8k + 6$, can be obtained in the following argument. Since a switching

changes the order of induced forest by at most 1, we have $\mathrm{Min}(I, 3^{p+q}) \leq \mathrm{Min}(I, 3^p) + \mathrm{Min}(I, 3^q) + 1$ for all even integers $p$ and $q$ with $4 \leq p \leq q$. Thus $5k+4 = \lceil \frac{5}{8}(8k+6) \rceil \leq \mathrm{Min}(I, 3^{8k+6}) \leq \mathrm{Min}(I, 3^4) + \mathrm{Min}(I, 3^{8(k-1)+10}) + 1 = 2 + 5(k-1) + 6 + 1 = 5k + 4$. Finally $5k + 3 = \lceil \frac{5}{8}(8k+4) \rceil \leq \mathrm{Min}(I, 3^{8k+4}) \leq \mathrm{Min}(I, 3^4) + \mathrm{Min}(I, 3^{8(k-1)+8}) + 1 = 2 + 5k + 1 = 5k + 3$. Therefore we obtain the following theorem and corollary.

**Theorem 3.2** *Let $n$ be an even integer with $n \geq 12$. Then*

$$\mathrm{Min}(I, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2(\mathrm{mod}\ 8), \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

□

**Corollary 3.3** *Let $n$ be an even integer with $n \geq 12$. Then*

$$\mathrm{Max}(\phi, 3^n) = \begin{cases} \frac{3}{8}n + \frac{1}{4} & \text{if } n \equiv 2(\mathrm{mod}\ 8), \\ \lfloor \frac{3}{8}n \rfloor & \text{otherwise.} \end{cases}$$

□

Let $H$ be a graph. A graph $G$ is called an *$H$-free graph* if $G$ does not contain $H$ as an induced subgraph. Let $X$ be a set of graphs. Then a graph $G$ is called an *$X$-free graph* if for every $H \in X$, $G$ is an $H$-free graph. In [14], there are five connected cubic graphs of order 8, all of which having maximum induced forests of order 5. Alon et al. proved in [1] that if $G$ is a $\{K_4, K_4'\}$-free graph with maximum degree 3. If $G$ is of order $n$ and of size $m$, then $I(G) \geq n - \frac{m}{4}$. Consequently, if $G$ is a cubic $\{K_4, K_4'\}$-free graph of order $n \geq 10$, then $I(G) \geq \frac{5n}{8}$. Zheng and Lu proved in [16] that $I(G) \geq \frac{2n}{3}$ for any connected cubic graph $G$ of order $n$ without triangles, except for two cubic graphs with $n = 8$. They also pointed out that this lower bound is best possible. It is easy to see that there exists cubic graph $G$ of order $n$ containing triangles and $I(G) \geq \frac{2n}{3}$. We extend their result by proving that $I(G) \geq \frac{2n}{3}$ for any connected cubic $K_4'$-free graph $G$ of order $n \geq 10$.

Let $P$ be a graph with $V(P) = \{v_0, v_1, \ldots, v_8\}$ and $E(P) = \{v_i v_{i+1} : i = 1, 2, \ldots, 8 \pmod 9\} \cup \{v_1 v_4, v_5 v_8, v_2 v_7, v_3 v_6\}$. Thus $P$ is a triangle-free graph of order 9 and of size 13. By Alon et al. [1], $I(P) \geq \lceil 9 - \frac{13}{4} \rceil = 6$. It is easy to find a set of 6 vertices of $P$, for example $\{v_0, v_1, v_2, v_3, v_5, v_6\}$, which is induced a forest. Therefore $I(P) = 6$.

By applying the result in [1] we find that if $G$ is a connected $K_4'$-free graph of order 8 and $\Delta(G) = 3$, then $I(G) = 5$ if and only if $G$ is a cubic graph.

**Lemma 3.4** *Let $G$ be a connected triangle-free graph of order $n$ and $\Delta(G) = 3$. If $G$ is not a cubic graph, then $I(G) \geq \frac{2n}{3}$.*

**Proof.** Suppose that $G$ does not contain a vertex of degree 1. Thus $G$ contains at least one vertex of degree 2. Let $k$ be the number of vertices of degree 2 in the graph $G$ and let $v_1, v_2, \ldots, v_k$ be the $k$ vertices of degree 2. Let $P_1, P_2, \ldots, P_k$ be $k$ graphs each of which is isomorphic to the graph $P$. The graph $G^*$ can be constructed from $G$ by adding an edge from $v_i (1 \leq i \leq k)$ to the vertex of degree 2 of $P_i (1 \leq i \leq k)$.

Therefore $G^*$ is a cubic triangle-free graph of order $n+9k$. Since $\frac{2(n+9k)}{3} \leq I(G^*) = I(G) + kI(P)$ and $I(P) = 6$, $I(G) \geq \frac{2n}{3}$. Suppose that $G$ contains a vertex $v$ of degree 1. Thus, by induction on $n$, we have $I(G) \geq I(G-v)+1 \geq \frac{2(n-1)}{3}+1 \geq \frac{2n}{3}$. □

**Lemma 3.5** *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let $G$ be an $X$-free graph of order $n$ with $\Delta(G) = 3$. Then $I(G) \geq \frac{2n}{3}$.*

**Proof.** Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let $G$ be an $X$-free graph of order $n$. Then, by Alon et al. [1], $I(G) \geq \frac{5n}{8}$. By calculation we found that $\lceil \frac{5n}{8} \rceil = \lceil \frac{2n}{3} \rceil$ for all $n$ with $4 \leq n \leq 10$ and $n \neq 8$. If $n = 8$, then $G$ is not cubic. Thus we also have that $I(G) \geq \frac{2n}{3}$. Thus the lemma is verified for all $n$ with $4 \leq n \leq 10$. Now suppose that $n \geq 11$. By Lemma 3.4, we may assume that $G$ contains a triangle $T$ with $V(T) = \{x, y, z\}$. If there exists a vertex in $V(T)$, say $x$ such that $d_G(x) = 2$, then by induction on $n$ there exists a maximum induced forest $F_1$ of $G - T$ with $|F_1| \geq \frac{2(n-3)}{3}$. Hence $F = F_1 \cup \{x, y\}$ is an induced forest of $G$ and $|F| \geq \frac{2n}{3}$. Suppose that for all triangles $T = \{x, y, z\}$ of $G$, $d_G(x) = d_G(y) = d_G(z) = 3$. Since $G$ is a $K_4$-free graph, $|N(T)| \geq 2$.

*Case 1.*
Suppose that $x$ and $y$ have a common neighbor $u$, and let $v$ be the neighbor of $z$. Since $G$ is a $K_4'$-free graph, $u$ and $v$ are not adjacent in $G$. Thus by induction on $n$, $G - T$ contains an induced forest of order at least $\frac{2(n-3)}{3}$. Since $d_{G-T}(u) \leq 1$, any maximum induced forest of $G - T$ must contain $u$. If there is a maximum induced forest $F_1$ of $G - T$ does not contain $u, v$-path, then $F_1 \cup \{y, z\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$. Suppose that for any maximum induced forest $F_1$ of $G - T$, $F_1$ contains $u, v$-path. Since $G' = G - T + uv$ satisfies conditions of the lemma, there is a maximum induced forest $F'$ of $G'$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{y, z\}$ is a maximum induced forest of $G$ of order at least $\frac{2n}{3}$. If $uv \in E(F')$, then $F = (F' - uv) \cup \{y, z\}$ is a maximum induced forest of $G$ of order at least $\frac{2n}{3}$.

*Case 2.*
Suppose that $x, y, z$ have different neighbors. Let $u, v, w$ be the neighbors of $x, y, z$ respectively. Put $G_1 = G[\{u, v, w\}]$. If $|E(G_1)| = 3$, then $G$ is not connected and $I(G) \geq 4 + I(G - G_1) \geq 4 + 2(n-6)/3 = 2n/3$. Suppose that $G_1$ is not a triangle and suppose further that there exist two vertices in $\{u, v, w\}$, say $u, v$, such that $uv \notin E(G)$ and $G' = G - T + uv$ satisfies conditions of the lemma. By induction on $n$, there exists a maximum induced forest $F_1$ of $G'$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F_1)$, then $F = F_1 \cup \{x, y\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$. If $uv \in E(F)$, then $F = (F_1 - uv) \cup \{x, y\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$. If $|E(H)| = 2$ and $uv \notin E(H)$, then $G' = G - T + uv$ satisfies conditions of the lemma. If $E(H) = \{vw\}$ and $G' = G - T + uw$ does not satisfy conditions of the lemma, then $G' = G - T + uv$ satisfies conditions of the lemma. Finally if $E(H) = \emptyset$, $G' = G - T + vw$ and $G' = G - T + uw$ do not satisfy conditions of the lemma, then $G' = G - T + uv$ satisfies conditions of the lemma. Thus the proof is complete.

□

We have the following theorem.

**Theorem 3.6** *Let $G$ be a connected cubic $K_4'$-free graph of order $n$, $n \geq 6$ and $n \neq 8$. Then $I(G) \geq \frac{2n}{3}$.* □

We proved in [10] the following theorem.

**Theorem 3.7** *Let $r \geq 3$ and $nr$ be even. Then*

$$\min(\phi, r^n) = \begin{cases} r-1 & if\ r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & if\ n \geq 2r. \end{cases}$$

□

Let $G$ be a connected $r$-regular graph and $S$ be a minimum decycling set of $G$. Since for any $v \in S$ there is a connected component $C$ of $G - S$ such that $v$ is adjacent to at least two vertices of $C$, there exists $u \in G - S$ such that $vu = e \in E(G)$ and $G - e$ is connected. Thus for two disjoint connected $r$-regular graphs $G$ and $H$ with minimum decycling set $S$ and $T$ of $G$ and $H$ respectively, there exist $u \in S$, $v \in G-S$, $x \in T$, $y \in H-T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$ and $G-e$, $H-f$ are connected. A connected $r$-regular graph $K = ((G-e) \cup (H-f)) + \{ux, vy\}$ satisfies

$$\phi(K) \leq \phi(G \cup H) = \phi(G) + \phi(H),$$

and the following corollary holds.

**Corollary 3.8** *Let $r \geq 3$ and $nr$ be even. Then*

$$\mathrm{Min}(\phi, r^n) = \begin{cases} r-1 & if\ r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & if\ n \geq 2r. \end{cases}$$

□

Thus the values of $\mathrm{Min}(\phi, r^n)$ for all $r$ and $n$ are already obtained. In particular the values of $\mathrm{Min}(\phi, 3^{2n})$ and $\mathrm{Max}(\phi, 3^{2n})$ are found for all $n$.

## 4. $\mathrm{Max}(\phi, r^n)$

We will discuss the problem of determining the values of $\mathrm{Max}(\phi, r^n)$ for $r \geq 4$ in this section. Note that $\mathcal{R}(r^n) = \mathcal{CR}(r^n)$ if and only if $r+1 \leq n \leq 2r+1$. Thus $\mathrm{Max}(\phi, r^n) = \max(\phi, r^n)$ for all $n \in \{r+1, r+2, \ldots, 2r+1\}$. In this case we have already obtained in [10] as stated in the following theorem.

**Theorem 4.1** *For $r \geq 4$, and $n = r + j$, $1 \leq j \leq r+1$, then*
(1) $\max(\phi, r^n) = n-2$, *if and only if* $j = 1$,
(2) $\max(\phi, r^n) = n-3$, *if and only if* $j = 2$,
(3) $\max(\phi, r^n) = n-4$, *for all even integers* $n = r+j$, $3 \leq j \leq r+1$,
(4) $\max(\phi, r^n) = n-4$, *for all odd integers* $n = r+j$, $3 \leq j \leq r+1$ *and* $n \geq f(j)$,

(5) $\max(\phi, r^n) = n - 5$, *for all odd integers* $n = r + j$, $3 \leq j \leq r + 1$ *and* $n < f(j)$, *where* $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 \pmod 4$, *and* $f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 \pmod 4$. □

Thus the values of $\mathrm{Max}(\phi, r^n)$ are already obtained for all $r$ and $n$ with $n \leq 2r + 1$. The problem of determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. In particular $\mathrm{Max}(\phi, r^n) = n - \mathrm{Min}(I, r^n)$.

Let $G$ be a $K_5$-free graph of order $n$, $\Delta(G) = 4$. Let $F$ be a maximal induced forest of $G$. We denote by $c(F)$ the number of cycles in $G - F$. A pair $(X, Y)$, where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $c((F - X) \cup Y) < c(F)$. In general we can define an interchangeable pair of vertices for a graph $G$ with $\Delta(G) > 4$ as follows. Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G) = \Delta > 4$. Let $F$ be a maximal induced forest of $G$. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G - F$. A pair $(X, Y)$, where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $k((F - X) \cup Y) < k(F)$.

Let $G$ be a $K_5$-free graph of order $n$ and $\Delta(G) = 4$. Thus for any maximal induced forest $F$ of $G$, $G - F$ is a union of cycles and paths. We choose a maximal induced forest $F$ of $G$ with minimum $c(F)$. In other word, the forest $F$ is chosen in such a way that it contains no interchangeable pair of vertices with respect to $F$. Suppose that $c(F) \geq 1$. Let $C$ be a cycle in $G - F$. Then each vertex of $C$ must be adjacent to exactly two vertices in $F$. Suppose that there exists a vertex $u \in F$, $d_F(u) \geq 2$, and $u$ is adjacent to a vertex $v \in V(C)$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to $F$. Thus for all cycles $C$ of $G - F$, each vertex $v \in V(C)$, $v$ must be adjacent to exactly two vertices $u_1, u_2 \in F$ with $d_F(u_1) = d_F(u_2) = 1$. By maximality of $F$, $u_1$ and $u_2$ must be in the same connected component of $F$. Since $F$ is a forest, there exists a unique path in $G[F]$ from $u_1$ to $u_2$. If $u_1$ and $u_2$ are not adjacent and there is a vertex $u \in F$ in the path such that $d_F(u) \geq 3$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to $F$. Therefore the connected component of $F$ containing $u_1$ and $u_2$ must be a path. Suppose that there exist exactly two vertices $v, w$ of $C$ adjacent to a vertex $u \in F$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to $F$. Finally suppose that there are three vertices $v, w, z$ of $C$ adjacent to a vertex $u \in F$, then the path $P$ in $G[F]$ containing $u$ has order at least 3 or the cycle $C$ has order at least 4, since otherwise $G$ would contain $K_5$. Let $u$ and $u'$ be the end vertices of $P$ in $G[F]$. Then $N_C(\{u, u'\}) = \{v, w, z\} \subseteq V(C)$. If $P$ has order at least 3, then $(\{u, u'\}, \{v, w\})$ is an interchangeable pair of vertices with respect to $F$. If $P$ has order 2 and $C$ has order at least 4, then $(\{u, u'\}, \{v, w, z\})$ is an interchangeable pair of vertices with respect to $F$. Thus the corresponding paths in $G[F]$ of vertices in $C$ are pairwise disjoint.

**Theorem 4.2** *Let* $G$ *be a* $K_5$*-free graph of order* $n$, $\Delta(G) = 4$. *Then* $\phi(G) \leq \frac{n}{2}$.

**Proof.** We may assume that $G$ is connected $K_5$-free graph of order $n$ and $\Delta(G) = 4$.

If $G$ contains a maximal induced forest $F$ with $G - F$ is a forest, then $\phi(G) \leq \frac{n}{2}$. Suppose for each maximal induced forest $F$ of $G$, $G - F$ contains at least one cycle. Choose a maximal induced $F$ of $G$ with minimum $c(F)$. Let $C$ be a cycle in $G - F$ with $V(C) = \{v_0, v_1, \ldots, v_{k-1}\}$ and $E(C) = \{v_i v_{i+1} : i = 0, 1, \ldots, k - 1(\text{mod } k)\}$. From above observation, for each $i = 0, 1, \ldots, k-1$, let $P(v_i)$ be the corresponding path in $G[F]$ containing $N_F(v_i) = \{u_{i1}, u_{i2}\}$ as its end vertices. Note that the paths $P(v_0), P(v_1), \ldots, P(v_{k-1})$ are pairwise disjoint. Moreover, since $G$ has no interchangeable pair of vertices with respect to $F$, for each $u_{ij} (0 \leq i \leq k-1, 1 \leq j \leq 2)$, there is a corresponding path $P(u_{ij})$ in $G - F$ and all the corresponding paths are pairwise disjoint.

*Case 1.*
If $k$ is even, we can form a new graph $G'$ in which $V(G') = V(G - C)$ and $E(G') = E(G - C) \cup \{u_{i1}u_{(i+1)1}, u_{i2}u_{(i+1)2} : i = 0, 2, 4, \ldots, k-2\}$. Since the corresponding paths $P(u_{ij})$ are pairwise disjoint, the graph $G'$ is a $K_5$-free graph of order $n - k$. By induction, $G'$ contains an induced forest $F'$ of order at least $\frac{n-k}{2}$. Since $G'[V(P(v_i)) \cup V(P(v_{i+1}))]$ is a cycle for all $i = 0, 2, 4, \ldots, k-2$, there exists $u \in V(P(v_i)) \cup V(P(v_{i+1}))$ such that $u \notin F'$. If $u \in V(P(v_i))$, then $F' \cup \{v_i\}$ is a forest of $G$. Similarly if $u \in V(P(v_{i+1}))$. Thus $\phi(G) \leq \frac{n}{2}$.

*Case 2.*
If $k$ is odd and there exists $i$ such that $P(v_i)$ has order at least three, then we can analogously form a graph $G'$ in which $V(G') = V(G - C)$, pairing the $k-1$ paths $P(v_j)$ with $j \neq i$, and adding $u_{i1}u_{i2}$ to the edge set of $G'$. The proof follows by similar argument as in Case 1.

*Case 3.*
If $k$ is odd and for each $i = 0, 1, 2, \ldots k-1$, $P(v_i)$ has order two, then since $G$ has no interchangeable pair of vertices, for each $i = 0, 1, 2, \ldots, k-1$ and $j = 1, 2$, there is a path $P(u_{ij})$ in $G - F$ and $P(u_{ij})$ has $N_{G-F}(u_{ij})$ as its end vertices. Put $N_{G-F}(u_{ij}) = \{v_{ij(1)}, v_{ij(2)}\}$. Moreover, the paths $P(u_{ij})$ are pairwise disjoint. We can now form a graph $G'$ in which $V(G') = V(G) - (C \cup N_F(V(C)))$ and $E(G') = E(G[V(G')] \cup E_1$, where $E_1 = \{v_{i1(1)}v_{i2(1)}, v_{i1(2)}v_{i2(2)} : i = 0, 1, 2, \ldots, k-1\}$. By induction on $n$, there exists an induced forest $F'$ of $G'$ with $|F'| \geq \frac{n-3k}{2}$. Since for each $i = 0, 1, 2, \ldots, k-1$, $F'$ can not contain all vertices in $V(P(u_{i1})) \cup V(P(u_{i2}))$, there exists $v \in V(P(u_{i1})) \cup V(P(u_{i2}))$ such that $v \notin F'$. If $v \in V(P(u_{i1}))$, then $F' \cup \{u_{i1}, v_i\}$ is an induced forest of $G$. Similarly if $v \in V(P(u_{i2}))$. Thus there is a set $X$ containing either $u_{i1}$ or $u_{i2}$, but not both, according to $F' \cup \{u_{i1}, v_i\}$ or $F' \cup \{u_{i2}, v_i\}$ is an induced forest of $G$. Therefore $F' \cup \{v_0, v_1, \ldots, v_{k-2}\} \cup X$ is an induced forest of $G$ of order at least $\frac{n-3k}{2} + 2k - 1 \geq \frac{n}{2}$. This completes the proof. □

**Corollary 4.3** $\text{Max}(\phi, 4^n) \leq \frac{n}{2}$. □

Let $G$ be a $K_{\Delta+1}$-free graph with $\Delta(G) = \Delta \geq 5$ and let $F$ be a maximal induced of $G$ with minimum $k(F)$. Then for each $v \in G - F$, there exists a connected component $T$ of $F$ such that $v$ is adjacent to at least two vertices of $T$. Thus $\Delta(G - F) \leq \Delta - 2$. Suppose that $k(F) \geq 1$. Let $K$ be a complete subgraph of $G - F$ of order $\Delta - 1$.

Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Thus for each $v_i$ there exists a connected component $P(v_i)$ of $G[F]$ such that $v_i$ is adjacent to exactly two vertices of $P(v_i)$. If there exists $u \in V(P(v_i))$ such that $uv_i \in E(G)$ and $d_F(u) \geq 2$, then $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to $F$. Thus for each $i = 1, 2, \ldots, \Delta - 1$, $v_i$ must be adjacent to exactly two vertices of degree one in $P(v_i)$. Suppose that there exists $u \in V(P(v_i))$ such that $d_F(u) \geq 3$, $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to $F$. Thus the corresponding $P(v_i)$ of $v_i$ in $K$ is a path. Furthermore all such paths are pairwise disjoint.

**Lemma 4.4** *Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G) = \Delta \geq 5$. Then $I(G) \geq \frac{2n}{\Delta}$ or there exists an induced forest $F$ of $G$ such that $G - F$ is a $K_{\Delta-1}$-free graph.*

**Proof.** With above observation in mind, suppose that for all maximal induced forests $F$ of $G$, $k(F) \geq 1$. Let $F$ be a maximal induced forest of $G$ with minimum $k(F)$ and let $K$ be a complete subgraph of $G - F$ of order $\Delta - 1$. Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Let $P(v_i)$ be defined as above and let $u_{i1}, u_{i2}$ be the two vertices with degree one of $P(v_i)$. We now form a graph $G'$ with $V(G') = V(G - K)$ and $E(G') = E(G[V(G')]) \cup X$, where $X = \{u_{11}u_{21}, u_{12}u_{22}, u_{31}u_{41}, u_{32}u_{42}\}$. By induction on $n$, $I(G') \geq \frac{2n-\Delta+1}{\Delta}$ or there exists an induced forest $F'$ of $G'$ such that $G' - F'$ is a $K_{\Delta-1}$-free graph. Since $F'$ does not contain all vertices of $P(v_1) \cup P(v_2)$ and likewise of $P(v_3) \cup P(v_4)$, there exist two vertices $x, y \in \{v_1, v_2, v_3, v_4\}$ such that $F'' = F' \cup \{x, y\}$ is an induced forest of $G$ and $G - F''$ is a $K_{\Delta-1}$-free graph. This means that if $G' - F'$ is a $K_{\Delta-1}$-free graph, then there exists a maximal induced forest $F''$ of $G$ such that $G - F''$ is a $K_{\Delta-1}$-free graph. Thus we may assume that $|F'| \geq \frac{2n-\Delta+1}{\Delta}$. Hence there exist two vertices $x, y \in \{v_1, v_2, v_3, v_4\}$ such that $F'' = F' \cup \{x, y\}$ is an induced forest of $G$ and $|F''| \geq \frac{2n-\Delta+1}{\Delta} + 2 \geq \frac{2n}{\Delta}$. This completes the proof. □

**Lemma 4.5** *Let $G$ be a connected $K_5$-free graph of order $n$ and $\Delta(G) = 5$. Then $I(G) \geq \frac{2n}{5}$.*

**Proof.** *Case 1.*
Suppose that for each maximal induced forest $F$ of $G$, $G - F$ contains $K_4$ as its component. Choose maximal induced forest $F$ of $G$ with minimum $k(F)$. Let $K$ be a copy $K_4$ in $G - F$ and $V(K) = \{v_0, v_1, v_2, v_3\}$. Since $G$ is connected and $G$ does not have an interchangeable pair of vertices with respect to $F$, for each vertex $v_i$, there is a path $P(v_i)$ of $F$ such that $v_i$ is adjacent to two vertices with degree one of $P(v_i)$ and for any two distinct vertices $v_i$ and $v_j$, $P(v_i)$ and $P(v_j)$ are disjoint. For each $v_i$, let $u_{i1}$ and $u_{i2}$ be the end vertices of $P(v_i)$, $i = 0, 1, 2, 3$. We now form a graph $G'$ with $V(G') = V(G - K)$ and $E(G') = E(G - K) \cup E_1$, where $E_1 = \{u_{01}u_{11}, u_{02}u_{12}, u_{21}u_{31}, u_{22}u_{32}\}$. By induction on $n$, $G'$ contains an induced forest $F'$ of $G'$ such that $|F'| \geq \frac{2(n-4)}{5}$. It is clear by forming the graph $G'$ that there exist two distinct vertices $v_i$ and $v_j$ such that $F' \cup \{v_i, v_j\}$ is an induced forest of $G$ and of order at least $\frac{2n}{5}$.

*Case 2.*

Suppose that for each induced forest $F$ of $G$, $G-F$ contains at least one $H \in \mathcal{CR}(3^8)$ as its component. Choose a maximal induced forest $F$ of $G$ in such a way that $F$ contains minimum number of copies of graphs in $H \in \mathcal{CR}(3^8)$. Let $K$ be a copy of graph in $\mathcal{CR}(3^8)$ in $G-F$ and put $V(K) = \{v_0, v_1, \ldots, v_7\}$. By choosing $F$ in this way, we have for each $v_i$, there is a path $P(v_i)$ of $F$ such that $v_i$ is adjacent to the end vertices of $P(v_i)$. Moreover for any two distinct $v_i$ and $v_j$, $P(v_i)$ and $P(v_j)$ are disjoint. Let $u_{i1}, u_{i2}$ be the end vertices of $P(v_i)$, $i = 0, 1, 2, \ldots, 7$. We now form a graph $G'$ with $V(G') = V(G-K)$ and $E(G') = E(G-K) \cup E_1$, where $E_1 = \{u_{i1}u_{(i+1)1}, u_{i2}u_{(i+1)2} : i = 0, 2, 4, 6\}$. By induction on $n$, $G'$ contains an induced forest $F'$ of order at least $\frac{2(n-8)}{5}$. It is clear by forming the graph $G'$ that there are at least four vertices $x_1, x_2, x_3, x_4 \in V(K)$ such that $F' \cup \{x_1, x_2, x_3, x_4\}$ forms an induced forest of $G$ of order at least $\frac{2n}{5}$.

*Case 3.*

Suppose that for each induced forest $F$ of $G$, $G-F$ contains at least one copy $K_4'$ as its induced subgraph. Choose a maximum induced forest $F$ of $G$ in such a way that $G-F$ contains minimum number of copies of $K_4'$. Let $K$ be a copy of $K_4'$ in $G-F$. Put $V(K) = \{v_0, v_1, v_2, v_3, v_4\}$ and $d_K(v_0) = 2$. For each $v_i, i = 1, 2, 3, 4$, there exists a connected component $P(v_i)$ of $F$ such that $v_i$ is adjacent to exactly two vertices of $P(v_i)$. Again let $\{u_{i1}, u_{i2}\}$ be two vertices of $P(v_i)$ that are adjacent by $v_i$, $i = 1, 2, 3, 4$. Since $K$ is not a cubic graph, $P(v_i)$ and $P(v_j)$ may not be disjoint. It is clear by choosing the minimum number of copies of $K_4'$ in $G-F$ that for each vertex $u$ of $P(v_i)$, $i = 1, 2, 3, 4$, there are at most two vertices of $K$ that are adjacent to $u$. Let $G'$ be a graph with $V(G') = V(G-K)$ and $E(G') = E(G-K) \cup E_1$. A graph $G'$ will be formed according to the following cases. We then apply induction to each such a forming of $G'$, there exists an induced forest $F'$ of $G'$ such that $F'$ together with two vertices of $K$ forms an induced forest of order at least $\frac{2n}{5}$. Put $N = \cup_{i=1}^{4} N_F(v_i)$. For two disjoint nonempty subsets $X, Y$ of $V(G)$ we denote $e(X, Y)$ the number of edges in $G$ joining between $X$ and $Y$. Note that $F$ was chosen as a maximal induced forest of $G$ with minimum $k(F)$, we have the following observation.

1. $4 \leq |N| \leq 8$.

2. $d_F(u) \in \{1, 2\}$, for all $u \in N$.

3. For each $u \in N$, there are at most two vertices in $\{v_1, v_2, v_3, v_4\}$ that are adjacent to $u$.

4. If $u \in N$ and $d_F(u) = 2$, then there is exactly one vertex in $\{v_1, v_2, v_3, v_4\}$ that is adjacent to $u$.

With the above observation in mind, suppose $|N| = 4$. Thus we may assume that $u_{11} = u_{21}$, $u_{12} = u_{22}$, $u_{31} = u_{41}$ and $u_{32} = u_{42}$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{31}, u_{12}u_{32}\}$.

Suppose $|N| = 8$ and suppose further that there exist $i$ and $i'$ with $i \neq i'$ such that $e(\{u_{i1}\}, \{u_{i'1}, u_{i'2}\}) = 2$. Then $u_{i1}, u_{i2}$ and $u_{i'1}, u_{i'2}$ are not adjacent. We can choose $E_1 = \{u_{i1}u_{i2}, u_{i'1}u_{i'2}\}$. Suppose that for $i$ and $i'$, $e(\{u_{i1}\}, \{u_{i'1}, u_{i'2}\}) \leq 1$. We may assume without loss of generality that for pairs $i, i'$ with $1 \leq i < i' \leq 4$, $u_{ij}$ and $u_{i'j}$

are not adjacent. Thus we can choose $E_1 = \{u_{11}u_{21}, u_{12}u_{22}, u_{31}u_{41}, u_{32}u_{42}\}$.

Suppose $4 < |N| < 8$ and suppose further that $u_{11} = u_{21}, u_{12} = u_{22}$. Thus $d_F(u_{11}) = d_F(u_{12}) = 1$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{41}, u_{12}u_{42}\}$. Finally suppose that any pair of $1 \leq i < i' \leq 4$, $|N_F(v_i) \cap N_F(v_{i'})| \leq 1$. Since $|N| < 8$, we may assume without loss of generality that $u_{11} = u_{21}$ and $u_{12} \neq u_{22}$. Thus $u_{11}, u_{12}, u_{22}$ lie in the same component of $F$. Since $d_F(u_{11}) = 1$, $d_F(u_{12}) = 2$ or $d_F(u_{22}) = 2$. Suppose that $d_F(u_{12}) = 2$ and $d_F(u_{22}) = 2$, then $\{u_{11}, u_{12}, u_{22}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$. We can choose $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{41}u_{22}u_{42}\}$ or $E_1 = \{u_{11}u_{31}, u_{11}u_{32}, u_{12}u_{42}u_{22}u_{41}\}$. Suppose that $d_F(u_{12}) = 2$ and $d_F(u_{22}) = 1$. Thus $\{u_{11}, u_{12}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$. If $\{u_{11}, u_{12}, u_{22}\} \cap \{u_{31}, u_{32}, u_{41}, u_{42}\} = \emptyset$, then we can choose $E_1$ as in the previous case. If $u_{22} \in \{u_{31}, u_{32}, u_{41}, u_{42}\}$, say $u_{22} = u_{32}$, we can choose $E_1 = \{u_{11}u_{41}, u_{11}u_{42}, u_{12}u_{32}\}$ or $E_1 = \{u_{11}u_{41}, u_{11}u_{42}, u_{12}u_{31}\}$.

*Case 4.*

Suppose that $G - F$ is an $X$-free graph, where $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$. If $|F| < \frac{2n}{5}$, then, by Lemma 3.5, $G - F$ contains an induced forest $F'$ of order at least $\frac{2(n-|F|)}{3} \geq \frac{2(n-\frac{2}{5}n)}{3} \geq \frac{2n}{5}$. This completes the proof. □

**Lemma 4.6** *Let $G$ be a $K_6$-free graph of order $n$ with $\Delta(G) = 5$. Then $I(G) \geq \frac{2n}{5}$.*

**Proof.** If $G$ does not contain $K_5$ as a subgraph, then the result follows from Lemma 4.5. Suppose that $G$ contains $K_5$ as a subgraph. Put $V(K) = \{v_1, v_2, \ldots v_5\}$. Since $G$ is a $K_6$-free graph, $N_{G-K}(K)$ contains at least two vertices. If there exists a maximum induced forest $F_1$ of $G - K$ such that $N_{G-K}(K) \not\subseteq F_1$ or $G[N_{F_1}(K)]$ is disconnected, then there exist $x, y \in V(K)$ such that $F_1 \cup \{x, y\}$ is an induced forest of $G$. By induction on $n$, we have $I(G) \geq |F_1| + 2 \geq \frac{2(n-5)}{5} + 2 = \frac{2n}{5}$. We now suppose that each maximum induced forest $F$ of $G - K$, $G[N_F(K)]$ is a connected component of $G[F]$. Suppose further that $N_F(K)$ contains exactly two elements $x, y \in F$. Put $L = K \cup \{x, y\}$ and $H = G[L]$. Thus $d_H(x) \geq 4$ or $d_H(y) \geq 4$. Thus $I(G) \geq I(L) + I(G-L) \geq 3 + \frac{2(n-7)}{5} \geq \frac{2n}{5}$. If $N_F(K)$ contains more than two elements, we can form a graph $G'$ with $V(G') = V(G) - K$ and $E(G') = E(G-K) \cup \{e\}$, where $e$ is a new edge connecting two vertices in $N_F(K)$. By induction on $n$, $G'$ contains an induced forest $F'$ of order at least $\frac{2(n-5)}{5}$. Since $F'$ can not contain all vertices of $N_F(K)$, there exist $v_i, v_j \in V(K)$ such that $F'' = F' \cup \{v_i, v_j\}$ is an induced forest of $G$ and $|F''| \geq \frac{2n}{5}$. □

As a direct consequence of Lemma 4.6 we have the following theorem.

**Theorem 4.7** *Let $G$ be a connected 5-regular graph of order $n \geq 12$. Then $\phi(G) \leq \frac{3n}{5}$.* □

**Lemma 4.8** *Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G) = \Delta \geq 5$. Then $I(G) \geq \frac{2n}{\Delta}$.*

**Proof.** The result follows for $\Delta = 4, 5$. Suppose that $\Delta \geq 6$, by Lemma 4.4 we have $I(G) \geq \frac{2n}{\Delta}$ or there exists an induced forest $F$ of $G$ such that $G - F$ is a $K_{\Delta-1}$-free

graph. If $|F| < \frac{2n}{\Delta}$, then $|G - F| > \frac{(\Delta-2)n}{\Delta}$. Since $G - F$ is a $K_{\Delta-1}$-free graph, we have $I(G) \geq I(G - F) \geq \frac{\frac{2(\Delta-2)n}{\Delta}}{\Delta-2} = \frac{2n}{\Delta}$. □

We have the following theorem.

**Theorem 4.9** *Let $G$ be a connected $r$-regular graph of order $n \geq 2r + 2$. Then $\phi(G) \leq \frac{n(r-2)}{r}$ for all $r \geq 4$.* □

Let $G$ be a connected $r$-regular graph of order $n = rq + t$, $0 \leq t \leq r - 1$, $r \geq 4$ and $q \geq 1$. Then by theorem 4.9, we have $I(G) \geq 2q + \lceil \frac{2t}{r} \rceil$. It is easy to construct a connected $r$-regular graph $G$ of order $n$ with $I(G) = 2q$ if $t = 0$, $IG) = 2q + 1$ if $t = 1, 2$ and $I(G) = 2q+2$ if $3 \leq t \leq r-1$. Consequently, we have $\text{Max}(\phi, r^n) = n-2q$ if $t = 0$, $\text{Max}(\phi, r^n) = n - 2q - 1$ if $t = 1, 2$, $\text{Max}(\phi, r^n) = n - 2q - 2$ if $2t > r$ and $\text{Max}(\phi, r^n) \in \{n - 2q - 2, n - 2q - 1\}$ if $3 \leq t \leq \frac{r}{2}$.

We close this paper with the following conjecture.

**Conjecture** $\text{Max}(\phi, r^n) = n - 2q - 2$ *if* $3 \leq t \leq \frac{r}{2}$, *for all* $r \geq 6$. □

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# Constrained switchings in cubic graphs

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**Abstract**

The graph $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ is a graph whose vertices are the graphs with degree sequence $\mathbf{d}$, two vertices are adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. It has been shown that the graph $\mathcal{R}(\mathbf{d})$ is connected. Let $\mathcal{CR}(d)$ be the set of connected graphs with degree sequence $\mathbf{d}$. Taylor [13] proved that the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathcal{CR}(d)$ is connected. Several connected subgraphs of $\mathcal{CR}(3^n)$ are obtained in this paper. As an application, we are able to obtain the interpolation and extremal results for the number of maximum induced forests in the classes of connected subgraphs of $\mathcal{CR}(3^n)$.

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## 1 Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph $G$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual $|S|$ denote the cardinality of a set $S$ and therefore we define $n = |V|$ to be the *order* of $G$ and $m = |E|$ the *size* of $G$. To simplify writing, we

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write $e = uv$ for the edge $e$ that joins the vertex $u$ to the vertex $v$. A *path* of length $k$ in a graph $G$, denoted by $P_k$, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of $G$ such that for all $i = 1, 2, \ldots, k-1$, $u_iu_{i+1}$ are edges of $G$. A $u,v$-path is a path which has $u$ as its first vertex and $v$ as its last vertex in the path. The *degree* of a vertex $v$ of a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph $G$ is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by $S$ in $G$. For a graph $G$ and $X \subseteq E(G)$, we denote $G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ and $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. For two disjoint graphs $G$ and $H$ (i.e. $V(G) \cap V(H) = \emptyset$), we denote $G \cup H$ their union, and define $pG$ the union of $p$ copies of $G$. For a graph $G$ and $s \in V(G)$, the *neighborhood* of $s$ in $G$ is defined by

$$\mathcal{N}(s) = \{v \in V(G) : sv \in E(G)\}.$$

If $S \subseteq V(G)$, then we define

$$\mathcal{N}(S) = \bigcup_{s \in S} \mathcal{N}(s).$$

If $F \subseteq V(G)$, we write $\mathcal{N}_F(S)$ for $\mathcal{N}(S) \cap F$. A graph $G$ is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph*.

Let $G$ be a graph of order $n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph $G$ is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph $G$. In this case, $G$ is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [6] and Hakimi [5]. We state their results in the following theorem.

**Theorem 1.1** *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

*Then* $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

■

Let $G$ be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from $G$ by a switching has the same degree sequence as $G$. The following theorem has been shown by Havel [6] and Hakimi [5].

**Theorem 1.2** *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* $G_1$ *and* $G_2$ *are any two realizations of* $\mathbf{d}$*, then* $G_2$ *can be obtained from* $G_1$ *by a finite sequence of switchings.*

■

As a consequence of Theorem 1.2, Eggleton and Holton [3] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

**Theorem 1.3** *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

■

The following theorem was shown by Taylor [13] in 1980.

**Theorem 1.4** *For a graphic degree sequence* $\mathbf{d}$*, let* $\mathcal{CR}(d)$ *be the set of all connected realizations of* $\mathbf{d}$*. Then the induced subgraph* $\mathcal{CR}(\mathbf{d})$ *of* $\mathcal{R}(\mathbf{d})$ *is connected.*

■

# 2 Connected subgraphs of $\mathcal{CR}(3^n)$

Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order $n$. Put $\mathbb{J}_1 = \mathcal{CR}(3^n)$. Let $\mathbb{J}_2$ be the class of all connected cubic $K_4'$-free graphs of order $n$, where $K_4'$ is a graph obtained from $K_4$ and a subdivision to an edge. Finally, let $\mathbb{J}_3$ be the class of all connected cubic triangle-free graphs of order $n$. It is clear that $\mathbb{J}_3 \subseteq \mathbb{J}_2 \subseteq \mathbb{J}_1$. Let $X_n = \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3, \mathbb{J}_1 - \mathbb{J}_2, \mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$.

The notation $\mathcal{CR}(3^n), \mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3$, and $X_n$ as described above will be used throughout this paper.

Let $\mathbb{J} \in X_n$ and $G \in \mathbb{J}$. A switching $\sigma$ is called a $\mathbb{J}$*-switching with respect to* $G$ if $G^\sigma \in \mathbb{J}$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a *sequence of* $\mathbb{J}$*-switchings with respect to* $G$ if for all $i = 1, 2, \ldots, t, G^{\sigma_1\sigma_2\cdots\sigma_i} \in \mathbb{J}$.

As a consequence of Taylor [13], one can see that if $G_1, G_2 \in \mathbb{J}_1$, then there exists a sequence of $\mathbb{J}_1$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G_1$ such that $G_1^{\sigma_1\sigma_2\cdots\sigma_t} = G_2$. Thus the subgraph of $\mathcal{R}(3^n)$ induced by $\mathbb{J}_1$ is connected.

Let $G$ be a connected graph of order at least three. A vertex $v \in V(G)$ is called a *cut vertex* of $G$ if $G - v$ contains at least 2 components. An edge $e \in E(G)$ is called a *cut edge* of $G$ if $G - e$ contains at least 2 components. A connected graph $G$ is 2-*connected* if $G$ contains no cut vertices and it is called 2-*edge connected* if it contains no cut edges. A maximal 2-connected subgraph of a connected graph $G$ is called a *block* of $G$. Thus a 2-connected graph is a block. It is well known that if $G$ is a connected graph containing two distinct blocks $B_1$ and $B_2$, then $E(B_1) \cap E(B_2) = \emptyset$ and $|V(B_1) \cap V(B_2)| \leq 1$. If $B_1$ and $B_2$ have a vertex $v$ in common, then $v$ is a cut vertex of $G$. Furthermore if $B$ is a block of order at least three, for any two distinct vertices of $B$, there is a cycle in $B$ containing the two vertices. Let $G$ be a graph and $r$ be a positive integer. An $r$-regular spanning subgraph of $G$ is called an $r$-*factor* of $G$. Petersen [8] proved the following theorem.

**Theorem 2.1** *If $G$ be a 2-connected cubic graph, then $G$ contains a 2-factor.*

■

For a 2-connected cubic graph $G$, $G$ may contain several 2-factors. If we choose one such 2-factor $F$ of $G$, then $F' = (V(G), E(G) - E(F))$ is a 1-factor of $G$.

A connected cubic graph $G$ which contains $K_4'$ as its subgraph is not 2-connected. We first consider the class $\mathbb{J}_1 - \mathbb{J}_2$ separately. Observe further that $\mathbb{J}_1 - \mathbb{J}_2 \neq \emptyset$ if and only if $n$ is even and $n \geq 10$.

**Theorem 2.2** *Let $G_1, G_2 \in \mathbb{J} = \mathbb{J}_1 - \mathbb{J}_2$. Then $G_1 \cong G_2$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G_1$ such that $G_1^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2$.*

**Proof.** Let $\mathbb{J} = \mathbb{J}_1 - \mathbb{J}_2$ and $G_1, G_2 \in \mathbb{J}$. Since $G_1$ and $G_2$ contain $K_4'$ as induced subgraph, there exist $e \in G_1$ and $f \in G_2$ such that $G_1 - e = G_1' \cup K_4'$ and $G_2 - f = G_2' \cup K_4'$. It turns out that the graphs $G_1'$ and $G_2'$ are connected and have the same degree sequence. Thus $G_1' \cong G_2'$ or by Taylor [13], there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $G_1'^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2'$. The sequence of switchings can be considered as a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G_1$ and $G_1^{\sigma_1\sigma_2\cdots\sigma_t} \cong G_2$. Thus the proof is complete. ■

Since the class of cubic graphs of order $n \leq 10$ are known in [12] and they are easy to verify as our purpose, we will consider from now on only the connected cubic graphs of order $n$ where $n \geq 12$. Let $\mathbb{J} \in X_n$ and $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$. We have the following theorems.

**Theorem 2.3** *Let $\mathbb{J} \in X_n$, $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$, and $G \in \mathbb{J}$. If $G$ contains a cut vertex, then there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ is 2-connected.*

**Proof.** Let $G \in \mathbb{J}$ and $G$ is not 2-connected. Since $G$ is cubic, $G$ contains at least three blocks. Let $S$ be the set of all cut vertices of $G$ and $v \in S$. Since $d_G(v) = 3$, there exists a component $G_i$ of $G - v$ such that $v$ is adjacent to exactly one vertex $u$ of $G_i$ and this vertex is also a cut vertex of $G$. Thus $|S| \geq 2$ and $|S| = 2$ if and only if $G$ contains exactly three blocks. We will proceed by induction on the number of blocks, $b(G)$, of a graph $G$. If $b(G) = 3$, there exists a cut set of vertices $S = \{u, v\}$ of $G$. Thus $uv \in E(G)$ and the blocks of $G$ are $B_1$, $B_2$, and $B_3 = G[\{u, v\}]$ such that $B_1$ and $B_2$ contain $u$ and $v$ respectively. It is clear that $B_1$ and $B_2$ have order at least 5.

If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$, then choose $e = ab \in E(B_1)$ and $f = cd \in E(B_2)$ such that $e$ is not adjacent to $u$ and $f$ is not adjacent to $v$. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to $G$ and $G^{\sigma(a,b;c,d)}$ is 2-connected.

If $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and $G \in \mathbb{J}$, then at least one of $B_i$ must contain a triangle as its subgraph. Suppose $B_1$ contains a triangle $T$. Choose $e = ab \in E(B_1)$ and $f = cd \in E(B_2)$ such that $e \notin E(T)$. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to $G$ and $G^{\sigma(a,b;c,d)}$ is a 2-connected graph containing a triangle.

Let $k$ be an integer greater than 2 and suppose that the theorem is true for all connected cubic graphs $H$ with $b(H) < k$. Let $G$ be a connected cubic graph with $b(G) = k$. Thus there exist two vertex disjoint blocks $B_1$ and $B_2$ of $G$, each of which has order at least 5. We can analogously define the switching $\sigma_1 = \sigma(a, b; c, d)$ as described above and the resulting graph $G_1 = G^{\sigma_1}$ reduces the number of blocks of $G$ by one. By induction, there exists a sequence of $\mathbb{J}$-switchings $\sigma_2, \sigma_3, \ldots, \sigma_t$ with respect to $G_1$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t} = G_1^{\sigma_2\sigma_3\cdots\sigma_t}$ is 2-connected. This completes the proof. ■

**Theorem 2.4** *Let $\mathbb{J} \in X_n$, $\mathbb{J} \neq \mathbb{J}_1 - \mathbb{J}_2$, and $G \in \mathbb{J}$. If $G$ is 2-connected, then $G$ is hamiltonian or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ is hamiltonian.*

**Proof.** Let $G$ be a 2-connected cubic graph. By Theorem 2.1, there is a 2-factor $F$ of $G$. Let $c(F)$ be the number of cycle in $F$. Thus we may

choose $F$ of $G$ with minimum $c(F)$. If $c(F) = 1$, then $G$ is hamiltonian. Suppose $c(F) = 2$ and $F = C_1 \cup C_2$.

If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$, then choose $e = ab \in E(C_1)$ and $f = cd \in E(C_2)$ such that $a$ is not adjacent to $c$ and $b$ is not adjacent to $d$. Thus $\sigma(a, b; c, d)$ is a $\mathbb{J}$-switching with respect to $G$ and $G^{\sigma(a,b;c,d)}$ is a hamiltonian graph.

If $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and $G \in \mathbb{J}$, then at least one of $C_i$ must contain three vertices which induces a triangle in $G$. Suppose $C_1$ contains three vertices which induces a triangle $T$ and $C_1$ has order at least 4. Choose $e = ab \in E(C_1)$ and $f = cd \in C_2$ such that $e \notin E(T)$. Thus $G^{\sigma(a,b;c,d)}$ is hamiltonian containing a triangle. Suppose $C_1$ has order 3 and $V(C_1) = \{x, y, z\}$. Then $C_2$ has order at least 9. Let $\{x', y', z'\} \in V(C_2)$ such that $xx', yy', zz' \in E(G)$. Since $G$ is not hamiltonian, $x'y', y'z', x'z' \notin E(C_2)$. Let $u \in V(C_2)$ and $ux' \in E(C_2)$. Thus there is a vertex $v \in V(C_2)$ such that $uv \in E(G)$. Since $x'y', x'z' \notin E(C_2)$, $u \notin \{y', z'\}$.

*Case 1.* Suppose that $y'v \notin E(G)$ or $z'v \notin E(G)$. Without loss of generality, we may assume that $y'v \notin E(G)$. Then $G^{\sigma(y,y';u,v)}$ is a hamiltonian graph containing a triangle.

*Case 2.* Suppose that $y'v \in E(G)$ and $z'v \in E(G)$. Since $\mathcal{N}(v) = \{u, y', z'\}$, there is at least one edge $e \in \{y'v, z'v\}$ satisfying $e \in E(C_2)$. Without loss of generality, we may assume that $z'v \in E(C_2)$. Therefore if $y'u \in E(G)$, then $G^{\sigma(x,x';u,v)}$ is a hamiltonian graph containing a triangle and if $y'u \notin E(G)$, then $G^{\sigma(y,y';v,u)}$ is a hamiltonian graph containing a triangle.

Suppose that $c(F) = k \geq 3$, we can analogously apply an appropriate $\mathbb{J}$-switching or a sequence of $\mathbb{J}$-switchings with respect to $G$ to obtain a 2-connected cubic graph $G'$ containing a 2-factor $F'$ with $c(F') < c(F)$. This completes the proof. ∎

Let $n$ be an integer and $n \geq 6$. Let $G(2n)$ be a graph of order $2n$ with $V(G(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(G(2n)) = \{v_i v_{i+1} : i = 0, 1, \ldots, 2n - 1 (\text{mod } 2n)\} \cup \{v_0 v_n, v_1 v_{n+1}\} \cup \{v_j v_{2n-j+1} : j = 2, 3, \ldots, n-1\}$.

It is clear that $G(2n)$ is a cubic triangle-free graph containing a hamiltonian cycle $C$ with $E(C) = \{v_i v_{i+1} : i = 0, 1, \ldots, 2n - 1 (\text{mod } 2n)\}$.

Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$ and let $G \in \mathbb{J}$. Suppose further that $G$ is hamiltonian. Up to isomorphism, we may assume that $G$ contains $C$ as its hamiltonian cycle. Thus the following sets are uniquely determined.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \ldots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and
$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

Since $e \in [L,R](G)$ if and only if $e \in [R,L](G)$, we have $[L,R](G) = [R,L](G)$ and $|[L,L](G)| = |[R,R](G)|$. Let $e = v_iv_j \in [L,L](G)$. Then there exists $f = v_sv_t \in [R,R](G)$. If $i \in \{0, 2n-1\}$ and $s \in \{1,2\}$, then $\sigma_1 = \sigma_1(v_i, v_j; v_t, v_s)$ is a $\mathbb{J}$-switching with respect to $G$. Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)| - 1$. If $i \in \{n+1, n+2\}$ and $s \in \{n-1, n\}$, then $\sigma_1 = \sigma_1(v_i, v_j; v_t, v_s)$ is a $\mathbb{J}$-switching with respect to $G$. Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)| - 1$. In other case, a switching $\sigma_1 = \sigma_1(v_i, v_j; v_s, v_t)$ is a $\mathbb{J}$-switching with respect to $G$. Thus the number of $[L,L](G^{\sigma_1})$ can be decreased by 2 from the number of $[L,L](G)$. With this observation, there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $[R,R](G^{\sigma_1\sigma_2\cdots\sigma_t}) = [L,L](G^{\sigma_1\sigma_2\cdots\sigma_t}) = \emptyset$.

**Theorem 2.5** *Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$. If $G \in \mathbb{J}$ and $[R,R](G) = \emptyset$, then $G \cong G(2n)$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \cong G(2n)$.*

**Proof.** If $v_0v_n \notin E(G)$, then there exist $i \in \{3, 4, \ldots, n-1\}$ and $j \in \{n+3, n+4, \ldots, 2n-1\}$ such that $v_0v_i, v_nv_j \in E(G)$. Then $G^{\sigma_1} = G^{\sigma(v_0,v_i;v_n,v_j)}$ is a $\mathbb{J}$-switching with respect to $G$ and $v_0v_n \in [L,R](G^{\sigma(v_0,v_i;v_n,v_j)})$. Similarly if $v_1v_{n+1} \notin E(G^{\sigma(v_0,v_i;v_n,v_j)})$, then there exists a $\mathbb{J}$-switching $\sigma_2$ with respect to $G_1 = G^{\sigma_1}$ such that $G_2 = G^{\sigma_1\sigma_2}$ and $v_0v_n, v_1v_{n+1} \in G_2$. If $v_2v_i \in E(G_2)$ and $i \neq 2n-1$, then there exists $j$, $3 \leq j \leq n-2$, such that $v_{2n-1}v_j \in E(G_2)$ and $v_2v_{2n-1} \in E(G_3) = E(G_2^{\sigma_3})$, where $\sigma_3 = \sigma(v_2, v_i; v_{2n-1}, v_j)$. Let $j$ be smallest integer in $\{3, 4, \ldots, n-1\}$ such that $v_jv_{2n-j+1} \notin E(G_3)$. Then there exist $i$ and $k$ such that $j < k < n$, $n+1 < i < 2n-j+1$ and $v_jv_i, v_kv_{2n-j+1} \in E(G_3)$. Thus $v_jv_{2n-j+1} \in E(G_4) = E(G_3^{\sigma_4})$, where $\sigma_4 = \sigma(v_j, v_i; v_{2n-j+1}, v_k)$. Similarly there exists a sequence of $\mathbb{J}$-switchings $\sigma_5, \sigma_6, \ldots, \sigma_t$ with respect to $G_4$ such that $G(2n) \cong G_4^{\sigma_5\sigma_6\cdots\sigma_t} = G^{\sigma_1\sigma_2\cdots\sigma_t}$. This completes the proof. ■

By Theorem 2.5, we have the following theorem.

**Theorem 2.6** *Let $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

Let $n$ be an integer and $n \geq 6$. Let $H(2n)$ be a graph of order $2n$ with $V(H(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(H(2n)) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 (\text{mod } 2n)\} \cup \{v_0v_2, v_1v_{2n-1}\} \cup \{v_{n-1}v_{n+1}, v_nv_{n+2}\} \cup \{v_jv_{2n-j+1} : j = 3, 4, \ldots, n-2\}$.

It is clear that $H(2n)$ is a cubic hamiltonian graph of order $2n$ containing a triangle. Again put $C$ as its hamiltonian cycle with $E(C) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 (\text{mod } 2n)\}$.

Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$ and let $G \in \mathbb{J}$. Suppose further that $G$ is hamiltonian. Up to isomorphism, we may suppose that $G$ contains $C$ as its hamiltonian cycle. Thus the following sets are uniquely determined.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \ldots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and
$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

Since $e \in [L, R](G)$ if and only if $e \in [R, L](G)$, we have $[L, R](G) = [R, L](G)$ and $|[L, L](G)| = |[R, R](G)|$. Suppose that $v_0v_2 \notin E(G)$. Thus there exist $i, j$ such that $v_0v_i, v_2v_j \in E(G)$ and $G_1 = G^{\sigma(v_0, v_i; v_2, v_j)}$ contains $v_0v_2$ as its edge. By using the same argument as described previously, there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G_t = G^{\sigma_1\sigma_2\cdots\sigma_t}$ and $[L, L](G_t) = \emptyset$. Again by using the same argument as described in the proof of Theorem 2.5 we have the following theorem.

**Theorem 2.7** *Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$. If $G \in \mathbb{J}$ and $[R, R](G) = \emptyset$, then $G = H(2n)$ or there exists a sequence of $\mathbb{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \cong H(2n)$.*

■

By Theorem 2.7, we have the following theorem.

**Theorem 2.8** *Let $\mathbb{J} \in \{\mathbb{J}_1 - \mathbb{J}_3, \mathbb{J}_2 - \mathbb{J}_3\}$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

Combining results in this section we can conclude the following theorem.

**Theorem 2.9** *Let $\mathbb{J} \in X_n$. Then the subgraph of $\mathcal{CR}(3^n)$ induced by $\mathbb{J}$ is connected.*

■

# 3 Interpolation and Extremal Results

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If $f$ is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, $f$ is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers $x$ and $y$ such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

Studying interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter $f$ and a subset $\mathbb{J}$ of $\mathbb{G}$, does $f$ interpolate over $\mathbb{J}$? If $f$ interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by

$$\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\} \text{ and } \max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}.$$

Thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f, \mathbb{J})$ and $\max(f, \mathbb{J})$ for the corresponding interpolation graph parameter $f$ and the set $\mathbb{J}$ and this part falls into the category of the extremal problems in graph theory.

*Acyclic graph* is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree.

Erdős et al. [4] first defined a graph parameter $t$ as follows. Let $G$ be a graph and $F \subseteq V(G)$. $F$ is called an *induced forest* of $G$ if $G[F]$ is acyclic. An induced forest $F$ of $G$ is *maximal* if for every $v \in V(G) - F$, $F \cup \{v\}$ is not an induced forest of $G$. Let $t(G)$ be defined as

$$t(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

Thus $t(G)$ is the maximum cardinality of induced forests of $G$. An induced forest $F$ of $G$ with $|F| = t(G)$ is call a *maximum induced forest* of $G$.

The second author proved in [9] that if $G$ is a graph and $\sigma$ is a switching on $G$, then $|t(G) - t(G^\sigma)| \leq 1$. With this result and the results in Section 2 we have the following theorem.

**Theorem 3.1** *Let $\mathbb{J} \in X_n$. Then $t$ is an interpolation graph parameter with respect to $\mathbb{J}$.*

■

We now answer the second part of interpolation theorem. In other words, we will find $\min(t, \mathbb{J})$ and $\max(t, \mathbb{J})$ for all $\mathbb{J} \in X_n$.

Let $G$ be a graph and $X, Y$ be disjoint nonempty subsets of $V(G)$. Denote by $e(X)$ the number of edges in $G[X]$ and $e(X, Y)$ the number of edges in $G$ connecting vertices in $X$ to vertices in $Y$.

Let $G$ be a cubic graph of order $n$ and $F$ be a maximum induced forest of $G$. Let $|F| = f$ and therefore $G - F$ has order $n - f$. An upper bound of $\max(t, \mathbb{J}_1)$ can be obtained by the following obvious identities.

1. $\frac{3n}{2} = |E(G)| = e(F) + e(F, V(G - F)) + e(V(G - F))$
2. $3(n - f) = e(F, V(G - F)) + 2e(V(G - F))$.

It follows that
$\frac{3n}{2} = e(F) + 3(n-f) - e(V(G-F)) \leq f - 1 + 3(n-f)$.
Therefore $f \leq \frac{3n-2}{4}$.
Since $\mathbb{J}_3 \subseteq \mathbb{J}_2 \subseteq \mathbb{J}_1$, $\max(t, \mathbb{J}_3) \leq \max(t, \mathbb{J}_2) \leq \max(t, \mathbb{J}_1) \leq \frac{3n-2}{4}$.

We will show in the next theorem that $\max(t, \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$ by constructing a connected cubic triangle-free graph $G$ of order $n$ and $t(G) = \lfloor \frac{3n-2}{4} \rfloor$. Note that several classes of graphs can be constructed to reach the bound even if we restrict that the graph is planar and $n \geq 8$. We now turn to consider the graph $G(2n)$ as we have constructed in Section 2. $G(2n)$ is a connected cubic triangle-free graph of order $2n$ if $n \geq 3$. We have the following theorem.

**Theorem 3.2** *If $n = 2m \geq 6$, then $t(G(2m)) = \lfloor \frac{3n-2}{4} \rfloor$.*

**Proof.** Let $G = G(2m) - \{v_0v_m, v_1v_{m+1}\}$ and $S = \{v_1, v_3, \ldots, v_m\}$ if $m$ is odd and $S = \{v_1, v_3, \ldots, v_{m-1}, v_m\}$ if $m$ is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. Since $v_1, v_m \in S$, adding edges $v_0v_m, v_1v_{m+1}$ to $G$ will not create any cycle in $F$. Therefore $t(G(2m)) = \lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. ■

**Corollary 3.3** $\max(t, \mathbb{J}_3) = \max(t, \mathbb{J}_2) = \max(t, \mathbb{J}_1) = \lfloor \frac{3n-2}{4} \rfloor$.

■

Let $G$ be a connected cubic graph of order $n$ containing $K_4'$ as its subgraph. Put $G' = G - K_4'$. Thus $G'$ has order $n-5$ and size $\frac{3n}{2} - 8$. Moreover $t(G) = t(G') + t(K_4') = t(G') + 3$. Let $F'$ be a maximum induced forest of $G'$ and $|F'| = f'$. We have
$\frac{3n}{2} - 8 = e(V(G')) \leq (f' - 1) + 3(n - 5 - f') = 3n - 16 - 2f'$.
It follows that $f' \leq \frac{3n-16}{4}$ and $t(G) \leq \frac{3n-16}{4} + 3 = \frac{3n-4}{4}$.

**Theorem 3.4** $\max(t, \mathbb{J}_1 - \mathbb{J}_2) = \lfloor \frac{3n-4}{4} \rfloor$.

**Proof.** It is clear that $\mathbb{J}_1 - \mathbb{J}_2 \neq \emptyset$ if and only if $n$ is even and $n \geq 10$. If $n = 10$, we can take $G$ as a graph obtained from two copies of $K_4'$ and joining the two vertices of degree 2. Let $n - 6 = 2m \geq 6$ and $H = G(2m)$. Thus by Theorem 3.2, we have $t(H) = \lfloor \frac{3(n-6)-2}{4} \rfloor$. In the proof of Theorem 3.2, let $e = v_0v_1 \in E(H)$ and define a graph $K$ with $V(K) = V(H) \cup \{x\}$ and $E(K) = (E(H) - e) \cup \{v_0x, xv_1\}$ and finally let $G$ be a graph obtained from $K$ and a copy of $K_4'$ by joining the two vertices of degree 2. Thus $t(G) = \lfloor \frac{3(n-6)-2}{4} \rfloor + 1 + 3 = \lfloor \frac{3(n-6)-2}{4} \rfloor + 4 = \lfloor \frac{3n-4}{4} \rfloor$.

■

Since $\mathbb{J}_2 - \mathbb{J}_3 \subseteq \mathbb{J}_1 - \mathbb{J}_3$, $\max(t, \mathbb{J}_2 - \mathbb{J}_3) \leq \max(t, \mathbb{J}_1 - \mathbb{J}_3) \leq \lfloor \frac{3n-2}{4} \rfloor$.

**Theorem 3.5** $\max(t, \mathbb{J}_2 - \mathbb{J}_3) = \max(t, \mathbb{J}_1 - \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$.

**Proof.** Let $m$ be an integer such that $n = 2m$. The graph $H(2m)$ contains a triangle. Let $S = \{v_1, v_3, \ldots, v_m\}$ if $m$ is odd and $S = \{v_1, v_3, \ldots, v_{m-1}, v_m\}$ if $m$ is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3n-2}{4} \rfloor$. Thus $\max(t, \mathbb{J}_2 - \mathbb{J}_3) = \max(t, \mathbb{J}_1 - \mathbb{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$.

■

Thus the values of $\max(t, \mathbb{J})$ are obtained for all $\mathbb{J} \in X_n$.

The problem of finding lower bounds of $t(G)$, where $G$ runs over a class of cubic graphs, have been investigated in the literature. First observe that if $G$ is a cubic graph of order $n$ and $F$ is a maximum induced forest of $G$, then $G - F$ is a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if $n$ is a multiple of 4. The second author proved in [10] that if $n = 4q+t$, $t = 0, 2$, then $\min(t, \mathcal{R}(3^n)) = 2q+t$. Liu and Zhao [7] proved that $t(G) \geq \frac{5}{8}n - \frac{1}{4}$ for any connected cubic graph $G$ of order $n$. This means that for any connected cubic graph of order $n$, the bound of the order of maximum induced forest can be improved from 50 percent of the number of vertices to 60 percent.

The second author proved in [11] the following result.

**Theorem 3.6** *Let $n$ be an even integer with $n \geq 12$. Then*

$$\min(t, \mathbb{J}_1) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2(\text{mod } 8), \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

■

Note that a graph $G$ that has been constructed in such a way that $t(G) = \min(t, \mathbb{J}_1)$ and $G$ contains $K_4'$ as its subgraph. Thus we have the following corollary.

**Corollary 3.7** *Let $n$ be an even integer with $n \geq 12$. If $\mathbb{J} \in \{\mathbb{J}_1, \mathbb{J}_1 - \mathbb{J}_2, \mathbb{J}_1 - \mathbb{J}_3\}$, then*

$$\min(t, \mathbb{J}) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2(\text{mod } 8), \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

■

Zheng and Lu [14] proved that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph $G$ of order $n$ without triangle, except for two cubic graphs with $n = 8$ and $t(G) = 5$. This means that if we look at the class of connected cubic graphs of order $n$ containing no triangle, then the bound can be improved from 60 percent of the number of vertices to 66 percent.

It is easy to see that there exists cubic graph $G$ of order $n$ containing triangles and $t(G) \geq \frac{2n}{3}$.

We are able to extend the result of Zheng and Lu [14] by proving that $t(G) \geq \frac{2n}{3}$ for any connected cubic $K_4'$-free graph $G$ of order $n \geq 10$.

Let $H$ be a graph. A graph $G$ is called an *$H$-free graph* if $G$ does not contain $H$ as an induced subgraph. Let $X$ be a set of graphs. Then a graph $G$ is called an *$X$-free graph* if for every $H \in X$, $G$ is an $H$-free graph. In [12], five connected cubic graphs of order 8 are given, all of which have maximum induced forests of order 5. Alon et al. [1] proved that let $G$ be a $\{K_4, K_4'\}$-free graph with maximum degree 3. If $G$ is of order $n$ and of size $m$, then $t(G) \geq n - \frac{m}{4}$. Consequently, if $G$ is a connected cubic $K_4'$-free graph of order $n \geq 10$, then $t(G) \geq \frac{5n}{8}$.

Observe that if $G$ is a connected $K_4'$-free graph of order 8 and $\Delta(G) = 3$, then $t(G) \geq 5$ and $t(G) = 5$ if and only if $G$ is a cubic graph.

**Theorem 3.8** *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let $G$ be an $X$-free graph of order $n$ with $\Delta(G) = 3$. Then $t(G) \geq \frac{2n}{3}$.*

**Proof.** Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let $G$ be an $X$-free graph of order $n$. By calculation we found that $\lceil \frac{5n}{8} \rceil = \lceil \frac{2n}{3} \rceil$ for all $n$ with $4 \leq n \leq 10$ and $n \neq 8$. For $n = 8$ we also have $t(G) \geq \frac{2n}{3}$. Thus the theorem holds for all $4 \leq n \leq 10$. Now suppose $n \geq 11$ and $G$ contains a triangle $T$ with $V(T) = \{x, y, z\}$. If there exists a vertex in $V(T)$, say $x$, such that $d_G(x) = 2$, then by induction on $n$ there exists a maximum induced forest $F_1$ of $G - T$ with $|F_1| \geq \frac{2(n-3)}{3}$. Hence $F = F_1 \cup \{x, y\}$ is an induced forest of $G$ and $|F| \geq \frac{2n}{3}$. Suppose for all triangles $T = \{x, y, z\}$ of $G$, $d_G(x) = d_G(y) = d_G(z) = 3$. Since $G$ is a $K_4$-free graph, $|N_G(T)| \geq 2$.

*Case 1.* Suppose $x$ and $y$ have a common neighbor $u$ and $v$ is a neighbor of $z$. Since $G$ is a $K_4'$-free graph, $u$ and $v$ are not adjacent in $G$. Thus by induction on $n$, $G - T$ contains an induced forest of order at least $\frac{2(n-3)}{3}$. Since $d_{G-T}(u) = 1$, any maximum induced forest of $G - T$ must contain $u$. If there is a maximum induced forest $F_1$ of $G - T$ which does not contain $u, v$-path, then $F_1 \cup \{y, z\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$. Suppose for any maximum induced forest $F_1$ of $G - T$, $F_1$ contains $u, v$-path. Since $G' = G - T + uv$ satisfies conditions of the theorem, there is a maximum induced forest $F'$ of $G'$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{y, z\}$ is a maximum induced forest of $G$ of order at least $\frac{2n}{3}$. If $uv \in E(F')$, then $F = (F' - uv) \cup \{y, z\}$ is a maximum induced forest of $G$ of order at least $\frac{2n}{3}$.

*Case 2.* Suppose $x, y, z$ have different neighbors $u, v$ and $w$ respectively. Since $n \geq 11$, $G[\{u, v, w\}]$ is not a triangle. Thus there exist two vertices in $\{u, v, w\}$, say $u, v$, such that $uv \notin E(G)$. We consider the following subcases.

*Subcase 2.1* Suppose that $G-T+uv$ contains $K_4$. Let $H$ be the copy of $K_4$ in $G-T+uv$. Thus $uv \in E(H)$ and $vw \notin E(G)$ . Since $G' = G-T-V(H)$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-7)}{3}$. Let $F'$ be a maximum induced forest of $G'$ and $F_1$ be a maximum induced forest of $H-uv$. Since $|F_1| = 3$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of $G$, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-7)}{3} + 2 + 3 = \frac{2n+1}{3} \geq \frac{2n}{3}$.

*Subcase 2.2* Suppose that $G-T+uv$ contains $K_4'$. Let $H$ be the copy of $K_4'$ in $G-T+uv$. Thus $uv \in E(H)$. Since $G' = G-T-V(H)-\{w\}$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-9)}{3}$. Let $F'$ be a maximum induced forest of $G'$ and $F_1$ be a maximum induced forest of $H-uv$. Since $|F_1| = 4$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of $G$, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-9)}{3} + 2 + 4 = \frac{2n}{3}$.

*Subcase 2.3* Suppose that $G-T+uv$ contains a cubic graph of order 8. Let $H$ be the cubic graph of order 8 in $G-T+uv$. Thus $uv \in E(H)$. Since $G' = G-T-V(H)$ satisfies the condition of the theorem, $t(G') \geq \frac{2(n-11)}{3}$. Let $F'$ be a maximum induced forest of $G'$ and $F_1$ be a maximum induced forest of $H-uv$. Since $|F_1| = 6$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of $G$, $t(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-11)}{3} + 2 + 6 = \frac{2n+2}{3} > \frac{2n}{3}$.

*Subcase 2.4* Suppose that $G-T+uv$ satisfies the condition of the theorem. Thus there exists a maximum induced forest $F'$ of $G-T$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{x,y\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$. If $uv \in E(F)$, then $F = (F_1 - uv) \cup \{x,y\}$ is an induced forest of $G$ of order at least $\frac{2n}{3}$.

Thus the proof is complete. ■

The following corollary is an immediate consequence of Theorem 3.8.

**Corollary 3.9** *Let $G$ be a connected cubic $K_4'$-free graph of order $n \neq 8$. Then $t(G) \geq \frac{2n}{3}$.*

■

According to the result of Zheng and Lu [14] that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph $G$ of order $n$ without triangles, except for two cubic graphs with $n = 8$. They mentioned in their paper that the lower bound is best possible but no proof of this was given in the paper.

We now construct a class of graphs to show that $\min(t, \mathbb{J}_3) = \min(t, \mathbb{J}_2) = \lceil \frac{2n}{3} \rceil$.

First observe the following facts:

1. $t(K_{3,3}) = 4$.
2. $t(Q_3) = 5$, where $Q_3$ is the 3-cube.

3. There is a switching $\sigma$ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $t((2K_4')^\sigma) = 7$. Let $K = (2K_4')^\sigma$.

4. If $e \in E(K_{3,3})$ and $f \in E(Q_3)$, then $t(K_{3,3}-e) = 4$ and $t(Q_3-f) = 6$. Let $P = K_{3,3} - e$ and $Q = Q_3 - f$.

5. Let $n$ be an even integer with $n \geq 12$. Write $n = 6q + t$, $t = 0, 2, 4$ and construct a connected cubic triangle-free graph according to the values of $t$:

   (a) If $t = 0$, construct graph $G$ of order $6q$ by taking $q$ copies of $P$ and joining $q$ appropriate edges between the $q$ copies of $P$.

   (b) If $t = 2$, construct graph $G$ of order $6q+2$ by taking $q-1$ copies of $P$ and a copy of $Q$ and then joining $q$ appropriate edges between them.

   (c) If $t = 4$, construct a graph $G$ of order $6q + 4$ by taking $q - 1$ copies of $P$ and a copy of $K$ and then joining $q$ appropriate edges between them.

It is easy to check that the graphs $G$ constructed above satisfying $t(G) = \lceil \frac{2n}{3} \rceil$. Thus we have the following theorem.

**Theorem 3.10** $\min(t, \mathbb{J}_2) = \min(t, \mathbb{J}_3) = \min(t, \mathbb{J}_2 - \mathbb{J}_3) = \lceil \frac{2n}{3} \rceil$.

■

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# The Decycling Number of Regular Graphs

N. Punnim[1]

**Abstract:** For a graph $G$ and $S \subseteq V(G)$, if $G - S$ is acyclic, then $S$ is called a *decycling set* or *feedback set* of $G$. The *decycling number* of $G$ is defined to be

$$\phi(G) := \min\{|S| : S \subseteq V(G) \text{ is a decycling set}\}.$$

This paper reviews some recent results on interpolation and extremal problems on the decycling number of graphs, with an emphasis on the decycling number of regular graphs. New results on this graph parameter are also presented together with some open problems in this area.

**Keywords:** Decycling set, decycling number, regular graphs
**2000 Mathematics Subject Classification:** 05C07, 05C70

## 1 Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph $G$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set $S$ and therefore we define $n = |V|$ to be the *order* of $G$ and $m = |E|$ the *size* of $G$. To simplify writing, we write $e = uv$ for the edge $e$ that joins the vertex $u$ to the vertex $v$. A *path* of length $k$ in a graph $G$, denoted by $P_k$, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of $G$ such that for all $i = 1, 2, \ldots, k-1$, $u_iu_{i+1}$ are edges of $G$. A $u, v$-path is a path which has $u$ as its first vertex and $v$ as its last vertex in the path. The *degree* of a vertex $v$ of a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph $G$ is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by $S$ in $G$. For a graph $G$ and $X \subseteq E(G)$, we denote $G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ and $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. Any two disjoint graph $G$ and $H$ we define $G \cup H$, their union, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs, since the

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operation "$\cup$" is associative. For a graph $G$ and $s \in V(G)$, the *neighborhood* of $s$ in $G$ is defined by

$$N(s) = \{v \in V(G) : sv \in E(G)\}.$$

If $S \subseteq V(G)$, then we define

$$N(S) = \bigcup_{s \in S} N(s).$$

If $F \subseteq V(G)$, we write $N_F(S)$ for $N(S) \cap F$. A graph $G$ is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph.*

Let $G$ be a graph of order $n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph $G$ is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph $G$. In this case, $G$ is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [15] and Hakimi [7]. We state their results in the following theorem.

**Theorem 1.1** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

*Then $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.*

Let $G$ be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from $G$ by a switching has the same degree sequence as $G$. The following theorem has been shown by Havel [15] and Hakimi [7].

**Theorem 1.2** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If $G_1$ and $G_2$ are any two realizations of $\mathbf{d}$, then $G_2$ can be obtained from $G_1$ by a finite sequence of switchings.*

As a consequence of Theorem 1.2, Eggleton and Holton [5] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

**Theorem 1.3** *The graph $\mathcal{R}(\mathbf{d})$ is connected.*

The following theorem was shown by Taylor [22] in 1980.

**Theorem 1.4** *For a graphic degree sequence $\mathbf{d}$, let $\mathcal{CR}(d)$ be the set of all connected realizations of $\mathbf{d}$. Then the induced subgraph $\mathcal{CR}(\mathbf{d})$ of $\mathcal{R}(\mathbf{d})$ is connected.*

## 2 The decycling number

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph $G$ is difficult even for some simply defined graphs. For a graph $G$, this minimum number is known as the *decycling number* of $G$, and is denoted by $\phi(G)$. The class of those graphs $G$ of which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if $G$ has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $K_{p,\,q} = p - 1$ if $p \leq q$, where $K_n$ denotes the complete graph of order $n$ and $K_{p,\,q}$ denotes the complete bipartite graph with partite sets of cardinality $p$ and $q$.

For a graph $G$ and $S \subseteq V(G)$, if $G - S$ is acyclic, then $S$ is called a *decycling set* or *feedback set* of $G$. The *decycling number* of $G$ is defined to be

$$\phi(G) := \min\{|S| : S \subseteq V(G) \text{ is a decycling set}\}.$$

There is a counterpart graph parameter of $\phi$ which has been investigated in the literature, which is defined by

$$t(G) := \max\{|F| : F \subseteq V(G) \text{ is an induced forest of} G\}.$$

It is clear that, for a graph $G$ of order $n$, $\phi(G) + t(G) = n$.

Let $\mathbb{G}$ be the class of all simple graphs. A function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G, H \in \mathbb{G}$ and $G \cong H$, then $f(G) = f(H)$.

Let $\mathbb{G}$ be the class of all simple graphs and $\mathbb{J} \subseteq \mathbb{G}$. A graph parameter $f$ is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if $G_1, G_2 \in \mathbb{J}$ and $a = f(G_1) < f(G_2) = b$, then for every integer $c$ between $a$ and $b$, there exists $G \in \mathbb{J}$ such that $f(G) = c$.

Studying interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter $f$ and a subset $\mathbb{J}$ of $\mathbb{G}$, does $f$ interpolate over $\mathbb{J}$? If $f$ interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$. Thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f, \mathbb{J})$ and $\max(f, \mathbb{J})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph $G$ contains spanning trees having $m$ and $n$ end-vertices, with $m < n$, does $G$ contain a spanning tree with $k$ end-vertices for every integer $k$ with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. Details can be found in [8], [9], [10], [11], [12], [13], [14].

We proved in [17] that for a graph $G$ and a switching $\sigma$ of $G$, $|\phi(G) - \phi(G^\sigma)| \leq 1$. Thus we have the following results.

1. $\phi$ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$.
2. $\phi$ is an interpolation graph parameter with respect to $\mathcal{CR}(\mathbf{d})$.
3. Let $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$. If the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathbb{J}$ is connected, then $\phi$ is an interpolation graph parameter as $\phi$.

4. We can replace $\phi$ by any graph parameter $f$ and obtain the interpolation result with respect to $\mathbb{J}$ if $|f(G) - f(G^\sigma)| \leq 1$ holds.

# 3 Extremal Problem

An extremal problem asks for minimum and maximum values of a function over a class of objects. Consider the following simple example.

**Example 3.1** In the class $\mathcal{C}_n$ of connected graphs of order $n$ and for $G \in \mathcal{C}_n$, $f(G)$ is defined to be the number of edges of $G$. Thus the minimum and maximum values of $f$ in $\mathcal{C}_n$ are $n-1$ and $\binom{n}{2}$, respectively.

It is easy to see that $f$ is an interpolation graph parameter with respect to $\mathcal{C}_n$.

**Remark 3.2** Proving that $A$ is the minimum of $f(G)$ for graphs in a class $\mathbb{J}$ requires showing two things:

1. $f(G) \geq A$ for all $G \in \mathbb{J}$.
2. $f(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of $f$.

Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

In graph theory, we use "extremal problem" for finding an optimal over a class of graphs. When seeking extremes in a single graph, such as the maximum size of an independent set, or maximum size of a bipartite subgraph, we have a different problem for each graph. To distinguish these from the earlier type of problem, we call them *optimization problem.*

Since an optimization problem has an instance for each graph, we usually can't list all solutions. We may seek a solution procedure or bounds on the answer in terms of other aspect of the input graph. For example if we consider the problem of finding a large bipartite subgraph, then it allows us to introduce the technique of constructive or "algorithmic" proof. (An algorithm is a procedure for performing some task.)

One way to prove that something exists is to build it. Such proofs can be viewed as algorithm. To complete an algorithmic proof, we must prove the algorithm terminate and yields the desired result. This may involve induction, contradiction, finiteness, etc.

The following theorem is an another example of an extremal problem.

**Theorem 3.3** *Every graph $G$ with $m$ edges has a bipartite subgraph of at least $\frac{m}{2}$ edges.*

Our next extremal problem doesn't start with bipartite, but it winds up there. In politics and warfare, seldom do two enemies have a common enemy; usually two of the three combine against the third. Given $n$ factions, how many pairs of enemies can there be if no two enemies have a common enemy?

In the language of graphs, we are seeking for the maximum number of edges in a graph with $n$ vertices with no triangle. Bipartite graphs have no triangles, but also

many non-bipartite graphs have no triangles. Using extremality, we will prove that the maximum is indeed achieved by a complete bipartite graph.

Let $H$ be a graph. A graph $G$ is called an *$H$-free graph* if $G$ has no induced subgraph isomorphic to $H$.

**Theorem 3.4** (Mantel (1907) cited from [23]) *The maximum number of edges in a triangle-free graph with $n$ vertices is $\lfloor n^2/4 \rfloor$.*

From Theorem 3.4, it is reasonable to ask the following question. What is the maximum number of edges that a $K_{r+1}$-free graph of order $n$ can have? Turán (1910-1976) answered this question. This famous result of Turán generalizes Theorem 3.4 and is viewed as the origin of *Extremal Graph Theory*. In order to prove Turán's result, we first give the following definition.

The *Turán graph* $T_{n,r}$ is the complete $r$-partite graph with $n$ vertices whose partite sets differ in size by at most 1. By the pigeonhole principle, some partite set has size at least $\lfloor n/r \rfloor$ and some has size at most $\lceil n/r \rceil$. Therefore, differing by at most 1 means that they all have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.

Let $a = \lfloor n/r \rfloor$. After putting $a$ vertices in each partite set, $b = n - ra$ remain, so $T_{n,r}$ has $b$ partite sets of size $a+1$ and $r-b$ partite sets of size $a$. Thus the defining condition on $T_{n,r}$ specifies a single isomorphism class.

**Lemma 3.5** *Among simple $r$-partite graphs with $n$ vertices, the Turán graph is the unique graph with the most edges.*

The following extremal theorem proved by Turán (1941, cited from [23]) and is considered as the origin of a new area of graph theory called extremal graph theory.

**Theorem 3.6** *Among the graphs of order $n$ with no $(r+1)$-clique, $T_{n,r}$ has the maximum number of edges.*

We solved in [17] the extremal problems for $\phi$ in the classes $\mathcal{R}(r^n)$ and $\mathcal{CR}(r^n)$ and with these notation,

$\min(\phi, r^n) = \min\{f(G) : G \in \mathcal{R}(r^n)\}$, $\max(\phi, r^n) = \max\{f(G) : G \in \mathcal{R}(r^n)\}$,
$\mathrm{Min}(\phi, r^n) = \min\{f(G) : G \in \mathcal{CR}(r^n)\}$, and $\mathrm{Max}(\phi, r^n) = \max\{f(G) : G \in \mathcal{CR}(r^n)\}$,

we obtained the following results.

**Theorem 3.7** *Let $r \geq 3$ and $nr$ be even. Then*

$$\min(\phi, r^n) = \begin{cases} r-1 & \text{if } r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & \text{if } n \geq 2r. \end{cases}$$

Let $G$ be a connected $r$-regular graph and $S$ be a minimum decycling set of $G$. Since for any $v \in S$ there is a connected component $C$ of $G-S$ such that $v$ is adjacent to at least two vertices of $C$, there exists $u \in G-S$ such that $vu = e \in E(G)$ and $G-e$ is connected. Thus for two disjoint connected $r$-regular graphs $G$ and $H$ with minimum decycling set $S$ and $T$ of $G$ and $H$ respectively, there exist $u \in S$, $v \in G-S$, $x \in T$, $y \in H-T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$ and $G-e$,

$H-f$ are connected. A connected $r$-regular graph $K=((G-e)\cup(H-f))+\{ux,vy\}$ satisfies

$$\phi(K)\leq\phi(G\cup H)=\phi(G)+\phi(H),$$

and the following corollary holds.

**Corollary 3.8** *Let $r\geq 3$ and $nr$ be even. Then*

$$\mathrm{Min}(\phi,r^n)=\begin{cases} r-1 & \text{if } r+1\leq n\leq 2r-1,\\ \lceil\frac{nr-2n+2}{2(r-1)}\rceil & \text{if } n\geq 2r.\end{cases}$$

Thus the values of $\mathrm{Min}(\phi,r^n)$ for all $r$ and $n$ are already obtained. In particular the values of $\mathrm{Min}(\phi,3^{2n})$ and $\mathrm{Max}(\phi,3^{2n})$ are found for all $n$.

Determining the values of $\mathrm{Max}(\phi,r^n)$ for $r\geq 4$ are more difficult. Note that $\mathcal{R}(r^n)=\mathcal{CR}(r^n)$ if and only if $r+1\leq n\leq 2r+1$. Thus $\mathrm{Max}(\phi,r^n)=\max(\phi,r^n)$ for all $n\in\{r+1,r+2,\ldots,2r+1\}$. In this case we have already obtained in [17] as stated in the following theorem.

**Theorem 3.9** *For $r\geq 4$, and $n=r+j$, $1\leq j\leq r+1$, then*
(1) $\max(\phi,r^n)=n-2$, *if and only if* $j=1$,
(2) $\max(\phi,r^n)=n-3$, *if and only if* $j=2$,
(3) $\max(\phi,r^n)=n-4$, *for all even integers* $n=r+j$, $3\leq j\leq r+1$,
(4) $\max(\phi,r^n)=n-4$, *for all odd integers* $n=r+j$, $3\leq j\leq r+1$ *and* $n\geq f(j)$,
(5) $\max(\phi,r^n)=n-5$, *for all odd integers* $n=r+j$, $3\leq j\leq r+1$ *and* $n<f(j)$,
*where* $f(j)=\frac{5}{2}(j-1)$ *if* $j\equiv 3(\mathrm{mod}\ 4)$, *and* $f(j)=1+\frac{5}{2}(j-1)$ *if* $j\equiv 1(\mathrm{mod}\ 4)$.

Thus the values of $\mathrm{Max}(\phi,r^n)$ are already obtained for all $r$ and $n$ with $n\leq 2r+1$. The problem of determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. In particular $\mathrm{Max}(\phi,r^n)=n-\mathrm{Min}(t,r^n)$.

We proved the following series of results in [19] to obtain the values of $\mathrm{Max}(\phi,r^n)$.

Let $G$ be a $K_5$-free graph of order $n$, $\Delta(G)=4$. Let $F$ be a maximal induced forest of $G$. We denote by $c(F)$ the number of cycles in $G-F$. A pair $(X,Y)$, where $X\subseteq F$ and $Y\subseteq G-F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F-X)\cup Y$ is a forest, $|(F-X)\cup Y|\geq|F|$, and $c((F-X)\cup Y)<c(F)$. In general we can define an interchangeable pair of vertices for a graph $G$ with $\Delta(G)>4$ as follows. Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G)=\Delta>4$. Let $F$ be a maximal induced forest of $G$. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G-F$. A pair $(X,Y)$, where $X\subseteq F$ and $Y\subseteq G-F$, is an *interchangeable pair of vertices with respect to* $F$ if $(F-X)\cup Y$ is a forest, $|(F-X)\cup Y|\geq|F|$, and $k((F-X)\cup Y)<k(F)$.

Let $G$ be a $K_5$-free graph of order $n$ and $\Delta(G)=4$. Thus for any maximal induced forest $F$ of $G$, $G-F$ is a union of cycles and paths. We choose a maximal induced forest $F$ of $G$ with minimum $c(F)$. In other word, the forest $F$ is chosen in such a way that it contains no interchangeable pair of vertices with respect to $F$. Suppose that $c(F)\geq 1$. Let $C$ be a cycle in $G-F$. Then each vertex of $C$ must be adjacent to exactly two vertices in $F$. Suppose that there exists a vertex $u\in F$, $d_F(u)\geq 2$, and $u$ is adjacent to a vertex $v\in V(C)$, then $(\{u\},\{v\})$ is an interchangeable pair of vertices with respect to $F$. Thus for all cycles $C$ of $G-F$, each vertex $v\in V(C)$, $v$ must be adjacent to exactly two vertices $u_1,u_2\in F$ with

$d_F(u_1) = d_F(u_2) = 1$. By maximality of $F$, $u_1$ and $u_2$ must be in the same connected component of $F$. Since $F$ is a forest, there exists a unique path in $G[F]$ from $u_1$ to $u_2$. If $u_1$ and $u_2$ are not adjacent and there is a vertex $u \in F$ in the path such that $d_F(u) \geq 3$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to $F$. Therefore the connected component of $F$ containing $u_1$ and $u_2$ must be a path. Suppose that there exist exactly two vertices $v, w$ of $C$ adjacent to a vertex $u \in F$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to $F$. Finally suppose that there are three vertices $v, w, z$ of $C$ adjacent to a vertex $u \in F$, then the path $P$ in $G[F]$ containing $u$ has order at least 3 or the cycle $C$ has order at least 4, since otherwise $G$ would contain $K_5$. Let $u$ and $u'$ be the end vertices of $P$ in $G[F]$. Then $N_C(\{u, u'\}) = \{v, w, z\} \subseteq V(C)$. If $P$ has order at least 3, then $(\{u, u'\}, \{v, w\})$ is an interchangeable pair of vertices with respect to $F$. If $P$ has order 2 and $C$ has order at least 4, then $(\{u, u'\}, \{v, w, z\})$ is an interchangeable pair of vertices with respect to $F$. Thus the corresponding paths in $G[F]$ of vertices in $C$ are pairwise disjoint.

**Theorem 3.10** *Let $G$ be a $K_5$-free graph of order $n$, $\Delta(G) = 4$. Then $\phi(G) \leq \frac{n}{2}$.*

**Corollary 3.11** $\mathrm{Max}(\phi, 4^n) \leq \frac{n}{2}$.

Let $G$ be a $K_{\Delta+1}$-free graph with $\Delta(G) = \Delta \geq 5$ and let $F$ be a maximal induced of $G$ with minimum $k(F)$. Then for each $v \in G - F$, there exists a connected component $T$ of $F$ such that $v$ is adjacent to at least two vertices of $T$. Thus $\Delta(G - F) \leq \Delta - 2$. Suppose that $k(F) \geq 1$. Let $K$ be a complete subgraph of $G - F$ of order $\Delta - 1$. Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Thus for each $v_i$ there exists a connected component $P(v_i)$ of $G[F]$ such that $v_i$ is adjacent to exactly two vertices of $P(v_i)$. If there exists $u \in V(P(v_i))$ such that $uv_i \in E(G)$ and $d_F(u) \geq 2$, then $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to $F$. Thus for each $i = 1, 2, \ldots, \Delta - 1$, $v_i$ must be adjacent to exactly two vertices of degree one in $P(v_i)$. Suppose that there exists $u \in V(P(v_i))$ such that $d_F(u) \geq 3$, $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to $F$. Thus the corresponding $P(v_i)$ of $v_i$ in $K$ is a path. Furthermore all such paths are pairwise disjoint.

**Lemma 3.12** *Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G) = \Delta \geq 5$. Then $t(G) \geq \frac{2n}{\Delta}$ or there exists an induced forest $F$ of $G$ such that $G - F$ is a $K_{\Delta-1}$-free graph.*

**Lemma 3.13** *Let $G$ be a connected $K_5$-free graph of order $n$ and $\Delta(G) = 5$. Then $t(G) \geq \frac{2n}{5}$.*

**Lemma 3.14** *Let $G$ be a $K_6$-free graph of order $n$ with $\Delta(G) = 5$. Then $t(G) \geq \frac{2n}{5}$.*

As a direct consequence of Lemma 3.14 we obtain the following theorem.

**Theorem 3.15** *Let $G$ be a connected 5-regular graph of order $n \geq 12$. Then $\phi(G) \leq \frac{3n}{5}$.*

**Lemma 3.16** *Let $G$ be a $K_{\Delta+1}$-free graph of order $n$ with $\Delta(G) = \Delta \geq 5$. Then $t(G) \geq \frac{2n}{\Delta}$.*

We have the following theorem.

**Theorem 3.17** *Let $G$ be a connected $r$-regular graph of order $n \geq 2r+2$. Then $\phi(G) \leq \frac{n(r-2)}{r}$ for all $r \geq 4$.*

Let $G$ be a connected $r$-regular graph of order $n = rq + s$, $0 \leq s \leq r-1$, $r \geq 4$ and $q \geq 1$. Then by Theorem 3.17, we have $t(G) \geq 2q + \lceil \frac{2s}{r} \rceil$. It is easy to construct a connected $r$-regular graph $G$ of order $n$ with $t(G) = 2q$ if $s = 0$, $IG) = 2q+1$ if $s = 1, 2$ and $t(G) = 2q+2$ if $3 \leq s \leq r-1$. Consequently, we have $\mathrm{Max}(\phi, r^n) = n-2q$ if $s = 0$, $\mathrm{Max}(\phi, r^n) = n-2q-1$ if $s = 1, 2$, $\mathrm{Max}(\phi, r^n) = n-2q-2$ if $2s > r$ and $\mathrm{Max}(\phi, r^n) \in \{n-2q-2, n-2q-1\}$ if $3 \leq s \leq \frac{r}{2}$.

Note that we improved an upper bound for $\mathrm{Max}(\phi, r^n)$ from $\max(\phi, r^n)$, for $r \geq 4$ and $n$ with $n \geq 2r+2$, and we have that

$\max(\phi, r^n) - \mathrm{Max}(\phi, r^n) = \frac{2n}{r(r+1)}$, roughly speaking.

Thus we can improve the upper bound for $\mathrm{Max}(\phi, r^n) \approx \frac{n(r-2)}{r}$ from $\max(\phi, r^n) \approx \frac{n(r-1)}{r+1}$.

**Conjecture** $\mathrm{Max}(\phi, r^n) = n-2q-2$ *if* $3 \leq s \leq \frac{r}{2}$, *for all* $r \geq 6$.

# 4 Cubic graphs

The interest of studying the decycling number of cubic graphs was motivated by the results of Zheng and Lu [24], Alon et al. [1], Liu and Zhao [16], and Bau and Beineke [2].

Note that the values of $\min(\phi, 3^n)$ have been already obtained in all situations and $\min(\phi, 3^n) = \mathrm{Min}(\phi, 3^n)$

The problem of finding upper bounds of $\phi(G)$, where $G$ runs over a class of cubic graphs, have been investigated in the literature. First observe that if $G$ is a cubic graph of order $n$ and $F$ is a maximum induced forest of $G$, then It is easy to see that $G-F$ is also a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if $n$ is a multiple of 4. We proved in [17] that if $n = 4q+t$, $t = 0, 2$, then $\max(\phi, 3^n) = 2q$.

The problems of finding $\mathrm{Max}(\phi, 3^n)$ are more difficult. We will consider such problems in terms of the graph parameter $t$. A *cubic tree* is a tree in which its vertices consisting of degree 1 or 3. Evidently if $T$ is a cubic tree of order $n$, then $n = 2k+2$, where $k$ is the number of vertices of degree 3 of $T$. Let $\mathbb{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of $K_4$ with one subdivided edge at every vertex of degree 1.

A lower bound for the order of maximum induced forest in connected cubic graphs has been obtained by Liu and Zhao [16] as stated in the following theorem.

**Theorem 4.1** *Let $G$ be a connected cubic graph of order $n \geq 12$. Then $t(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathbb{T}$ and $t(G) \geq \frac{5}{8}n$ if $G \notin \mathbb{T}$.*

We have determined in [19] the value of $\mathrm{Min}(t, 3^n)$ by observing the following situation. First observe that if $G \in \mathbb{T}$, then $G$ has order $8k+10$, where $k$ is the number

of vertices of degree 3 in the corresponding cubic tree. Thus $t(G) = \mathrm{Min}(t, 3^{8k+10}) = 5k + 6$. We now consider a cubic graph of order $8k + 8$. Let $C$ be a cubic graph of order $8k + 8$. Then by Theorem 4.1, $t(C) \geq \frac{5}{8}(8k + 8) = 5(k + 1)$. A cubic graph $T$ obtained by taking cubic tree with $k$ vertices of degree 3, replacing $k-1$ of the vertices by a triangle and attaching a copy of $K$ at every vertex of degree 1. Thus $T$ has order $8k + 8$ and $t(T) = 5(k + 1)$. Thus $\mathrm{Min}(I, 3^{8k+8}) = 5(k + 1)$. The value of $\mathrm{Min}(t, 3^n)$, $n = 8k + 4, 8k + 6$ can be obtained in the following argument. Since a switching changes the order of induced forest by at most 1, we have $\mathrm{Min}(t, 3^{p+q}) \leq \mathrm{Min}(t, 3^p) + \mathrm{Min}(t, 3^q) + 1$ for all even integers $p$ and $q$ with $4 \leq p \leq q$. Thus $5k+4 = \lceil \frac{5}{8}(8k+6) \rceil \leq \mathrm{Min}(t, 3^{8k+6}) \leq \mathrm{Min}(t, 3^4) + \mathrm{Min}(t, 3^{8(k-1)+10}) + 1 = 2 + 5(k-1) + 6 + 1 = 5k + 4$. Finally $5k + 3 = \lceil \frac{5}{8}(8k + 4) \rceil \leq \mathrm{Min}(t, 3^{8k+4}) \leq \mathrm{Min}(t, 3^4) + \mathrm{Min}(t, 3^{8(k-1)+8}) + 1 = 2+5k+1 = 5k+3$. Therefore we obtained in [19] the following theorem and corollary.

**Theorem 4.2** *Let $n$ be an even integer with $n \geq 12$. Then*

$$\mathrm{Min}(t, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2(\mathrm{mod}\ 8), \\ \lceil \frac{5}{8}n \rceil & otherwise. \end{cases}$$

**Corollary 4.3** *Let $n$ be an even integer with $n \geq 12$. Then*

$$\mathrm{Max}(\phi, 3^n) = \begin{cases} \frac{3}{8}n + \frac{1}{4} & if\ n \equiv 2(\mathrm{mod}\ 8), \\ \lfloor \frac{3}{8}n \rfloor & otherwise. \end{cases}$$

In [21], there are five connected cubic graphs of order 8, all of which having maximum induced forests of order 5. Alon et al. proved in [1] that if $G$ is a $\{K_4, K_4'\}$-free graph with maximum degree 3. If $G$ is of order $n$ and of size $m$, then $t(G) \geq n - \frac{m}{4}$. Consequently, if $G$ is a cubic $\{K_4, K_4'\}$-free graph of order $n \geq 10$, then $t(G) \geq \frac{5n}{8}$. Zheng and Lu proved in [24] that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph $G$ of order $n$ without triangles, except for two cubic graphs with $n = 8$. They also pointed out that this lower bound is best possible. It is easy to see that there exists cubic graph $G$ of order $n$ containing triangles and $t(G) \geq \frac{2n}{3}$. We have extended their result by proving that $t(G) \geq \frac{2n}{3}$ for any connected cubic $K_4'$-free graph $G$ of order $n \geq 10$ as stated in the following results.

**Theorem 4.4** *Let $G$ be a connected triangle-free graph of order $n$ and $\Delta(G) = 3$. If $G$ is not a cubic graph, then $t(G) \geq \frac{2n}{3}$.*

**Theorem 4.5** *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let $G$ be an $X$-free graph of order $n$ with $\Delta(G) = 3$. Then $t(G) \geq \frac{2n}{3}$.*

**Theorem 4.6** *Let $G$ be a connected cubic $K_4'$-free graph of order $n$, $n \geq 6$ and $n \neq 8$. Then $t(G) \geq \frac{2n}{3}$.*

We recently constructed in [4] a class of connected triangle-free graphs $\mathbb{J}_n$ of order $n$ to show that $\min(t, \mathbb{J}_n) = \lceil \frac{2n}{3} \rceil$ as follows.

First observe that

1. $t(K_{3,3}) = 4$.
2. $t(Q_3) = 5$, where $Q_3$ is the 3-cube.
3. There is a switching $\sigma$ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $t((2K_4')^\sigma) = 7$. Put $K = (2K_4')^\sigma$.
4. If $e \in E(K_{3,3})$ and $f \in E(Q_3)$, then $t(K_{3,3} - e) = 4$ and $t(Q_3 - f) = 6$. Put $P = K_{3,3} - e$ and $Q = Q_3 - f$.
5. Let $n$ be an even integer with $n \geq 12$. Write $n = 6q+t$, $t = 0, 2, 4$ and construct a connected cubic triangle-free graph according to the values of $t$

   (a) if $t = 0$, construct graph $G$ of order $6q$ by taking $q$ copies of $P$ and joining $q$ appropriate edges between the $q$ copies of $P$.

   (b) if $t = 2$, construct graph $G$ of order $6q+2$ by taking $q-1$ copies of $P$ and a copy of $Q$ and then joining $q$ appropriate edges between them.

   (c) if $t = 4$, construct a graph $G$ of order $6q+4$ by taking $q-1$ copies of $P$ and a copy of $K$ and then joining $q$ appropriate edges between them.
6. It is easy to check that the graphs $G$ constructed above satisfying $t(G) = \lceil \frac{2n}{3} \rceil$.

Thus we have the following theorem.

**Theorem 4.7** [4] $\min(t, \mathbb{J}_n) = \lceil \frac{2n}{3} \rceil$.

# 5 Cubic planar graphs

In the class of connected cubic graphs of order $2n$, we have already proved that in [18]

$$\phi(3^{2n}) = \{\phi(G) : G \in \mathcal{CR}(3^{2n})\} = [\mathrm{Min}(\phi, 3^{2n}), \mathrm{Max}(\phi, 3^{2n})],$$

where

$$\mathrm{Min}(\phi, 3^{2n}) = \lceil \tfrac{n+1}{2} \rceil$$

and

$$\mathrm{Max}(\phi, 3^{2n}) = \begin{cases} \frac{3}{4}n + \frac{1}{4} & \text{if } n \equiv 1 (\mathrm{mod}\ 4), \\ \lfloor \frac{3}{4}n \rfloor & \text{otherwise.} \end{cases}$$

For each $c \in \phi(3^{2n})$, let $\mathcal{CR}(3^{2n};\ \phi = c) = \{G \in \mathcal{CR}(3^{2n}) : \phi(G) = c\}$.

We consider the problem of determining whether the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{CR}(3^{2n};\ \phi = c)$ is connected or not. If it is connected, then for any two graphs $G_1, G_2 \in \mathcal{CR}(3^{2n};\ \phi = c)$ there is a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for each $i = 1, 2, \ldots, t$, $G_1^{\sigma_1\sigma_2\cdots\sigma_i} \in \mathcal{CR}(3^{2n};\ \phi = c)$ and $G_1^{\sigma_1\sigma_2\cdots\sigma_t} = G_2$.

We proved in [18] that $\mathcal{CR}(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil)$ is connected. Nothing has been done for other values of $c$.

That is the class of cubic graphs with smallest decycling number induced a connected subgraph. This provides an algorithm to finding all cubic graphs with smallest decycling number in the class of connected cubic graphs and this also leads to an answering to the Problem 1 asked by Bau and Beineke [2].

The following problems asked by Bau and Beineke [2].

**Problem 1** Which cubic graphs $G$ of order $2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

**Problem 2** Which cubic planar graphs $G$ of order $2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

We will give an outline of proof to answer Problem 2. Details can be found in [20].

Let us first consider the following example.

**Example 5.1** $\text{Min}(\phi, 3^{34}) = \frac{17+1}{2} = 9$ and $\text{Max}(\phi, 3^{34}) = \frac{3 \cdot 17}{4} + \frac{1}{4} = 13$.

Thus $\mathcal{CR}(3^{34})$ can be partitioned into five classes $C_i (9 \leq i \leq 13)$, where

$C_i = \{G \in \mathcal{CR}(3^{34}) : \phi(G) = i\}$ and $C_i \neq \emptyset$, for all $i = 9, 10, 11, 12, 13$.

For any integer $n \geq 2$ and for each $i \in [\text{Min}(\phi, 3^{2n}), \text{Max}(\phi, 3^{2n})]$ we have $C_i = \{G \in \mathcal{CR}(3^{2n}) : \phi(G) = i\} \neq \emptyset$.

Let $\mathcal{P}(3^{2n})$ be the class of all connected cubic planar graphs of order $2n$. We can ask the following questions.

1. Is $\phi$ an interpolation graph parameter with respect to $\mathcal{P}(3^{2n})$?
2. What is the values of $\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\}$ and $\max\{\phi(G) : G \in \mathcal{P}(3^{2n})\}$?
3. Is the subgraph of the graph $\mathcal{CR}(3^{2n})$ induced by $\mathcal{P}(3^{2n})$ connected?
4. Which cubic planar graphs $G$ of order $2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$? (Problem 2)
5. Is the subgraph $\mathcal{P}(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil)$ connected?

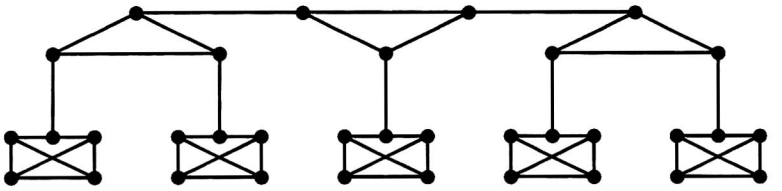
Question 3 implies Question 1.

That is if the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{P}(3^{2n})$ is connected and with the fact that $|\phi(G) - \phi(G^\sigma)| \leq 1$, then $\phi$ is an interpolation graph parameter with respect to $\mathcal{P}(3^{2n})$.

Answering Question 3 seems to be difficult. We will answer Question 1 directly.

First let us consider the following examples and the proofs of Question 1 and Question 2 are easily obtained.

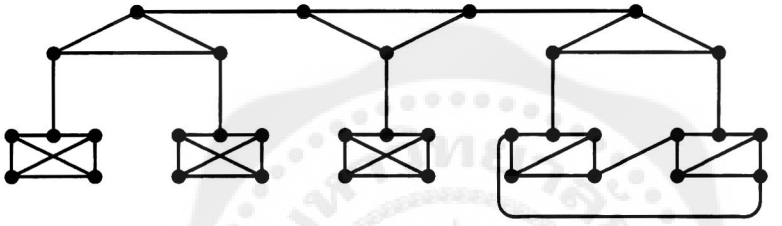
**Example 5.2** Graph $G_1$



$= G_1$

$G_1$ of order 34 and $\phi(G_1) = 13 = \text{Max}(\phi, 3^{34})$.

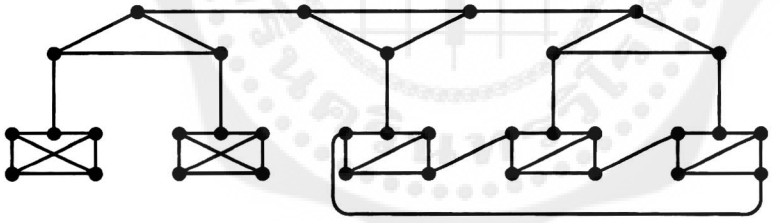
**Example 5.3** Graph $G_2$



$= G_2$

$G_2$ of order 34 and $\phi(G_2) = 12 = \text{Max}(\phi, 3^{34}) - 1$.

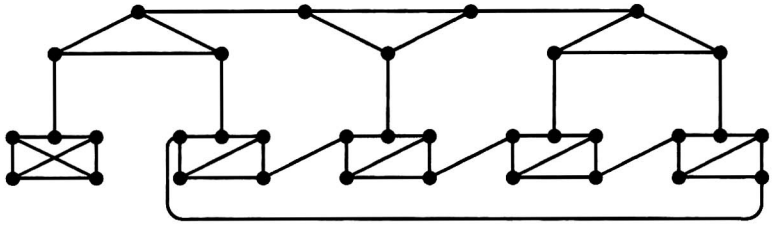
**Example 5.4** Graph $G_3$



$= G_3$

$G_3$ of order 34 and $\phi(G_3) = 11 = \text{Max}(\phi, 3^{34}) - 2$.

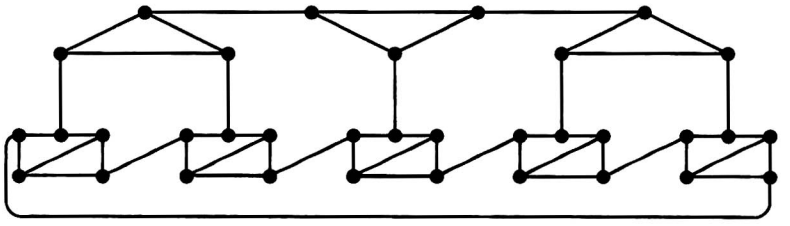
**Example 5.5** Graph $G_4$



$= G_4$

$G_4$ of order 34 and $\phi(G_4) = 10 = \text{Max}(\phi, 3^{34}) - 3$.

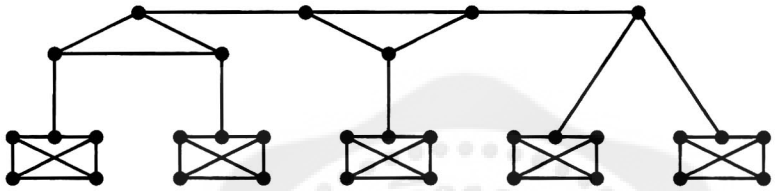
**Example 5.6** Graph $G_5$



$= G_5$

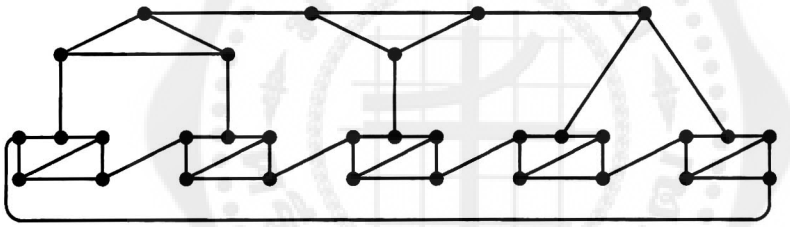
$G_5$ of order 34 and $\phi(G_5) = 9 = \text{Max}(\phi, 3^{34}) - 4 = \text{Min}(\phi, 3^{34})$.

**Example 5.7** Consider the following graphs $G_1$ and $G_5$.



$= G_1$

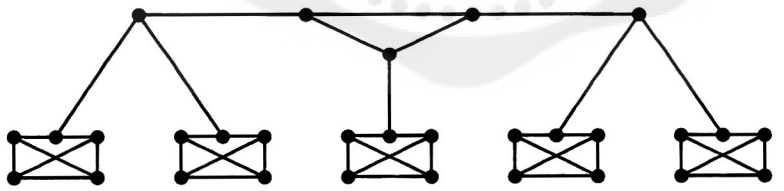
Cubic planar graph $G_1$ of order $8k + 8 = 32$, $\phi(G) = 12 = \text{Max}(\phi, 3^{32})$



$= G_5$

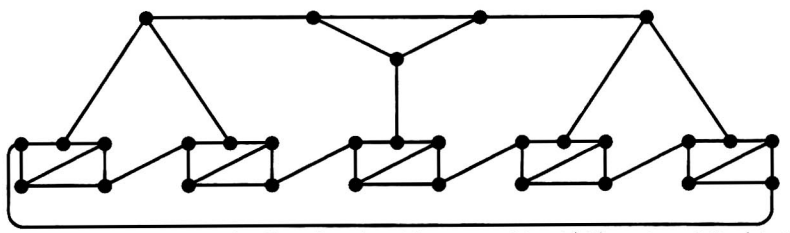
Cubic planar graph $G_5$ of order $8k + 8 = 32$, $\phi(G) = 9 = \text{Min}(\phi, 3^{32})$

**Example 5.8** Consider the following graphs $G_1$ and $G_5$.



$= G_1$

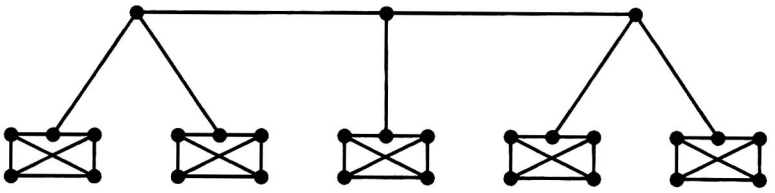
Cubic planar graph $G_1$ of order $8k + 6 = 30$, $\phi(G) = 11 = \text{Max}(\phi, 3^{30})$



$= G_5$

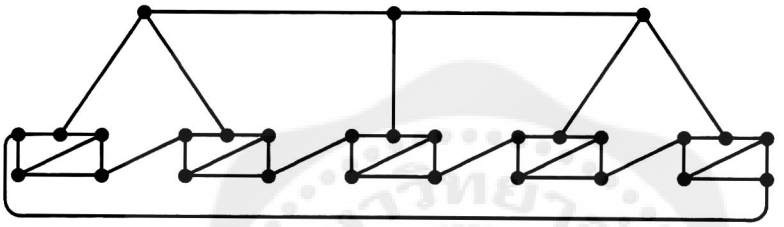
Cubic planar graph $G_5$ of order $8k + 6 = 30$, $\phi(G) = 8 = \text{Min}(\phi, 3^{30})$

**Example 5.9** Consider the following graphs $G_1$ and $G_5$.



$= G_1$

Cubic planar graph $G_1$ of order $8k+4 = 28$, $\phi(G) = 10 = \text{Max}(\phi, 3^{28})$



$= G_5$

Cubic planar graph $G_5$ of order $8k+4 = 28$, $\phi(G) = 8 = \text{Min}(\phi, 3^{28})$

Thus we have proved the following results:

**Theorem 5.10** (Question 1) *$\phi$ is an interpolation graph parameter with respect to $\mathcal{P}(3^{2n})$.*

**Theorem 5.11** (Question 2)

$$\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\} = \text{Min}(\phi, 3^{2n}) \text{ and}$$
$$\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\} = \text{Min}(\phi, 3^{2n}).$$

We proved in [18] the following result.

Let $G \in \mathcal{CR}(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil)$ and $S$ be a minimum decycling set of $G$. Put $F = V(G) - S$. Then

1. $e(S) = 0$, if $n$ is odd and $e(S) \leq 1$, if $n$ is even,

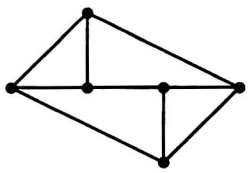
2. if $n$ is odd, then $G[F]$ is a tree,

3. if $n$ is even and $e(S) = 1$, then $G[F]$ is a tree,

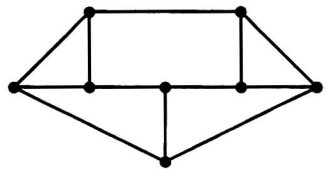
4. if $n$ is even and $e(S) = 0$, then $G[F]$ has 2 connected components.

**Theorem 5.12** *There exists $G \in \mathcal{P}(3^{2n})$ such that $\phi(G) = \lceil \frac{n+1}{2} \rceil$ and $G$ contains a path of order $\lfloor \frac{3n-1}{2} \rfloor$ as its induced forest. Furthermore the induced subgraph of the corresponding decycling set of $G$ contains at most one edge and it contains no edge if and only if $n$ is odd.*

A proof of above theorem can be obtained easily by introducing the graphs $G_6$ and $G_8$ as in the following figures and the definition of *super subdivision.*

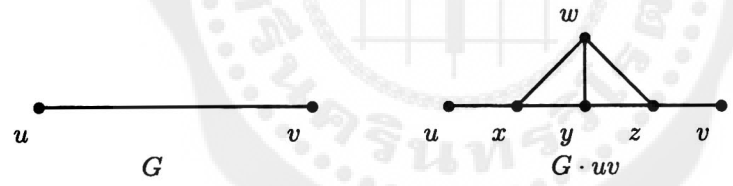


$G_6$

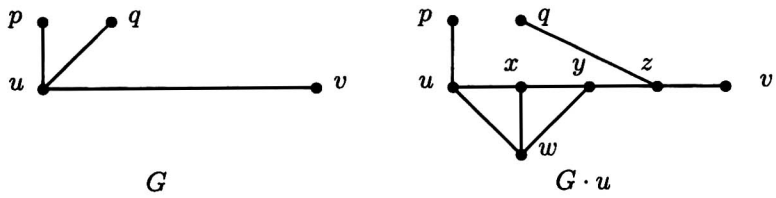


$G_8$

**Definition 5.13** Let $G$ be a connected cubic planar graph and $F$ is an induced forest of $G$. If $u, v \in F$ and $uv \in E(G)$, define a super subdivision with respect to $uv$ of $G$ to be a graph $G \cdot uv$ obtained from $G$ by inserting three vertices $x, y, z$ on the edge $uv$ and joining $x, y, z$ to a new vertex $w$. Let $u \in F$, $d_F(u) = 1$. Then there exists a unique vertex $v \in F$ that is adjacent to $u$ and there exist two vertices $p, q \in V(G-F)$ such that $p$ and $q$ are adjacent to $u$. Define a super subdivision with respect to $u$ of $G$ to be a graph $G \cdot u$ obtained from $G$ as follows: Inserting three vertices $x, y, z$ on the edge $uv$ to produce a path $uxyzv$ in $F$, deleting $pu$, and joining $pz$, and finally joining $u, x, y$ to a new vertex $w$.




super subdivision




Let $\mathcal{P}(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil)$ be the class of cubic planar graphs $G$ of order $2n$ and $\phi(G) = \lceil \frac{n+1}{2} \rceil$. We obtained in [4] an algorithm of finding all cubic planar graphs $G$ of order $2n$ with decycling number $\lceil \frac{n+1}{2} \rceil$. In addition, we proved the following theorem.

**Theorem 5.14** *The subgraph* $\mathcal{P}(3^{2n};\ \phi = \frac{n+1}{2})$ *is connected.*

## 6 Acknowledgements

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# Almost Hamiltonian Cubic Graphs

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**Abstract**

A Hamiltonian walk in a connected graph $G$ of order $n$ is a closed spanning walk of minimum length in $G$. For a connected graph $G$, let $h(G)$ be the length of a Hamiltonian walk in $G$ and call it the *Hamiltonian number* of $G$. Let $i$ be a non-negative integer. A connected graph $G$ of order $n$ is called an *i-Hamiltonian* if $h(G)$ $n+i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an *almost Hamiltonian graph*. We prove in this paper that for an even integer $n \geq 10$ there exists an almost Hamiltonian cubic graph of order $n$. Let $P(k, m)$ be the generalized Petersen graph of order $2k$. We show that $P(k, m)$ is an almost Hamiltonian graph if and only if $m-2$ and $k \equiv 5 \pmod 6$. For a cubic graph $G$, we define $G^*$ to be the graph obtained from $G$ by replacing each vertex of $G$ to a triangle, matching the vertices of the triangle to the former neighbors of the replaced vertex. We show that $G$ is Hamiltonian if and only if $G^*$ is Hamiltonian and if $G$ is almost Hamiltonian then $G^*$ is 2-Hamiltonian.

***Key words:***
*Hamiltonian walk, Hamiltonian number, and cubic graph.*

## 1. Introduction

While certainly not every connected graph of order at least 3 contains a Hamiltonian cycle, every connected graph does contain a closed spanning walk (in which all vertices are encountered, possibly more than once). If $G$ is a connected graph of size $m$, there is always a closed spanning walk of length at most $2m$. In [6, 7] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph $G$, defined as a closed spanning walk of minimum length in $G$. They denoted the length of a Hamiltonian walk in $G$ by $h(G)$. Therefore, for a connected graph $G$ of order $n \geq 3$, it follows that $h(G) = n$ if and only if $G$ is Hamiltonian. Hamiltonain walks were studied further by T. Asano, T. Nishizeki, and T. Watanabe [2, 3], J. C. Bermond [4], and P. Vacek [9]. Thus $h$ may be considered as a measure of how far a given graph is from being Hamiltonian.

In [5] an alternative way to define the length $h(G)$ of a Hamiltonian walk in a connected graph $G$ was presented. A Hamiltonian graph $G$ contains a spanning cycle $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$. Thus Hamiltonian graphs of order $n \geq 3$ are those graphs for which there is a cyclic ordering $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i-1})$ is the distance between $v_i$ and $v_{i+1}$ for $1 \leq i \leq n$. For a connected graph $G$ of order $n \geq 3$ and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the elements of $V(G)$, the number $d(s)$ is defined as $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$. Therefore, $d(s) \geq n$ for each cyclic ordering $s$ of the elements of $V(G)$. The *Hamiltonian number* $h(G)$ of $G$ is defined in [5] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings $s$ of elements of $V(G)$. It was shown in [5] that the Hamiltonian number of a connected graph $G$ is, in fact, the length of a Hamiltonian walk in $G$.

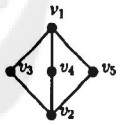


Figure 1: A graph $G$ with $h(G) = 6$

To illustrate these concepts, consider the graph $G = K_{2,3}$ of Figure 1. For the cyclic orderings $s_1 : v_1, v_2, v_3, v_4, v_5, v_1$ and $s_2 : v_1, v_3, v_2, v_4, v_5, v_1$ of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since $G$ is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h(G) = 6$.

Let $i$ be a non-negative integer. A connected graph G of order $n$ is called an *i-Hamiltonian* if $h(G) = n + i$. Thus a 0-Hamiltonian graph is Hamiltonian. An *almost Hamiltonian* graph is a graph $G$ of order $n$ and $h(G) = n + 1$. Thus $K_{2,3}$ is an example of an almost Hamiltonian graph.

The following results are known (see [5, 7]).

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**Theorem A** *For every connected graph $G$ of order $n \geq 2$,*

$$n \leq h(G) \leq 2n - 2.$$

*Moreover,*

1. *$h(G) = 2n - 2$ if and only if $G$ is a tree and*
2. *for every pair $n, k$ of integers with $3 \leq n \leq k \leq 2n-2$, there exists a connected graph $G$ of order $n$ having $h(G) = k$.*

+

Thus for a connected graph $G$ of order $n$, $G$ is an $i$-Hamiltonian graph for some $i = 0, 1, 2, \cdots, n-2$. Moreover, for integers $n$ and $i$ with $n \geq 3$ and $0 \leq i \leq n-2$, there is an $i$-Hamiltonian graph $G$ of order $n$.

Let $P(k, m)$ be a generalized Petersen graph such that $V(P(k, m)) = \{u_i, v_i : i = 0, 1, \cdots, k-1\}$ and $E(P(k, m)) = \{u_iu_{i+1}, v_iv_{i+m}, u_iv_i : i = 0, 1, 2, \cdots, k-1\}$ where addition is taken modulo $k$ and $m \leq \frac{k}{2}$. In [1] Alspach completed the determination of the parameters $k$, $m$ for which $P(k, m)$ is Hamiltonian as stated in the following theorem.

**Theorem B** *The generalized Petersen graph $P(k, m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5 \pmod 6$.*

+

## 2. Almost Hamiltonian cubic graphs

We have seen that a generalized Petersen graph is not a Hamiltonian graph if and only if $m = 2$ and $k \equiv 5 \pmod 6$. Thus in this case $h(P(k, m)) \geq 2k + 1$. We will show in the next theorem that $P(k, m)$ is an almost Hamiltonian graph if and only if $m = 2$ and $k \equiv 5 \pmod 6$.

**Theorem 2.1** *Let $P(k, m)$ be a generalized Petersen graph. Then*

$$h(P(k,m)) = \begin{cases} 2k+1 & \text{if } m = 2 \text{ and } k \equiv 5 \pmod 6, \\ 2k & \text{otherwise.} \end{cases}$$

**Proof.** Let $m = 2$ and $k \equiv 5 \pmod 6$. By Theorem B, it is suffice to show that $h(P(k,2)) = 2k+1$. Consider a closed spanning walk $W : v_0, v_2, \cdots, v_{k-1}, v_1, v_3, \cdots, v_{k-2}, u_{k-2}, u_{k-3}, u_{k-4}, \ldots, u_1, u_0, u_{k-1}, u_0, v_0$ of $P(k,2)$. It is clear that $W$ has length $2k+1$. Thus $h((P(k, m)) = 2k+1$.

+

It was shown in [8] that all connected cubic graphs of order $n$, where $4 \leq n \leq 8$, are Hamiltonian. It was also shown in [8] that the Petersen graph $P(5,2)$ and the Tietze graph (denoted by $T_{12}$) are the only 2-connected cubic graph of order 10 and 12, respectively, that are not Hamiltonian. They are, in fact, almost Hamiltonian cubic graphs of respective order. Note that the $T_{12}$ is obtained from $P(5,2)$ by replacing one vertex of $P(5,2)$ to a triangle, matching the vertices of the triangle to the former neighbors of the replaced vertex. Thus $T_{12}$ contains a triangle. Let $G$ be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from $G$ by replacing $v$ to a triangle, matching the vertices of the triangle to the former neighbors of $v$. Thus $G * v$ is also a cubic graph containing a triangle.

**Theorem 2.2** *Let $G$ be a cubic graph of order $n \geq 4$ and $v \in V(G)$. Then $G$ is Hamiltonian if and only if $G * v$ is Hamiltonian.*

**Proof.** Let $G$ be a cubic graph and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $v = v_1$. Thus $G * v$ is the graph with $V(G * v) = (V(G) - v) \cup \{x_1, y_1, z_1\}$, $\{x_1, y_1, z_1\}$ induced a triangle in $G * v$ and $y_1v_2, v_nz_1 \in E(G * v)$.

Suppose that $G$ is Hamiltonian. Without loss of generality we may assume that $C : v_1, v_2, \cdots, v_n, v_1$ is a Hamiltonian cycle of $G$. Thus

$$C_v : z_1, x_1, y_1, v_2, v_3, \cdots, v_n, z_1$$

is a Hamiltonian cycle of $G * v$.

Conversely, suppose that $G * v$ is Hamiltonian and let

$$C_v : u_1, u_2, \cdots, u_{n+2}, u_1$$

be a Hamiltonian cycle of $G * v$. If $x_1$ is not a neighbor of $y_1$ and $z_1$ in $C_v$, then $d_{G*v}(x_1) \geq 4$. Thus $x_1$ is a neighbor of $y_1$ or $z_1$ in $C_v$. It is also true for $y_1$ and $z_1$. Thus $x_1, y_1, z_1$ must appear as consecutive vertices in $C_v$. Deleting the three vertices and replacing by $v_1$, we obtain a Hamiltonian cycle of $G$.

+

Let $G$ be a cubic graph of order $n$ with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $G^1 = G * v_1$ and for $1 \leq i \leq n-1$, put $G^{i+1} = G^i * v_{i+1}$. Thus from Theorem 2.2 we have the following corollary.

**Corollary 2.3** *Let $G$ be a cubic graph of order $n$. Then $G$ is Hamiltonian if and only if $G^i$ is Hamiltonian for all $1 \leq i \leq n$.*

+

Now we consider when $G$ is not Hamiltonian cubic graph. Thus $G * v$ is not Hamiltonian by Theorem 2.2. Let $K'_4$ be the graph obtained from $K_4$ and a subdivision to an edge of $K_4$ (see Figure 2).

$K'_4$ =

Figure 2 : Graph $K'_4$

Let $G$ be a graph obtained from three copies of $K'_4$ and connecting three vertices of degree two to a new vertex $v$.

Thus $G$ is cubic of order 16 with $h(G) = 21$ but $h(G * v) = 24$. That is $G$ is a 5-Hamiltonian while $G * v$ is 6-Hamiltonian. Note that $G * w$ is 5-Hamiltonian, for each vertex $w$ of G difference from $v$.

**Theorem 2.4** *For an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order n.*

**Proof.** The Petersen graph $P(5,2)$ is the unique almost Hamiltonian cubic graph of order 10 and the Tietze graph $T_{12}$ is also the unique almost Hamiltonian cubic graph of order 12 and $T_{12} = G * v$, where $G = P(5,2)$ and $v \in V(P(5,2))$. Let $u_1, v_1, w_1$ be the induced triangle of $T_{12}$. Let $T_{14} = T_{12} * v_1$. Thus for an integer $i \geq 1$, let $u_i, v_i, w_i$ be the induced triangle of $T_{12+2(i-1)}$ and $T_{12+2i} = T_{12+2(i-1)} * v_i$. By assuming that the graph $T_{12+2(i-1)}$ is almost hamiltonian, we have that $h(T_{12+2i}) \leq 12 + 2i + 1$. By Theorem 2.2 we have that $T_{12+2i}$ is not Hamiltonian. Therefore $h(T_{12+2i}) = 12 + 2i + 1$ and $T_{12+2i}$ is almost Hamiltonian.

A Hamiltonian graph is necessary 2-connected. The same result is also hold in the class of almost Hamiltonian cubic graphs. The following results can be considered as a characterization of cubic graphs for being almost Hamiltonian.

**Theorem 2.5** *Let G be a connected cubic graph of order $n \geq 10$. If G is almost Hamiltonian, then G is 2-connected.*

**Proof.** Suppose $G$ is not 2-connected and $v$ is a cut vertex of $G$. Since $G$ is cubic, there exists a vertex $u$ such that $u$ is also a cut vertex of $G$ and $u$ is adjacent to $v$. Furthermore, $uv$ is a cut edge of $G$. Let $G - e = G_1 \cup G_2$. It follows that $h(G) \geq h(G_1) + h(G_2) + 2 \geq n + 2$. The proof is complete.

**Theorem 2.6** *Let G be a connected non-Hamiltonian cubic graph of order $n \geq 10$. Then G is an almost Hamiltonian graph if and only if for every Hamiltonian walk W of G, W contains a cycle of order n -1.*

**Proof.** Suppose $h(G) = n + 1$. Let $v_1, v_2, \cdots, v_{n+2} = v_1$ be a Hamiltonian walk of length $n + 1$. Thus there exist $v_i$ and $v_j$ with $1 \leq i < j \leq n$ and $v_i = v_j$ and all other vertices are distinct. Without loss of generality we may assume that $i = 1$. If $j \geq 4$, then $d(v_1) \geq 4$. Thus $j = 3$ and $v_3, v_4, \cdots, v_{n+2} = v_3$ is a cycle in $G$ of length $n - 1$. Suppose $G$ contains a cycle $v_1, v_2, \cdots, v_n = v_1$ of length $n - 1$. Let $v \in V(G) - \{v_1 = v_n, v_2, v_3, \cdots, v_{n-1}\}$. Thus there exists an integer $k$ with $1 \leq k \leq n - 1$ such that $v_k$ is adjacent to $v$. We now form a Hamiltonian walk $v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_n = v_1$ and this walk has length $n + 1$. Therefore $h(G) = n + 1$.

Let $G$ be a Hamiltonian cubic graph. We have shown in Theorem 2.2 that for every $v \quad V(G)$, $G * v$ is Hamiltonian and vise versa. We have also mentioned that there is a 5-Hamiltonian graph $G$ and $v \quad V(G)$ such that $G * v$ is 6-Hamiltonian.

Let $G$ be a connected cubic graph of order $n$ with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $G^* = G^n$. Figure 3 shows the graphs $P(5,2)$ and $P^*(5,2)$.

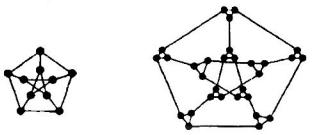

Figure 3: Graphs $P(5,2)$ and $P^*(5,2)$

**Theorem 2.7** *If G is an almost Hamiltonian cubic graph of order n, then $h(G^*) = 3n + 2$.*

**Proof.** By Theorem 2.2, it follows that $h(G^*) \geq 3n + 1$. Assume, to the contrary, that $h(G^*) = 3n + 1$. By Theorem 2.6, let $C : x_1, x_2, \cdots, x_{3n-1}, x_1$ be a cycle of length $3n - 1$ of $G^*$, where $V(G^*) = \{x_1, x_2, \cdots, x_{3n}\}$. Without loss of generality we may assume that $x_{3n}$ is adjacent to $x_1$. Since $G^*$ is non-Hamiltonian, $x_{3n}x_2, x_{3n}x_{3n-1} \notin E(G^*)$. Since $G^*$ is cubic, there exist $i, j$ with $1 < i < j < 3n - 1$ such that $\{x_{3n}, x_i, x_j\}$ induced a triangle in $G^*$. Since $G^*$ is cubic, $j = i + 1$ and $G^*$ is Hamiltonian. This is a contradiction.

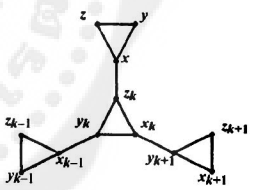


Figure 4: Part of $G^*$

In order to show that $h(G^*) = 3n + 2$, we will construct a Hamiltonian walk of $G^*$ of length $3n + 2$. In the proof of Theorem 2.6, let

$$W : v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_{n+1} = v_1$$

be a Hamiltonian walk of $G$.

For each $i$, $1 \leq i \leq n$, we replace vertices $v_i$ and $v$ in $W$ by triangles $x_i, y_i, z_i$ and $x, y, z$ respectively, and then arrange them in such a way that $x_i$ is adjacent to $y_{i+1}$, for all $i = 1, 2, \cdots, n - 1$. Without loss of generality we may assume that $z_k$ is adjacent to $x$ as shown in Figure 4. Thus the Hamiltonian walk

$W : z_1, x_1, y_2, z_2, x_2, \cdots, y_{k-1}, z_{k-1}, x_{k-1}, y_k, z_k, x, y, z, x, z_k, x_k, y_{k+1}, z_{k+1}, x_{k+1}, \cdots, y_{n-1}, z_{n-1}, x_{n-1}, y_n, z_n = z_1$

has length $3n + 2$.

+

The following result can be obtained as a direct consequence of Theorem 2.7.

**Corollary 2.8** $h(P^*(k, 2)) = 6k + 2$, *for every positive integer k with k* 5 (mod 6).

+

## 3. Conclusion

A Hamiltonian walk in a connected graph $G$ of order $n$ is a closed opening walk of minimum length in $G$. Let $h(G)$ be the length of a Hamiltonian walk in $G$. The graph parameter $h$ is called the Hamiltonian number of G. Thus $h(G)$ may be considered as a measure of how far the graph $G$ is from being Hamiltonian. A connected graph $G$ of order $n$ is called an $i$-Hamiltonian if $h(G) = n + i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an almost Hamiltonian graph. Some characterizations of almost Hamiltonian cubic graphs are obtained in this paper. In other words, we proved that a cubic graph $G$ of order $n$ is almost Hamiltonian if and only if $G$ is 2-connected containing a cycle of length $n - 1$. In particular, we proved that the generalized Petersen graph $P(k, m)$ is almost Hamiltonian if and only if $m = 2$ and $k$ 5(mod 6). Let $G$ be a cubic graph of order $n$. we denote $G^*$ the graph obtained from $G$ by replacing each vertex of $G$ to a triangle, matching the vertices of the triangle to the former neighbors. We proved that $G$ is Hamiltonian if and only if $G^*$ is Hamiltonian and if $G$ is almost Hamiltonian, then $G^*$ is 2-Hamiltonian.

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# The Decycling Number of Cubic Planar Graphs*

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**Abstract.** Bau and Beineke [2] asked the following questions:
1 Which cubic graphs $G$ of order $2n$ have decycling number $\phi(G) = \lceil \frac{n+1}{2} \rceil$?
2 Which cubic planar graphs $G$ of order $2n$ have decycling number $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

We answered the first question in [10]. In this paper we prove that if $\mathcal{P}(3^{2n})$ is the class of all connected cubic planar graphs of order $2n$ and $\phi\left(\mathcal{P}(3^{2n})\right) = \left\{\phi(G) : G \in \mathcal{P}(3^{2n})\right\}$, then there exist integers $a_n$ and $b_n$ such that there exists a graph $G \in \mathcal{P}(3^{2n})$ with $\phi(G) = c$ if and only if $c$ is an integer satisfying $a_n \leq c \leq b_n$. We also find all corresponding integers $a_n$ and $b_n$. In addition, we prove that if $\mathcal{P}\left(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil\right)$ is the class of all connected cubic planar graphs of order $2n$ with decycling number $\lceil \frac{n+1}{2} \rceil$ and $G_1, G_2 \in \mathcal{P}\left(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil\right)$, then there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots, t-1$, $G_1^{\sigma_1\sigma_2\cdots\sigma_i} \in \mathcal{P}\left(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil\right)$ and $G_2 = G_1^{\sigma_1\sigma_2\cdots\sigma_t}$.

## 1 Introduction

We consider only finite simple graphs. For the most part, our notation and terminology follow that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set $S$ and define $n = \nu(G) = |V|$ to be the *order* of $G$ and $m = \varepsilon(G) = |E|$ the *size* of $G$. We write $e = uv$ for an edge $e$ that joins vertex $u$ to vertex $v$. A *path* of order $k$ in a graph $G$, denoted by $P_k$, is a sequence of distinct vertices $u_1u_2\cdots u_k$ of $G$ such that for all $i = 1, 2, \ldots, k-1$, $u_iu_{i+1}$ is an edge of $G$. The *degree* of a vertex $v$ of a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph $G$ is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by $S$ in $G$ and we also use the notation $\varepsilon(S)$ for $\varepsilon(G[S])$. For a graph $G$ and $X \subseteq E(G)$, we denote $G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V(G)$, $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ and $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$

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and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs $G$ and $H$, $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs. For a graph $G$ and $s \in V(G)$, the *neighborhood* of $s$ in $G$ is defined by $N(s) = \{v \in V(G)\colon sv \in E(G)\}$. If $S \subseteq V(G)$, then we define $N(S) = \bigcup_{s \in S} N(s)$. If $F \subseteq V(G)$, we write $N_F(S)$ for $N(S) \cap F$. A graph $G$ is called *$r$-regular* if all of its vertices have degree $r$. A 3-regular graph is called a *cubic graph.* Let $G$ be a graph of order $n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of $G$, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph $G$ is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph $G$. In this case, $G$ is called a *realization* of $\mathbf{d}$. An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [6] and Hakimi [5]. We state their result in the following theorem.

**Theorem 1.** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

*Then $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.*

Let $G$ be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Define

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation* on $G$. For a graph $G$, put $G_0 = G$. We define $G^{\sigma_1\sigma_2\cdots\sigma_t}$ recursively by for each $i = 1, 2, \ldots, t$, $G_i = G_{i-1}^{\sigma_i}$, where $\sigma_i$ is a switching on $G_{i-1}$. It is easy to see that the graph $G_i$ has the same degree sequence as $G$. The following theorem was proved by Havel [6] and later by Hakimi [5] independently.

**Theorem 2.** *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If $G$ and $H$ are any two realizations of $\mathbf{d}$, then $H$ can be obtained from $G$ by a finite sequence of switchings.*

As a consequence of Theorem 2, Eggleton and Holton [4] defined in 1978 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the set of all non-isomorphic graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

**Theorem 3.** *The graph $\mathcal{R}(\mathbf{d})$ is connected.*

The following theorem was obtained by Taylor [11] in 1980.

**Theorem 4.** *For a graphic degree sequence $\mathbf{d}$, let $\mathcal{CR}(d)$ be the set of all connected realizations of $\mathbf{d}$. Then the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by $\mathcal{CR}(\mathbf{d})$ is connected.*

## 2 The Decycling Number

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph $G$ is difficult even for some simply defined graphs. For a graph $G$, this minimum number is known as the *decycling number* of $G$, and is denoted by $\phi(G)$. Let $\mathbb{G}$ be the class of all graphs. A function $f\colon \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G$, $H \in \mathbb{G}$ and $G \cong H$, then $f(G) = f(H)$. Let $\mathbb{J} \subseteq \mathbb{G}$. A graph parameter $f$ is called an *interpolation graph parameter with respect to* $\mathbb{J}$ or $f$ *interpolates over* $\mathbb{J}$ if $G$, $H \in \mathbb{J}$ and $a = f(G) < f(H) = b$, then for every integer $c$ between $a$ and $b$, there exists $K \in \mathbb{J}$ such that $f(K) = c$. If $f$ interpolates over $\mathbb{J}$, then $f(\mathbb{J}) := \{f(G)\colon G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) := \min\{f(G)\colon G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) := \max\{f(G)\colon G \in \mathbb{J}\}$. Let $\mathbf{d} = r^n$ be a graphic regular degree sequence and let $f$ be a graph parameter. We use the following notation: $\min(f, r^n) = \min(f, \mathcal{R}(r^n))$, $\max(f, r^n) = \max(f, \mathcal{R}(r^n))$, $\mathrm{Min}(f, r^n) = \min(f, \mathcal{CR}(r^n))$, and $\mathrm{Max}(f, r^n) = \max(f, \mathcal{CR}(r^n))$.

We proved in [8] that for a graph $G$ and a switching $\sigma$ on $G$, $|\phi(G) - \phi(G^\sigma)| \leq 1$. Thus we have the following results.

**1** $\phi$ interpolates over $\mathcal{R}(\mathbf{d})$.
**2** $\phi$ interpolates over $\mathcal{CR}(\mathbf{d})$.
**3** Let $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$. If the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by $\mathbb{J}$ is connected, then $\phi$ interpolates over $\mathbb{J}$.

Let $G$ be a graph and $F \subseteq V(G)$. $F$ is called an *induced forest* of $G$ if $G[F]$ contains no cycle. An induced forest $F$ of $G$ is *maximal* if for every $v \in V(G-F)$, $G[F \cup \{v\}]$ contains a cycle. Let $t(G)$ be defined as

$$t(G) := \max\{|F|\colon F \text{ is an induced forest of } G\}.$$

It is clear that, for a graph $G$ of order $n$, $\phi(G) + t(G) = n$. Consequently, $t$ interpolates over $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$ if and only if $\phi$ does.

## 3 Cubic Graphs

The interest of studying the decycling number of cubic graphs was motivated by the results of Zheng and Lu [12], Alon et al. [1], Liu and Zhao [7], and Bau and Beineke [2]. We found in [8] the values of $\min(\phi, 3^{2n})$ in all situations and proved that $\min(\phi, 3^{2n}) = \mathrm{Min}(\phi, 3^{2n})$.

The problem of finding upper bounds of $\phi(G)$, where $G$ runs over a class of cubic graphs, has been investigated in the literature. First observe that if $G$ is a cubic graph of order $n$ and $F$ is a maximum induced forest of $G$, then it is easy to see that $G - F$ is also a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if $n$ is a multiple of 4. We proved in [8] that if $n = 4q + t$, $t = 0, 2$, then $\max(\phi, 3^n) = 2q$.

The problem of finding $\mathrm{Max}(\phi, 3^n)$ is more difficult. We have considered such problem in term of the graph parameter $t$ in [9]. A *cubic tree* is a tree whose

vertices consisting of degree 1 or 3. Evidently, if $T$ is a cubic tree of order $n$, then $n = 2k+2$, where $k$ is the number of vertices of degree 3 of $T$. Let $\mathbb{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of $K_4$ with one subdivided edge at every vertex of degree 1.

A lower bound for the order of maximum induced forest in connected cubic graphs has been obtained by Liu and Zhao [7] as stated in the following theorem.

**Theorem 5.** *Let $G$ be a connected cubic graph of order $2n \geq 12$. Then $t(G) = \frac{5}{4}n - \frac{1}{4}$ if $G \in \mathbb{T}$ and $t(G) \geq \frac{5}{4}n$ if $G \notin \mathbb{T}$.*

By Theorem 5 and with the fact that a switching changes the value of $\phi$ by at most 1 in the resulting graph, we proved in [9] the following results.

**Theorem 6.** *Let $n$ be an integer with $n \geq 6$. Then*

$$\mathrm{Max}(\phi, 3^{2n}) = \begin{cases} \frac{3}{4}n + \frac{1}{4}, & \text{if } n \equiv 1 \,(\mathrm{mod}\, 4), \\ \lfloor \frac{3}{4}n \rfloor, & \text{otherwise.} \end{cases}$$

## 4 Cubic Planar Graphs

Note that $\phi$ interpolates over $\mathcal{CR}(3^{2n})$ with

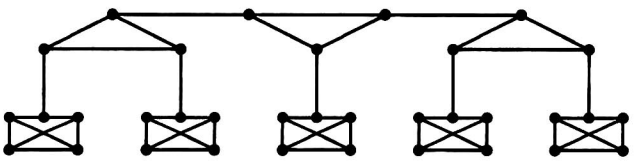
$$\mathrm{Min}(\phi, 3^{2n}) = \lceil \tfrac{n+1}{2} \rceil \text{ and } \mathrm{Max}(\phi, 3^{2n}) = \begin{cases} \frac{3}{4}n + \frac{1}{4}, & \text{if } n \equiv 1 \,(\mathrm{mod}\, 4), \\ \lfloor \frac{3}{4}n \rfloor, & \text{otherwise.} \end{cases}$$

For each $c \in \phi(3^{2n})$, let $\mathcal{CR}(3^{2n};\ \phi = c) = \{G \in \mathcal{CR}(3^{2n})\colon \phi(G) = c\}$.

We answered Question 1 in [10] by determining all cubic graphs $G$ of order $2n$ and $\phi(G) = \lceil \frac{n+1}{2} \rceil$. In addition, we proved that the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{CR}\left(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil\right)$ is connected.

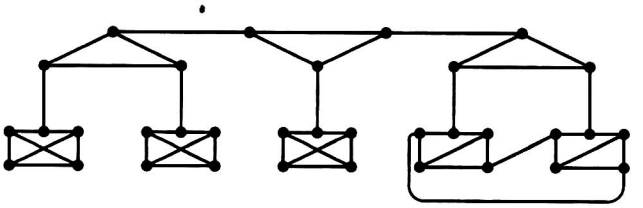
Let $\mathcal{P}(3^{2n})$ be the class of all connected cubic planar graphs of order $2n$. For small $n$, the difference between $\mathrm{Max}(\phi, 3^{2n})$ and $\mathrm{Min}(\phi, 3^{2n})$ is relatively small and it is easy to show by construction that $\phi(\mathcal{P}(3^{2n})) = \phi(\mathcal{CR}(3^{2n}))$, for all $n \leq 13$. We now consider the interpolation property of $\phi$ with respect to $\mathcal{P}(3^{34})$. Since $\mathrm{Min}(\phi, 3^{34}) = \frac{17+1}{2} = 9$ and $\mathrm{Max}(\phi, 3^{34}) = \frac{3 \cdot 17}{4} + \frac{1}{4} = 13$, $\mathcal{CR}(3^{34})$ can be partitioned into five classes $C_i (9 \leq i \leq 13)$, where $C_i = \{G \in \mathcal{CR}(3^{34})\colon \phi(G) = i\}$ and $C_i \neq \emptyset$, for all $i = 9, 10, 11, 12, 13$. Consider the following example.

*Example 1.* Planar graphs $G^1$, $G^2$, $G^3$, $G^4$, $G^5$



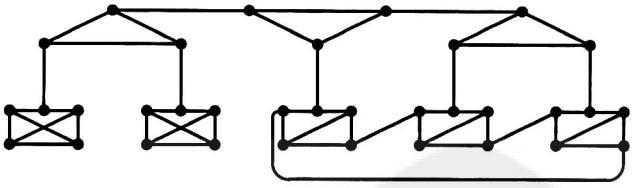
$= G^1$

$G^1$ of order $34 = 8k + 10$ and $\phi(G^1) = 13 = \mathrm{Max}(\phi, 3^{34})$.



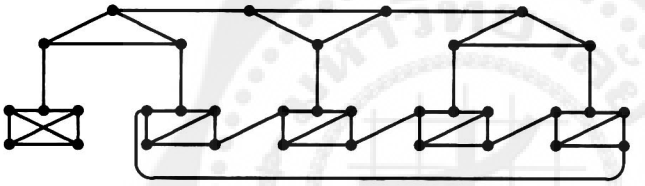
$= G^2$

$G^2$ of order $34 = 8k + 10$ and $\phi(G^2) = 12 = \text{Max}(\phi, 3^{34}) - 1$.



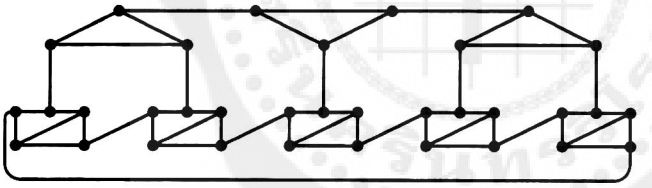
$= G^3$

$G^3$ of order $34 = 8k + 10$ and $\phi(G^3) = 11 = \text{Max}\,(\phi, 3^{34}) - 2$.



$= G^4$

$G^4$ of order $34 = 8k + 10$ and $\phi(G^4) = 10 = \text{Max}(\phi, 3^{34}) - 3$.



$= G^5$

$G^5$ of order $34 = 8k + 10$ and $\phi(G^5) = 9 = \text{Max}(\phi, 3^{34}) - 4 = \text{Min}(\phi, 3^{34})$.

Thus for each $i = 9, 10, 11, 12, 13$, $C_i \cap \mathcal{P}(3^{34}) \neq \emptyset$. Therefore $\phi$ interpolates over $\mathcal{P}(3^{34})$. The same result can be obtained in $\mathcal{P}(3^{2n})$ for $n \in \{14, 15, 16\}$.

In general if we write $2n = 8k + 2i$, $i = 2, 3, 4, 5$ and $k \geq 4$, then we can construct a cubic planar graph $G_i$ of order $8k + 2i$ by a cubic tree $T_i$ of order $2k + 2$, where $k$ is the number of vertices of degree 3, replacing $k - 5 + i$ vertices of degree 3, by a triangle and attaching a copy of $K_4$ with one subdivided edge at every vertex of degree 1. Thus $G_i \in \mathcal{P}(3^{2n})$ and $\phi(G_i) = 3k - 1 + i = \text{Max}(\phi, 3^{2n})$. Hence $\max(\phi, \mathcal{P}(3^{2n})) = \text{Max}(\phi, 3^{2n})$. By forming appropriate sequence of switchings to $G_i$ as described in above example and with the fact that a switching changes the value of $\phi$ of the resulting graph by at most 1, we have the following theorem.

**Theorem 7.** *$\phi$ interpolates over $\mathcal{P}(3^{2n})$, with*

$$\min\{\phi(G): G \in \mathcal{P}(3^{2n})\} = \mathrm{Min}(\phi, 3^{2n}), \text{ and}$$
$$\max\{\phi(G): G \in \mathcal{P}(3^{2n})\} = \mathrm{Max}(\phi, 3^{2n}).$$

Let $G \in \mathcal{CR}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ and $S$ be a minimum decycling set of $G$. Let $F = V(G - S)$. We proved in [10] the following results.

**1** $\varepsilon(S) = 0$ if $n$ is odd and $\varepsilon(S) \leq 1$ if $n$ is even,
**2** if $n$ is odd, then $G[F]$ is a tree,
**3** if $n$ is even and $\varepsilon(S) = 1$, then $G[F]$ is a tree,
**4** if $n$ is even and $\varepsilon(S) = 0$, then $G[F]$ has 2 components.

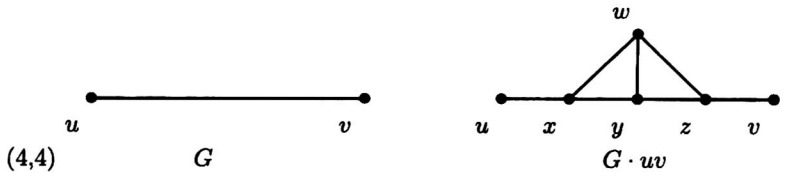
**Lemma 1.** *There exists $G \in \mathcal{P}(3^{2n})$ such that $\phi(G) = \lceil \frac{n+1}{2} \rceil$ and $G$ contains a path of order $\lfloor \frac{3n-1}{2} \rfloor$ as its induced forest. Furthermore the induced subgraph of the corresponding decycling set of $G$ contains at most one edge and it contains no edge if and only if $n$ is odd.*

A proof of above lemma can be obtained easily by introducing the graphs $G_6$ and $G_8$ of order 6 and 8, respectively, as showed in the following figures and the definitions of *super subdivision*.

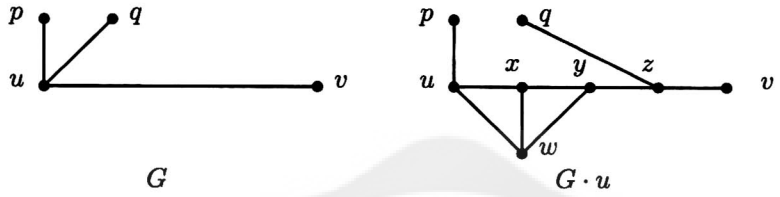
$G_6$ $G_8$

**Definition 1.** *Let $G$ be a connected cubic planar graph and let $F$ be an induced forest of $G$. If $u, v \in F$ and $uv \in E(G)$, define a super subdivision with respect to $uv$ of $G$ to be a graph $G \cdot uv$ obtained from $G$ by inserting three vertices $x, y, z$ on the edge $uv$ and joining $x, y, z$ to a new vertex $w$.*

**Definition 2.** *Let $u \in F$, $d_F(u) = 1$. Then there exists a unique vertex $v \in F$ that is adjacent to $u$ and there exist two vertices $p, q \in V(G - F)$ such that $p$ and $q$ are adjacent to $u$. Define a super subdivision with respect to $u$ of $G$ to be a graph $G \cdot u$ obtained from $G$ by inserting three vertices $x, y, z$ on the edge $uv$ to produce a path $uxyzv$ in $F$, then deleting $qu$, and adding the edge $qz$, and finally joining $u, x, y$ to a new vertex $w$.*




Super subdivision w. r. t. edge $uv$




Super subdivision w. r. t. vertex $u$

Note that the concept of "super subdivision" may be defined by the concept of a contraction. A contraction that maps the free vertex, all three of its neighbors and a neighboring vertex $x$ on the tree to the vertex $x$.

Let $\mathcal{P}\left(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil\right)$ be the class of cubic planar graphs $G$ of order $2n$ and $\phi(G) = \lceil \frac{n+1}{2} \rceil$. Let $G \in \mathcal{P}\left(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil\right)$. Then for any minimum decycling set $S$ of $G$, $G-S$ is a tree or a union of two trees. Since $G$ is cubic, $\Delta(G-S) \leq 3$. Let $n_3(G-S)$ be the number of vertices of degree 3 in $G-S$ and $n_3(G)$ be defined by

$$n_3(G) = \min\{n_3(G-S)\colon S \text{ is a minimum decycling set of } G\}.$$

For nonnegative integer $i$, put

$$\mathcal{P}_i(2n) = \left\{G \in \mathcal{P}\left(3^{2n};\ \phi = \lceil \frac{n+1}{2} \rceil\right) : n_3(G) = i\right\}.$$

We first consider the class $\mathcal{P}_0(2n)$ and prove that the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{P}_0(2n)$ is connected. In order to prove the result, we need to introduce some notation and terminology which will be used from now on.

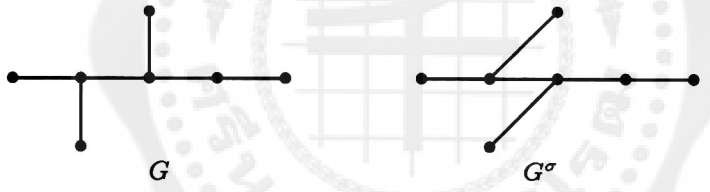
Let $G \in \mathcal{P}_0(2n)$ and put $N = \lfloor \frac{3n-1}{2} \rfloor$. Thus there exists a minimum decycling set $S$ of $G$ such that $G - S = F_N = P_l \cup P_{N-l}$, where $l \geq N - l \geq 0$, $P_l = v_1 v_2 \cdots v_l$, and $P_{N-l} = v_{l+1} v_{l+2} \cdots v_N$. For a planar embedding of $G$, put $F_N$ in the horizontal direction and the vertices in $S$ are arranged either above or below $F_N$. Thus the vertices in $S$ can be partitioned into at most three subsets, $U$, $L$, $D$, according to the following rules.

**1** For $s \in S$, $s \in U$ if $s$ lies above $F_N$ and all edges from $s$ to $F_N$ lie above $F_N$.
**2** For $s \in S$, $s \in L$ if $s$ lies below $F_N$ and all edges from $s$ to $F_N$ lie below $F_N$.
**3** Put $D = S - (U \cup L)$.

With a planar embedding we can also partition the edge set from $S$ to $F_N$ into $U'$ and $L'$ according to it is incident to a vertex of $F_N$ above or below $F_N$. It is clear that an edge from $s \in U$ to a vertex of $F_N$ is in $U'$ and likewise an edge from $s \in L$ to a vertex of $F_N$ is in $L'$ while an edge from $s \in D$ is either in $U'$ or $L'$ depending on the direction to its end. Note that for each planar embedding of $G \in \mathcal{P}_0(2n)$, the sets $U'$ and $L'$ are uniquely determined. Thus we can write $U'(G)$ and $L'(G)$ for $U'$ and $L'$ if we work with more than one graph.

**Definition 3.** *Let $G \in \mathcal{P}_i(2n)$. A switching $\sigma$ on $G$ is called a $\mathcal{P}_i$-switching with respect to $G$ if $G^\sigma \in \mathcal{P}_j(2n)$, for some $j \leq i$. A sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ is called a sequence of $\mathcal{P}_i$-switchings with respect to $G$ if for all $k = 1, 2, \ldots, t$ there exists $j$ such that $j \leq i$ and $G^{\sigma_1\sigma_2\cdots\sigma_k} \in \mathcal{P}_j(2n)$.*

A $\mathcal{P}_0$-switching plays a crucial role in proving the main results. We introduce a special type of $\mathcal{P}_0$-switching called *elementary switching*. Let $G \in \mathcal{P}_0(2n)$. We choose a planar embedding of $G$ as described above. Let $x$, $y$ be distinct vertices of $S$, $x$ is adjacent to $v_i$ and $y$ is adjacent to $v_{i+1}$, where $i \neq l$. If $xv_i \in U'$ and $yv_{i+1} \in L'$ or $xv_i \in L'$ and $yv_{i+1} \in U'$, then $\sigma(x, v_i; v_{i+1}, y)$ is a $\mathcal{P}_0$-switching with respect to $G$. This special type of $\mathcal{P}_0$-switching with respect to $G$ will be used throughout and we call it an *elementary switching with respect to $F_N$*. A sequence of elementary switchings can be similarly defined.




$\mathcal{P}_0$-switching: Elementary switching

By the definition of an elementary switching with respect to $F_N$ as described above, we assume that $x$ and $y$ are distinct. If $x = y$ and $x$ lies above $F_N$, then we can not apply the switching but we can redraw either an edge $xv_i$ or $xv_{i+1}$ so that the two edges lie above $F_N$. Similarly when $x$ lies below $F_N$. For a planar embedding of $G$, we want to avoid this situation. Thus for a planar embedding of $G$, if $x \in S$ is adjacent to $v_i$ and $v_{i+1}$ $(i \neq l)$, then we draw $xv_i$, $xv_{i+1}$ and $x$ on the same side of $F_N$. Clearly if $\sigma$ is an elementary switching with respect to $F_N$, then the resulting graph $G^\sigma \in \mathcal{P}_0(2n)$ with $S$ as its minimum decycling set and $F_N$ as its induced forest. The following lemmas are useful and they are direct consequences of applying a sequence of elementary switchings.

**Lemma 2.** *Let $G \in P_0(2n)$. Suppose that there are $k$ edges in $U'$ from $S$ to $P_l - \{v_1, v_l\}$ and there are $h$ edges in $L'$ from $S$ to $P_l - \{v_1, v_l\}$. Then there*

*exists a sequence of elementary switchings* $\sigma_1, \sigma_2, \ldots, \sigma_t$ *with respect to* $F_N$ *such that* $G_t = G^{\sigma_1\sigma_2\cdots\sigma_t}$ *contains* $k$ *edges in* $U'(G_t)$ *from* $S$ *to* $\{v_2, v_3, \ldots, v_{k+1}\}$ *and* $h$ *edges in* $L'(G_t)$ *from* $S$ *to* $\{v_{k+2}, v_{k+3}, \ldots v_{k+h+1}\}$. *It is clear that* $k+h = l-2$.

The result in Lemma 2 is also true if $P_l$ is replaced by $P_{N-l}$. Moreover, it is also true that there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $F_N$ such that $G_t = G^{\sigma_1\sigma_2\cdots\sigma_t}$ contains $h$ edges in $L'(G_t)$ from $S$ to $\{v_2, v_3, \ldots, v_{h+1}\}$ and $k$ edges in $L'(G_t)$ from $S$ to $\{v_{h+2}, v_{h+3}, \ldots v_{h+k+1}\}$.

**Lemma 3.** *Let* $G \in \mathcal{P}_0(2n)$. *If* $e = sv_i \in U'$, $f = sv_j \in L'$, $1 \leq i < j \leq l$ *and* $s$ *is not adjacent to any internal vertex between* $v_i$ *and* $v_j$, *then there exists a sequence of elementary switchings* $\sigma_1, \sigma_2, \ldots, \sigma_t$ *with respect to* $F_N$ *such that* $s$ *is adjacent to consecutive vertices of* $P_l$ *in* $G^{\sigma_1\sigma_2\cdots\sigma_t}$.

*Proof.* We have already mentioned that if $e = sv_i \in U'$ and $f = sv_j \in L'$, $1 \leq i < j \leq l$, then $i + 2 \leq j$. Suppose that $j = i + 2$, there is a vertex $s' \in S$ such that $s \neq s'$ and $s'$ is adjacent to $v_{i+1}$. If $s'v_{i+1} \in U'$, then $\sigma(s, v_j; v_{i+1}, s')$ is an elementary switching and $s$ is adjacent to $v_i$ and $v_{i+1}$ in $G^{\sigma(s, v_j; v_{i+1}, s')}$. Similarly if $s'v_{i+1} \in L'$. Suppose $j = i + k$ with $k \geq 3$. Let $e_k$ denote an edge connecting with a vertex in $S$ and $v_k$, $k = i, i + 1, \ldots, j - 1$. Let $k$ be the first integer such that $i < k < j$ and $e_k \in L'$. Thus there exists a sequence of $k - i$ switchings $\sigma_1, \sigma_2, \ldots, \sigma_{k-i}$ with respect to $F_N$ such that $s$ is adjacent to $v_{i+1}$ and $v_j$ in $G^{\sigma_1\sigma_2\cdots\sigma_{k-i}}$. If $k = j$, then there exists a sequence of $j - i - 2$ switchings $\sigma_1, \sigma_2, \ldots, \sigma_{j-i-2}$ with respect to $F_N$ such that $s$ is adjacent to $v_i$ and $v_{i+1}$ in $G^{\sigma_1\sigma_2\cdots\sigma_{j-i-2}}$. Thus the proof is complete by continuing the process if necessary. □

The result in Lemma 3 is also true if $P_l$ is replaced by $P_{N-l}$. As a direct consequence of Lemma 3, if $s \in S$ is adjacent to $v_i$ and $v_j$, where $v_i$ and $v_j$ are in the same path, while $sv_i$ and $sv_j$ are in different sets, then there is a sequence of elementary switchings with respect to $F_N$ applying to $G$ and $s$ is adjacent to consecutive vertices of the path in the resulting graph. Thus we can redraw one of the edges so that $s$ and the two edges lie on the same side of $F_N$. It can happen that there is $s \in U$, $s$ is adjacent to $v_j$ and $sv_j \in U'$ but $sv_j$ lies above and below $F_N$.

**Lemma 4.** *Let* $G \in \mathcal{P}_0(2n)$. *If* $e = sv_i \in U'$, $f = sv_j \in U'$, $1 \leq i < j \leq l$ *but* $sv_j$ *lies above and below* $F_N$ *and* $s$ *is not adjacent to any internal vertex between* $v_i$ *and* $v_j$, *then there exists a sequence of elementary switchings* $\sigma_1, \sigma_2, \ldots, \sigma_t$ *with respect to* $F_N$ *such that* $s$ *is adjacent to consecutive vertices of* $P_l$ *in* $G^{\sigma_1\sigma_2\cdots\sigma_t}$.

*Proof.* If there is an integer $k$, $i < k < j$ such that $e_k \in L'$, then by Lemma 3 there exists a sequence of elementary switchings with respect to $F_N$ applying to $G$ so that $s$ is adjacent to $v_i$ and $v_{j'}$ with $i < j'$ and for all $k$, $i \leq k \leq j'$, $e_k \in U'$ in the resulting graph. Thus we can redraw the edge $sv_{j'}$ so that $sv_{j'} \in L'$. The proof follows by Lemma 3. □

Thus if $F_N = P_l \cup P_{N-l}$ and $N - l \geq 1$, then $n$ is even and for every $s \in S$, $s$ is adjacent to three vertices in $F_N$. Therefore for every $s \in S$, $s$ is adjacent to $P_l$ or $P_{N-l}$ by at least two vertices. If $s$ is adjacent to three vertices in $P_l$, then there exists a sequence of $\mathcal{P}_0$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $F_N$ such that $s$ is adjacent to three consecutive vertices in $P_l$ of $G^{\sigma_1\sigma_2\cdots\sigma_t}$. Similarly if $s$ is adjacent to three vertices in $P_{N-l}$. If $s$ is adjacent to two vertices in $P_l$ and one in $P_{N-l}$, then there exists a sequence of $\mathcal{P}_0$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_r$ with respect to $F_N$ such that $s$ is adjacent to two consecutive vertices in $P_l$ of $G_r = G^{\sigma_1\sigma_2\cdots\sigma_r}$. Thus there exists $i\,(1 \leq i < l)$ such that $s$ is adjacent to $v_i$ and $v_{i+1}$, and $s$, $sv_i$ and $sv_{i+1}$ lie in the same side of $F_N$. Suppose that $s$ lies above $F_N$ and $s$ is adjacent to $v_k$ for some $k\,(l+1 \leq k \leq N)$. If $sv_k \in L'(G_r)$, then there exists a sequence of switchings $\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_t$ with respect to $F_N$ such that for each $h$ $(l+1 \leq h < k)$, $e_k \in L'(G_t)$, where $G_t = G^{\sigma_1\sigma_2\cdots\sigma_t}$. Similarly if $sv_k \in U'(G_r)$ and $sv_k$ lies above and below $F_N$. Thus we have the following theorem.

**Theorem 8.** *Let $G \in \mathcal{P}_0(2n)$. If the planar embedding of $G$ is chosen in such a way that the corresponding set $D$ has minimum cardinality, then $D = \emptyset$ or there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $F_N$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ has a planar embedding with empty corresponding set $D$.*

**Definition 4.** *Let $G \in \mathcal{P}_0(2n)$. A vertex $s \in S$ is called a* free vertex with respect to $F_N$ *if $s$ is adjacent to three consecutive vertices in $F_N$.*

We have the following theorem.

**Theorem 9.** *If $G \in \mathcal{P}_0(2n)$, then there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $F_N$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ contains a free vertex with respect to $F_N$.*

*Proof.* By Theorem 8 we may assume that there is a planar embedding of $G$ such that the corresponding set $D$ is empty.

*Case 1.* Suppose $\varepsilon(S) = 0$. Let $s \in U$ and $\mathbb{N}_G(s) = \{v_i, v_j, v_k\}$ with $1 \leq i < j < k \leq N$. We choose $s \in S$ so that $k - i$ is minimum. If there exists $x \in U$ such that $x$ is adjacent to $v_p$ for some $p$ with $i < p < j$ or $j < p < k$, then $\mathbb{N}_{F_N}(x) \subseteq \{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\}$ or $\mathbb{N}_{F_N}(x) \subseteq \{v_{j+1}, v_{i+2}, \ldots, v_{k-1}\}$. This is a contradiction to the choice of $s$. Thus we can assume that for every $p$ with $i < p < j$ or $j < p < k$, $v_p$ is adjacent to a vertex in $L$. By Lemma 4, there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $P_N$, such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ contains $s$ as a free vertex with respect to $F_N$.

*Case 2.* Suppose $\varepsilon(S) = 1$ and $s_1s_2 \in E(G)$. Thus $F_N = P_N$. Suppose $s_1 \in U$ and $s_1$ is adjacent to $v_i$ and $v_j$ with $i < j$ and if there exists $s \in U$ such that $s$ is adjacent to $v_p$ where $i < p < j$, then the result follows as in Case 1. Thus if $s \in S$ and $s \neq s_1$ and $s$ is adjacent to $v_p$, then $s \in L$. Thus we can assume that $s_1$ and $s_2$ are adjacent to consecutive vertices in $P_N$. Let $s \in S - \{s_1, s_2\}$ and $\mathbb{N}_G(s) = \{v_{i_1}, v_{i_2}, v_{i_3}\}$, where $i_1 < i_2 < i_3$. Choose $s$ in such a way that $i_3 - i_1$

is minimum. It follows by Case 1 that $s$ is a free vertex with respect to $P_N$ or there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_h$ with respect to $P_N$ such that $G_h = G^{\sigma_1\sigma_2\cdots\sigma_h}$ contains $s$ as a free vertex with respect to $P_N$. □

**Lemma 5.** *Let $n$ be an even integer and $G \in \mathcal{P}_0(2n)$. Then $G$ contains $P_N$ as its induced forest or there exists a sequence of $\mathcal{P}_0$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t}$ contains $P_N$ as its induced forest.*

*Proof.* Let $S$ be a minimum decycling set of $G$ and let $G[S]$ be an empty graph. Let $F_N = G - S = P_l \cup P_{N-l}$, where $l \geq N - l \geq 1$, $P_l = v_1v_2\cdots v_l$, and let $P_{N-l} = v_{l+1}v_{l+2}\cdots v_N$ be an induced forest of $G$ of order $N = \frac{3n}{2} - 1$. By Theorem 8 we can assume that there is a planar embedding of $G$ with $F_N$ in a horizontal direction with empty corresponding set $D$. Further, by Theorem 9 we can assume that $G$ contains $s \in S$ as a free vertex with respect to $F_N$. Since $s$ is a free vertex with respect to $F_N$, it can freely belong to $U$ or $L$. Thus, without loss of generality, there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_h$ with respect to $F_N$ such that in $G_h = G^{\sigma_1\sigma_2\cdots\sigma_h}$ and $s$ is adjacent to $v_{l-2}, v_{l-1}, v_l$. Let $s_1 \in S$ where $s_1$ is adjacent to $v_l$. Thus $s \neq s_1$. We put $s$ and $s_1$ in different sets, say $s \in U$ and $s_1 \in L$. Let $p, q \in S$ that are adjacent to $v_{l+1}$ and at least one of which, say $p$, belongs to $U$. Then $\sigma(s, v_l; p, v_{l+1})$ is a $\mathcal{P}_0$-switching of $G_h$ and $G_h^{\sigma(s, v_l; p, v_{l+1})} \in \mathcal{P}_0(2n)$ having $P_N$ as its induced forest. Now suppose that $p, q \in L$ and $p$ lies on the left of $q$ in the planar embedding of $G_h$. Then $G_h^{\sigma} \in \mathcal{P}_0(2n)$ has $P_N$ as its induced forest, where $\sigma = \sigma(s_1, v_l; p, v_{l+1})$ if $s_1 \neq p$ and $\sigma = \sigma(s_1, v_l; q, v_{l+1})$ if $s_1 = p$. □

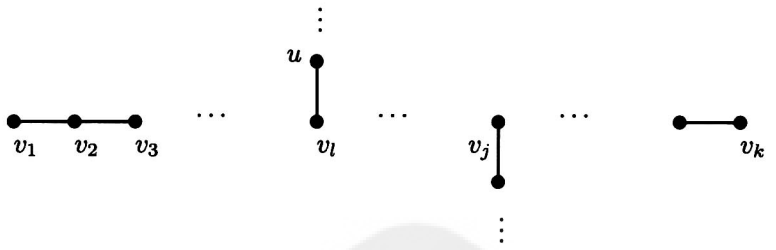
**Theorem 10.** *Let $G_1, G_2 \in \mathcal{P}_0(2n)$. Then $G_1 \cong G_2$ or there exists a sequence of $\mathcal{P}_0$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $G_1$ such that $G_2 = G_1^{\sigma_1\sigma_2\cdots\sigma_t}$.*

*Proof.* Up to isomorphism there is only one graph in $\mathcal{P}_0(4)$. Suppose that $n > 2$. Let $G_1, G_2 \in \mathcal{P}_0(2n)$. By Lemma 5 we can assume that both $G_1$ and $G_2$ have $P_N = v_1v_2\cdots v_N$ as their induced paths. By Theorem 9, we may assume that each of $G_1$ and $G_2$ contains a free vertex. Thus there exist $G_1'$ and $G_2'$ such that $G_1$ and $G_2$ are super subdivisions of $G_1'$ and $G_2'$ with either by edges or vertices, respectively. By induction on $n$, there exists a sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $P_{N-3}$ such that $G_2' = G_1'^{\sigma_1\sigma_2\cdots\sigma_t}$. Since a free vertex of $G \in \mathcal{P}_2(2n)$ with respect to $P_N$ can be in $U$ or $L$, the sequence of elementary switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to $P_{N-3}$ as described above can be regarded as a sequence of elementary switchings with respect to $P_N$ and $G_2 = G_1^{\sigma_1\sigma_2\cdots\sigma_t}$. This completes the proof. □

We now consider the class $\mathcal{P}_i(2n)$ when $i \geq 1$. That is for a graph $G \in \mathcal{P}_i(2n)$, $G$ contains an induced forest $F_N$ of order $N = \lfloor \frac{3n-1}{2} \rfloor$ with $n_3(F_N) = i$. Note that $F_N$ is a tree if $n$ is odd or $n$ is even and $\varepsilon(S) = 1$, where $S = V(G - F_N)$, otherwise $F_N$ is a union of two trees.

**Theorem 11.** *Let $G \in \mathcal{P}_i(2n)$, $i \geq 1$ and let $F_N$ be an induced forest of $G$ of order $N = \lfloor \frac{3n-1}{2} \rfloor$ and $n_3(F_N) = i$. Then there exists a sequence of $\mathcal{P}_i$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \in \mathcal{P}_{i-1}$.*

*Proof.* Let $G \in \mathcal{P}_i(2n)$, $i \geq 1$ and let $F_N$ be an induced forest of $G$ of order $N = \lfloor \frac{3n-1}{2} \rfloor$ and $n_3(F_N) = i$. Let $P_k = v_1 v_2 \cdots v_k$ be a longest path of $F_N$ containing a vertex $v_l$ of degree 3 in the forest $F_N$. We choose a planar embedding of $G$ by placing $F_N$ in such a way that $v_1$ lies in the leftmost vertex of $F_N$ and placing the path as in the following figure.




Suppose that $l$, $2 \leq l \leq k-1$, is the least integer such that $d_{F_N}(v_l) = 3$. Suppose further that $v_l$ is adjacent to a vertex $u$ in $F_N$ that lies above $P_k$ as in above figure. Since $d_{F_N}(v_1) = 1$, there exist $p$, $q \in S = V(G - F_N)$ that are adjacent to $v_1$. Suppose further that $p$, $q$ lie above $P_k$ and $p$ lies on the left of $q$. We now consider the following cases.

*Case 1.* If $l = 2$, then $\sigma(p, v_1; v_2, u)$ is a $\mathcal{P}_i$-switching and $G^{\sigma(p, v_1; v_2, u)} \in \mathcal{P}_{i-1}(2n)$.

*Case 2.* Suppose that $l \geq 3$. Note that if there is a vertex $v \in S$ such that $v$ is adjacent to $v_j$ for some $j = 2, 3, \ldots, i-1$ and ends under $P_k$. Choose $j$ to be maximum with this property, then we can analogously apply a sequence of elementary switchings with respect to $P_k$. Then we obtain the resulting graph $G'$ with decycling set $S$ and an induced forest $F_N = V(G' - S)$ containing $P_k$ as a longest path. Therefore, there exists $k$, $2 \leq k \leq l-1$, such that $d_{F_N}(v_k) = 3$.

Thus we may assume that every vertex in $S$ that is adjacent to $v_j$ $j = 2, 3, \ldots, l-1$, ends at $v_j$ above $P_k$.

If there exists $v \in S$, $v \neq p$, and $v \neq q$ such that $v$ is adjacent to $v_2$, then $\sigma(p, v_1; v_2, v)$ is a $\mathcal{P}_i$-switching, $G^{\sigma(p, v_1; v_2, v)} \in \mathcal{P}_i(2n)$, and $p$ is adjacent to $v_2$ under $P_k$. Thus there exists a sequence of $\mathcal{P}_i$-switchings that transforms $G$ to $G'$ with decycling set $S$, and an induced forest $F_N = V(G' - S)$ containing $P_k$ as its longest path. Therefore, there exists $k$, $2 \leq k \leq l-1$, such that $d_{F_N}(v_k) = 3$.

Thus we can assume that $q$ is adjacent to $v_2$. Let $x$, $y \in F_N$ that are adjacent to $p$. Thus $\sigma(p, x; v_2, q)$ or $\sigma(p, y; v_2, q)$ is a $\mathcal{P}_i$-switching. If $\sigma(p, x; v_2, q)$ is a $\mathcal{P}_i$-switching, then $G^{\sigma(p, x; v_2, q)}$ contains an edge $pv_2$ which lies under $P_k$. Thus there exists a sequence of $\mathcal{P}_i$-switchings that transforms $G$ to $G'$ with decycling set $S$ and an induced forest $F_N = V(G' - S)$ containing $P_k$ as its longest path. Therefore, there exists $k$, $2 \leq k \leq i-1$, such that $d_{F_N}(v_k) = 3$.

Therefore in any case there exists a sequence of $\mathcal{P}_i$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $G^{\sigma_1 \sigma_2 \cdots \sigma_t} \in \mathcal{P}_{i-1}$. The proof is complete. □

The following corollaries are immediate consequence of the preceding theorem.

**Corollary 1.** *Let $G \in \mathcal{P}_i(2n)$, $i \geq 1$ and let $F_N$ be an induced forest of $G$ of order $N = \lfloor \frac{3n-1}{2} \rfloor$. Then there exists a sequence of $\mathcal{P}_i$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $G^{\sigma_1\sigma_2\cdots\sigma_t} \in \mathcal{P}_0$.*

**Corollary 2.** *The subgraph $\mathcal{P}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ is connected.*

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# Interpolation theorems in jump graphs

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**Abstract**

For positive integers $m$ and $n$ $(1 \leq m \leq \binom{n}{2})$, let $\mathbb{J}_m(n)$ be the class of all distinct subgraphs of $K_n$ of size $m$. Let $G, H \in \mathbb{J}_m(n)$. Then $G$ is said to be obtained from $H$ by an edge jump if there exist four distinct vertices $u, v, w$, and $x$ of $K_n$ such that $e = uv \notin E(H)$, $f = wx \in E(H)$ and $\sigma(e, f)H := H + e - f = G$. The minimum number of edge jumps required to transform $H$ to $G$ is the jump distance from $H$ to $G$. The graph $\mathbb{J}_m(n)$ is that graph having $\mathbb{J}_m(n)$ as its vertex set where two vertices of $\mathbb{J}_m(n)$ are adjacent if and only if the jump distance between the corresponding subgraphs is 1. Let $\mathbb{C}_m(n)$ be the subset of $\mathbb{J}_m(n)$ consisting of all connected graphs. We prove in this paper that the graph $\mathbb{J}_m(n)$ is connected and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{C}_m(n)$ is also connected. Several graph parameters are proved to interpolate over the class $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$. Algorithms for determining the extreme values for the chromatic number $\chi$ and the clique number $\omega$ are also provided.

## 1 Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Bondy and Murty [1]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_m\}$. We use $|S|$ to denote the cardinality of a set $S$ and define $n = \nu(G) = |V|$ to be the *order* of $G$ and $m = \varepsilon(G) = |E|$ the *size* of $G$. We write $e = uv$ for an edge $e$ that joins vertex $u$ to vertex $v$. A *path* of order $k$ in a graph $G$, denoted by $P_k$, is a sequence of distinct vertices $u_1u_2 \cdots u_k$ of $G$ such that for all $i = 1, 2, \cdots, k-1$, $u_iu_{i+1}$ is an edge of $G$. The *degree* of a vertex $v$ of

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a graph $G$ is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree and the minimum degree of a graph $G$ is usually denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph of $G$ induced by $S$. We also use the notation $\varepsilon(S)$ for $\varepsilon(G[S])$. For a graph $G$ and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from $G$ by removing all edges in $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V(G)$, $G - X$ is the graph obtained from $G$ by removing all vertices in $X$ and all edges incident with vertices in $X$. For a graph $G$ and $X \subseteq E(\overline{G})$, $G + X$ denotes the graph obtained from $G$ by adding all edges in $X$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs $G$ and $H$, $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs.

For positive integers $m$ and $n$ $(1 \leq m \leq \binom{n}{2})$, let $\mathbb{J}_m(n)$ be the class of all distinct subgraphs of $K_n$ of size $m$. Let $G, H \in \mathbb{J}_m(n)$. Then $G$ is said to be obtained from $H$ by an *edge jump* if there exist four distinct vertices $u, v, w$, and $x$ of $K_n$ such that $e = uv \notin E(H)$, $f = wx \in E(H)$ and $\sigma(e, f)H := H + e - f = G$. The minimum number of edge jumps required to transform $H$ to $G$ is the jump distance from $H$ to $G$. *The graph* $\mathbb{J}_m(n)$ is that graph having $\mathbb{J}_m(n)$ as its vertex set and where two vertices are adjacent if and only if the jump distance between the corresponding subgraphs is 1. Let $\mathbb{C}_m(n)$ be the subset of $\mathbb{J}_m(n)$ consisting of all connected graphs. It is clear that the graph $\mathbb{J}_m(n)$ has order $\binom{N}{m}$, where $N = \binom{n}{2}$.

Note that the operation $\sigma(e, f)$ is well defined on $G$ if and only if $\sigma(f, e)$ is well defined on $\sigma(e, f)G$ and $\sigma(f, e)\sigma(e, f)G = G$. Thus the graph $\mathbb{J}_m(n)$ is simple.

The concept of the edge jump is not new. Harary et al. [6, 7, 8, 9, 10, 11, 12] introduced this concept in late 1980 and called it an *edge exchange*. They used this concept to prove interpolation problems raised by Chartrand. Structure of specific classes of jump graphs were also studied by Chartrand et al. [2, 3].

**Theorem 1.1** *Let $G, H \in \mathbb{J}_m(n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that*
$H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1}) \cdots \sigma(e_1, f_1)G$.

**Proof.** Let $G, H \in \mathbb{J}_m(n)$. If $G \neq H$, then $E(G) - E(H) \neq \emptyset$. Let $E(G) - E(H) = \{f_1, f_2, \cdots, f_t\}$ and $E(H) - E(G) = \{e_1, e_2, \cdots, e_k\}$. Since $G$ and $H$ are of the same size, it follows that $t = k$. Since $e_1 \notin E(G)$ and $f_1 \in E(G)$, it follows that $\sigma(e_1, f_1)$ is a well defined edge jump on $G$ and $|E(G_1) \cap E(H)| = |E(G) \cap E(H)| + 1$, where $G_1 = \sigma(e_1, f_1)G$. Further, for all $i = 2, 3, \cdots, t$, $\sigma(e_i, f_i)$ is a well defined operation on $G_{i-1}$ and $|E(G_i) \cap E(H)| = |E(G_{i-1}) \cap E(H)| + 1$, where $G_i = \sigma(e_i, f_i)G_{i-1}$. Thus $G_t = H$ and the proof is complete. □

**Corollary 1.2** *The graph $\mathbb{J}_m(n)$ is connected.*

**Theorem 1.3** *Let $G, H \in \mathbb{C}_m(n)$. Then $G = H$ or there is a finite sequence of edge jumps $\sigma(e_1, f_1)$, $\sigma(e_2, f_2)$, $\cdots$, $\sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$,*

$\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1})\cdots\sigma(e_1, f_1)G \in \mathbb{C}_m(n)$ *and*

$$H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1})\cdots\sigma(e_1, f_1)G.$$

**Proof.** Note that $\mathbb{C}_m(n) \neq \emptyset$ if and only if $m \geq n-1$. If $m = n-1$, then $\mathbb{C}_m(n)$ contains all trees of order $n$. We first prove the theorem when $m = n-1$. Let $T_1, T_2 \in \mathbb{C}_{n-1}(n)$ and $T_1 \neq T_2$. Then $E(T_1) - E(T_2) \neq \emptyset$, $E(T_2) - E(T_1) \neq \emptyset$, and $|E(T_1) - E(T_2)| = |E(T_2) - E(T_1)|$. For any $e_1 \in E(T_2) - E(T_1)$, $T_1 + e_1$ contains a cycle. Thus there exists $f \in E(T_1) - E(T_2)$, say $f_1$, such that $T_1 + e_1 - f_1$ is a tree. In this case $|E(T_1 + e_1 - f_1) \cap E(T_2)| = |E(T_1) \cap E(T_2)| + 1$. Thus there exists a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$, $\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1})\cdots\sigma(e_1, f_1)G \in \mathbb{C}_{n-1}(n)$ and $T_2 = \sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1})\cdots\sigma(e_1, f_1)T_1$. Let $G, H \in \mathbb{C}_m(n)$, $m \geq n$. Let $T_1$ and $T_2$ be spanning trees of $G$ and $H$, respectively. Then, by the previous argument, we may assume that $T_1 = T_2$. Thus $E(G) \cap E(H)$ contains $E(T_1)$. Put $E(G) - E(H) = \{f_1, f_2, \cdots, f_t\}$ and $E(H) - E(G) = \{e_1, e_2, \cdots, e_t\}$. Thus there is a finite sequence of edge jumps $\sigma(e_1, f_1), \sigma(e_2, f_2), \cdots, \sigma(e_t, f_t)$ such that for all $i = 1, 2, \cdots, t$, $\sigma(e_i, f_i)\sigma(e_{i-1}, f_{i-1})\cdots\sigma(e_1, f_1)G \in \mathbb{C}_m(n)$ and $H = \sigma(e_t, f_t)\sigma(e_{t-1}, f_{t-1})\cdots\sigma(e_1, f_1)G$. □

**Corollary 1.4** *The graph $\mathbb{C}_m(n)$ is connected.*

Let $\mathbb{G}$ be the class of all simple graphs. A function $\pi : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $\pi(G) = \pi(H)$ for all isomorphic graphs $G$ and $H$. A graph parameter $\pi$ is called an *interpolation graph parameter over* $\mathbb{J} \subseteq \mathbb{G}$ if there exist integers $x$ and $y$ such that

$$\{\pi(G) : G \in \mathbb{J}\} := [x, y] := \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

If $\pi$ is an interpolation graph parameter over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J}) := \min\{\pi(G) : G \in \mathbb{J}\}$ and $\max(\pi, \mathbb{J}) := \max\{\pi(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathbb{J}_m(n)$ we write $\min(\pi; m, n)$ and $\max(\pi; m, n)$ for $\min(\pi, \mathbb{J}_m(n))$ and $\max(\pi, \mathbb{J}_m(n))$, respectively, and in the case where $\mathbb{J} = \mathbb{C}_m(n)$ we write $\mathrm{Min}(\pi; m, n)$ and $\mathrm{Max}(\pi; m, n)$ for $\min(\pi, \mathbb{C}_m(n))$ and $\max(\pi, \mathbb{C}_m(n))$, respectively.

## 2 Interpolation Theorems

Studying interpolation theorems for graph parameters may be divided into two parts. The first part deals with the following question: Given a graph parameter $\pi$ and a subset $\mathbb{J}$ of $\mathbb{G}$, does $\pi$ interpolate over $\mathbb{J}$? If $\pi$ interpolates over $\mathbb{J}$, then $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$. The second part of the interpolation theorems for graph parameters is to find the values of $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph $G$ contains spanning trees having $m$ and $n$ end-vertices, with $m < n$, does $G$ contain a spanning tree with $k$ end-vertices for every integer $k$ with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. The details can be found in [6, 7, 8, 9, 10, 11, 12]. In [16], several graph parameters were proved to interpolate over the class of all graphs with the same degree sequence and presented the two parts of the interpolation theorems.

For any integers $m$ and $n$, it was shown in the previous section that the graph $\mathbb{J}_m(n)$ is connected and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{C}_m(n)$ is also connected. Let $\pi$ be a graph parameter. Thus we have the following theorems.

**Theorem 2.1** *For a graph $G \in \mathbb{J}_m(n)$ and an edge jump $\sigma(e, f)$, if*

$$|\pi(G) - \pi(\sigma(e, f)G)| \leq 1,$$

*then $\pi$ is an interpolation graph parameter over $\mathbb{J}_m(n)$.*

**Theorem 2.2** *For a graph $G \in \mathbb{C}_m(n)$ and an edge jump $\sigma(e, f)$, if*

$$|\pi(G) - \pi(\sigma(e, f)G)| \leq 1,$$

*then $\pi$ is an interpolation graph parameter over $\mathbb{C}_m(n)$.*

**Theorem 2.3** *Let $\mathbb{J} \subseteq \mathbb{J}_m(n)$ and the subgraph of the graph $\mathbb{J}_m(n)$ induced by $\mathbb{J}$ be connected. For a graph $G \in \mathbb{J}$ and an edge jump $\sigma(e, f)$, if $|\pi(G) - \pi(\sigma(e, f)G)| \leq 1$, then $\pi$ is an interpolation graph parameter over $\mathbb{J}$.*

## 3 Interpolation Graph Parameters

We will now present various graph parameters and prove interpolation results on the corresponding graph parameters with respect to $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$. We first state the definition of $\chi$. A *$k$-coloring* of a graph $G = (V, E)$ is a partition of its vertex set $V$ as $V_1 \cup V_2 \cup \cdots \cup V_k$ such that no two vertices in $V_i$ $(1 \leq i \leq k)$ are adjacent. The $V_i$'s are called the *color classes.* A function $c : V \to \{1, 2, \cdots, k\}$ such that $c(v) = i$ for each $v \in V_i (1 \leq i \leq k)$ is called a *color function.* If $G$ has a $k$-coloring, it is said to be *$k$-colorable* and the minimum integer $k$ for which $G$ is $k$-colorable is called the *chromatic number* of $G$ and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that $G$ is *$k$-chromatic.*

**Remark 3.1** In a proper coloring, each color class contains no edge, so $G$ is $k$-colorable if and only if $G$ is a $k$-partite graph. Thus a graph is 2-colorable if and only if it is bipartite. Thus a graph containing an odd cycle must be at least 3-colorable.

**Theorem 3.2** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\chi(G) - \chi(\sigma(e,f)G)| \leq 1$.*

**Proof.** Let $G$ be a graph. If $e \notin E(G)$, then $\chi(G) \leq \chi(G+e) \leq \chi(G)+1$ and if $f \in E(G)$, then $\chi(G+e)-1 \leq \chi(G+e-f) \leq \chi(G+e)$. Since either $\chi(G+e) = \chi(G)$ or $\chi(G+e) = \chi(G)+1$, it follows that $\chi(G) \leq \chi(G+e-f) \leq \chi(G)+1$. Hence $|\chi(G) - \chi(\sigma(e,f)G)| \leq 1$. □

A maximal complete subgraph of a graph $G$ is called a *clique* of $G$. The maximum order of clique of $G$ is called the *clique number* of $G$ and is denoted by $\omega(G)$.

In general there is no formula for the chromatic number of a graph. Determining the chromatic number of even a relatively small graph is often a challenging problem. However, lower bounds for the chromatic number of a graph $G$ can be given in term of the clique number of $G$. That is $\chi(G) \geq \omega(G)$, for any graph $G$.

**Theorem 3.3** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\omega(G) - \omega(\sigma(e,f)G)| \leq 1$.*

**Proof.** Let $G$ be a graph. If $e \notin E(G)$, then $\omega(G) \leq \omega(G+e) \leq \omega(G)+1$ and if $f \in E(G)$, then $\omega(G+e)-1 \leq \omega(G+e-f) \leq \omega(G+e)$. Since either $\omega(G+e) = \omega(G)$ or $\omega(G+e) = \omega(G)+1$, it follows that $\omega(G) \leq \omega(G+e-f) \leq \omega(G)+1$. Hence $|\omega(G) - \omega(\sigma(e,f)G)| \leq 1$. □

A graph containing no cycle as its subgraph is called an *acyclic graph*. An acyclic graph is called a *forest*. Thus each component of an acyclic graph is a tree.

Let $G$ be a graph and $F \subseteq V(G)$. This subset $F$ is called an *induced forest* of $G$ if the subgraph of $G$ induced by $F$ contains no cycle. The maximum cardinality of an induced forest of a graph $G$ is called the *forest number* of $G$ and is denoted by $I(G)$. That is

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

We have the following theorem.

**Theorem 3.4** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|I(G) - I(\sigma(e,f)G)| \leq 1$.*

**Proof.** Let $G$ be a graph and $F$ be a maximum induced forest of $G$. For any $e \notin E(G)$ we see that the subgraph of $G+e$ induced by $F$ contains at most one cycle. Thus $I(G+e) \geq I(G)-1$. It is clear that any induced forest $F'$ of $G+e$ is an induced forest of $G$. Thus $I(G+e) \leq I(G)$. Therefore $I(G)-1 \leq I(G+e) \leq I(G)$. Let $f \in E(G)$. Then $I(G+e) \leq I(G+e-f) \leq I(G+e)+1$. Since either $I(G+e) = I(G)$ or $I(G+e) = I(G)-1$, it follows that $I(G)-1 \leq I(G+e-f) \leq I(G)$. Hence $|I(G) - I(\sigma(e,f)G)| \leq 1$. □

There is a counterpart graph parameter of $I$ called the *decycling number*. Let $G$ be a graph. The minimum number of vertices of $G$ whose removal eliminates all cycles in

the graph $G$ is called the *decycling number* of $G$ and is denoted by $\phi(G)$. Evidently, for a graph $G$ of order $n$, $\phi(G) + I(G) = n$. Thus we obtain the following corollary.

**Corollary 3.5** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\phi(G) - \phi(\sigma(e,f)G)| \leq 1$.*

A subset $U$ of the vertex set $V(G)$ of a graph $G$ is said to be an *independent set* of $G$ if the subgraph of $G$ induced by $U$ contains no edge. An independent set of $G$ with maximum number of vertices is called a *maximum independent set* of $G$. The number of vertices of a maximum independent set of $G$ is called the *independence number* of $G$ and is denoted by $\alpha_0(G)$.

It is clear that $\alpha_0(G) = \omega(\overline{G})$, for any graph $G$. Observe that a graph $G \in \mathbb{J}_m(n)$ if and only if $\overline{G} \in \mathbb{J}_{\binom{n}{2}-m}(n)$. Thus we have the following corollary as a direct consequence of Theorem 3.5.

**Corollary 3.6** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\alpha_0(G) - \alpha_0(\sigma(e,f)G)| \leq 1$.*

A vertex of a graph $G = (V,E)$ is said to cover the edges incident with it. A *vertex cover* of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the *vertex covering number* of $G$ and is denoted by $\beta_0(G)$.

A subset $M$ of the edge set $E(G)$ of a graph $G$ is an *independent edge set* or *matching* of $G$ if no two distinct edges in $M$ have a common vertex. A matching $M$ of $G$ is *maximum* if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality of a maximum matching of $G$ is called the *matching number* of $G$ and is denoted by $\alpha_1(G)$.

There is an analogous covering concept for edges. An edge of a graph $G$ is said to cover two vertices incident with it. An *edge cover* of a graph $G$ is a set of edges covering all the vertices of $G$. The minimum cardinality of an edge cover of $G$ is called the *edge covering number* of $G$ and is denoted by $\beta_1(G)$.

**Theorem 3.7** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\alpha_1(G) - \alpha_1(\sigma(e,f)G)| \leq 1$.*

**Proof.** Let $G$ be a graph and $M$ be a matching of $G$. For any $e \notin E(G)$ we see that $M$ is also a matching of $G+e$. Thus $\alpha_1(G+e) \geq \alpha_1(G)$. On the other hand, for any matching $M'$ of $G+e$ we see that $M'$ is a matching of $G$ if $e \notin M'$ and $M'-e$ is a matching of $G$ if $e \in M'$. Thus $\alpha_1(G) - 1 \leq \alpha(G+e) \leq \alpha_1(G)$. Let $f \in E(G)$. Then $\alpha_1(G+e) - 1 \leq \alpha_1(G+e-f) \leq \alpha_1(G+e)$. Since either $\alpha_1(G+e) = \alpha_1(G)$ or $I(G+e) = I(G) - 1$, it follows that $\alpha_1(G) - 1 \leq \alpha_1(G+e-f) \leq \alpha_1(G)$. Hence $|\alpha_1(G) - \alpha_1(\sigma(e,f)G)| \leq 1$. □

Let $n$ be a positive integer. The *star* of order $n+1$ is the complete bipartite graph $K_{n,1}$. The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest number of stars. This is equivalent to our next graph parameter.

A *dominating set* of a graph $G$ is a subset $D$ of $V(G)$ such that each vertex of $V(G) - D$ is adjacent to at least one vertex of $D$. The *domination number*, $\gamma(G)$, is the cardinality of a minimal dominating set with the least number of elements.

**Theorem 3.8** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\gamma(G) - \gamma(\sigma(e,f)G)| \leq 1$.*

**Proof.** Let $G$ be a graph and $D$ be a dominating set of $G$. For any $e \notin E(G)$ we see that $D$ is also a dominating set of $G+e$. Thus $\gamma(G+e) \leq \gamma(G)$. On the other hand, for any dominating set $D'$ of $G+e$, where $e = uv$, we see that $D'$ is a dominating set of $G$ if either $u, v \in D'$ or $u, v \notin D'$, otherwise $D' \cup \{u\}$ or $D' \cup \{v\}$ is a dominating set of $G$. Thus $\gamma(G) - 1 \leq \gamma(G+e) \leq \gamma(G)$. Let $f \in E(G)$. Then $\gamma(G+e) \leq \gamma(G+e-f) \leq \gamma(G+e) + 1$. Since either $\gamma(G+e) = \gamma(G)$ or $\gamma(G+e) = \gamma(G) + 1$, it follows that $\gamma(G) \leq \gamma(G+e-f) \leq \gamma(G) + 1$. Hence $|\gamma(G) - \gamma(\sigma(e,f)G)| \leq 1$. □

Gallai [5] and Norman-Rabin [14] proved the following results concerning relationship between $\alpha_0$ and $\beta_0$, and between $\alpha_1$, and $\beta_1$, respectively.

**Theorem 3.9** *For any graph $G$ of order $n$, $\alpha_0 + \beta_0 = n$.*

**Theorem 3.10** *For any graph $G$ of order $n$ and $\delta \geq 1, \alpha_1 + \beta_1 = n$.*

As a direct consequence of the previous two theorems, we have the following corollary.

**Corollary 3.11** *Let $G \in \mathbb{J}_m(n)$ and $\sigma(e,f)$ be an edge jump on $G$. Then $|\beta_0(G) - \beta_0(\sigma(e,f)G)| \leq 1$. and $|\beta_1(G) - \beta_1(\sigma(e,f)G)| \leq 1$.*

The following theorems are obtained as consequences of our results.

**Theorem 3.12** *$\chi$, $\omega$, $I$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$ and $\gamma$ are interpolation graph parameters over $\mathbb{J}_m(n)$.*

**Theorem 3.13** *$\chi$, $\omega$, $I$, $\phi$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$ and $\gamma$ are interpolation graph parameters over $\mathbb{C}_m(n)$.*

**Theorem 3.14** *Let $\pi \in \{\chi, \omega, I, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then there exist integers $a := \min(\pi; m, n)$ and $b := \max(\pi; m, n)$ such that there exists $G \in \mathbb{J}_m(n)$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

**Theorem 3.15** *Let $\pi \in \{\chi, \omega, I, \phi, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then there exist integers $a := \mathrm{Min}(\pi; m, n)$ and $b := \mathrm{Max}(\pi; m, n)$ such that there exists $G \in \mathbb{C}_m(n)$ with $\pi(G) = c$ if and only if $c$ is an integer satisfying $a \leq c \leq b$.*

## 4 Extremal Problems for Graph Parameters

We have already proved interpolation property of various graph parameters in Section 3. Thus for an interpolation graph parameter $\pi$ over $\mathbb{J} \subseteq \mathbb{G}$, the values of $\{\pi(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$. The problem of finding $\min(\pi, \mathbb{J})$ and $\max(\pi, \mathbb{J})$ is so called the extremal problem in graph theory.

An *extremal problem* asks for minimum and maximum values of a function over a class of objects.

**Remark 4.1** Proving that $A = \min\{\pi(G) : G \in \mathbb{J}\}$ requires showing two things:

1. $\pi(G) \geq A$ for all $G \in \mathbb{J}$.
2. $\pi(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of $\pi$.

Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

A study of extremal problems for graph parameters was motivated by Dirac's conjecture as state in the following.

In the graph-theoretic colloquium at Smolenice in 1963, Dirac conjectured that the chromatic number of a proper regular subgraph of a complete $n$-gon is at most $\frac{3n}{5}$. Erdős and Gallai answered this conjecture immediately and presented their result during the conference. Their article was entitled "Solution to a problem of Dirac" [4], and it appeared in the proceedings of the symposium, Smolenice, in 1964. In fact, the result was more than Dirac asked for. In addition, they found all regular graphs reaching the upper bound as stated in the following theorem.

**Theorem 4.2** *An $r$-regular graph $G$ of order $n > r + 1$ has chromatic number $k \leq \frac{3n}{5}$, with equality holds if and only if the complementary graph $\overline{G}$ of $G$ is a union of 5-cycles.*

In [15] we generalized the result of the above theorem by introducing the notion of $F(j)$-graphs and using these graphs to obtain generalized results.

Let $j$ be a positive integer. An *$F(j)$-graph* is a $(j-1)$-regular graph $G$ of minimum order $f(j)$ satisfying $\chi(\overline{G}) > f(j)/2$.

It is easy to see that an $F(3)$-graph is $C_5$ and $f(3) = 5$. We will see that $F(j)$-graphs, $j \geq 5$, are not unique.

We have found $F(j)$-graphs for all odd integers $j$ as we state in the following theorem.

**Theorem 4.3** *For odd integers $j$ with $j \geq 3$, we have $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 (\mathrm{mod}\ 4)$ and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 (\mathrm{mod}\ 4)$.*

The following result can be found in [15] which is a generalization of Theorem 4.2.

**Theorem 4.4** *Any $r$-regular graph of order $n$ with $n - r = j$ odd and $j \geq 3$ has chromatic number at most $\frac{f(j)+1}{2f(j)} \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.*

As we have already proved the interpolation property of $\chi$ with respect to $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$ in Section 3, it is natural to ask the same for $\min(\chi; m, n)$, $\max(\chi; m, n)$, $\mathrm{Min}(\chi; m, n)$ and $\mathrm{Max}(\chi; m, n)$. We will consider this question in this section.

First, we state some elementary facts concerning the chromatic number of graphs of order $n$ and size $m$.

1. Every $k$-chromatic graph with $n$ vertices has at least $\binom{k}{2}$ edges. Equality holds for the graph $K_k \cup (n-k)K_1$.
2. If the connectivity is concerned, one can ask for the minimum size among connected $k$-chromatic graphs with $n$ vertices. This question was answered in [18]: the minimum size of such a graph is at least $\binom{k}{2}+n-k$. Equality holds for a graph $G$ with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{v_1v_i : 2 \leq i \leq n\} \cup \{v_iv_j : 2 \leq i < j \leq k\}$.
3. Mantel proved in [13] that the maximum size of a triangle-free graph of order $n$ is $\lfloor \frac{n^2}{4} \rfloor$. Equality holds for the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Mantel's theorem provides the maximum number of edges that a 2-chromatic graph of order $n$ can have. On the other hand the minimum number of edges in a 2-chromatic graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is $n - 1$. Turán [17] extended the result of Mantel by introducing the *Turán graph*. The famous result of Turán is viewed as the origin of extremal graph theory.

The *Turán graph* $T_{n,r}$ is the complete $r$-partite graph of order $n$ whose partite sets differ in size by at most 1.

**Theorem 4.5** *Among the graphs of order $n$ containing no complete subgraph of order $r+1$, $T_{n,r}$ has the maximum number of edges.*

In order to apply Turán's theorem in our context, we would like to state the following facts.

1. If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ consists of $t$ partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.
2. Let $G \in \mathbb{J}_m(n)$. If $\omega(G) \leq r$, then $m \leq \varepsilon(T_{n,r})$.
3. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

4. Let $t(n,r) = \varepsilon(T_{n,r})$. Then for a fixed $n$, by using elementary arithmetic, we have $t(n,r-1) < t(n,r)$ for all $r$, $2 \leq r \leq n$. In fact $t(n,r) - t(n,r-1) \geq \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

We have the following theorems.

**Theorem 4.6** *Let $n, m$ and $k$ be positive integers satisfying $n \geq k \geq 3$ and $\binom{k}{2} \leq m < \binom{k+1}{2}$. Then* $\max(\chi; n, m) = k$.

**Proof.** By above observation, we see that every $k$-chromatic graph has at least $\binom{k}{2}$ edges. Thus for $n, m$ and $k$ satisfying conditions of the theorem, $\max(\chi; n, m) \leq k$. Observe that if $n = k$, then $m = \binom{k}{2}$. Thus $\max(\chi; k, \binom{k}{2}) = k$. Suppose $n > k$. Since $\binom{k}{2} \leq m < \binom{k+1}{2}$, it follows that $m - \binom{k}{2} < \binom{k+1}{2} - \binom{k}{2} = k$. Let $G$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{v_i v_j : 1 \leq i < j \leq k\} \cup \{v_n v_i : 1 \leq i \leq m - \binom{k}{2}\}$. It is easy to see that $G \in \mathbb{J}_m(n)$ and $\chi(G) = k$. Thus $\max(\chi; n, m) = k$. □

**Theorem 4.7** *Let $n, m$ and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n,k)$. Then* $\min(\chi; n, m) = k$.

**Proof.** By definition of $T_{n,r}$ and above observation, we find that if $t(n,k-1) < m \leq t(n,k)$, them $\min(\chi; m, n) \geq k$. Let $G$ be a graph of order $n$ obtained from removal $t(n,k) - m$ edges from $T_{n,k}$. Thus $G \in \mathbb{J}_m(n)$ and $G$ is a subgraph of $T_{n,k}$. Therefore $\chi(G) \leq \chi(T_{n,k}) = k$. This proves that $\min(\chi; m, n) = k$. □

As we have mentioned earlier, the graph parameters $\chi$ and $\omega$ are closely related in the sense that $\chi(G) \geq \omega(G)$. However, it is known that for any positive integer $t$, there exists a graph $G$ such that $\chi(G) - \omega(G) \geq t$. But this situation does not occur from the results of the previous two theorems. The results are still true if we replace $\chi$ by $\omega$ as state in the following corollaries.

**Corollary 4.8** *Let $n, m$ and $k$ be positive integers satisfying $n \geq k$ and $\binom{k}{2} \leq m < \binom{k+1}{2}$. Then* $\max(\omega; n, m) = k$.

**Corollary 4.9** *Let $n, m$ and $k$ be positive integers satisfying $t(n, k-1) < m \leq t(n,k)$. Then* $\min(\omega; n, m) = k$.

Since $\mathbb{C}_m(n) \subseteq \mathbb{J}_m(n)$, one sees that $\min(\chi; m, n) \leq \mathrm{Min}(\chi; m, n)$ and $\mathrm{Max}(\chi; m, n) \leq \max(\chi; m, n)$. A graph $G$ that we constructed in the proof of Theorem 4.7 contains enough edges to be connected. Thus $\min(\chi; m, n) = \mathrm{Min}(\chi; m, n)$.

**Corollary 4.10** *Let $n, m$ and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n,k)$. Then* $\mathrm{Min}(\chi; n, m) = k$.

**Corollary 4.11** *Let $n, m$ and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n, k)$. Then* $\mathrm{Min}(\omega; n, m) = k$.

Results can be obtained similarly as stated in the following theorems.

**Theorem 4.12** *Let $n, m$ and $k$ be positive integers satisfying $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. Then* $\mathrm{Max}(\chi; n, m) = k$.

**Theorem 4.13** *Let $n, m$ and $k$ be positive integers satisfying $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. Then* $\mathrm{Max}(\omega; n, m) = k$.

Thus all extreme values of $\chi$ and $\omega$ in $\mathbb{J}_m(n)$ and $\mathbb{C}_m(n)$ are obtained in all situations.

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# A 3-$(v, 4, 1)$ PACKING CONSTRUCTION

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**Abstract**

The packing number $D(v, k, t)$ is the maximum number of blocks for any $t$-$(v, k, 1)$ packing. A construction for 3-$(v, 4, 1)$ is given, which when applied to $v \equiv 5 (\mathrm{mod}\ 6)$ produces packings which can improve the packing number for $2v + 1$ in certain cases. In some cases, it can achieve the bound e.g. it is shown that $D(27, 4, 3) = 702$.

## 1 Introduction

Let $v \geq k \geq t$. A $t$-$(v, k, \lambda)$ *packing* is a pair $(X, \mathcal{B})$, where $X$ is a $v$-set of elements (*points*) and $\mathcal{B}$ is a collection of $k$-subsets of $X$ (*blocks*), such that every $t$-subset of points occurs in at most $\lambda$ blocks in $\mathcal{B}$. The *packing number* $D_\lambda(v, k, t)$ is the maximum number of blocks in any $t$-$(v, k, \lambda)$ packing.. A $t$-$(v, k, \lambda)$ packing $(X, \mathcal{B})$ is *optimal* if $|\mathcal{B}| = D_\lambda(v, k, t)$. If $\lambda = 1$, we usually write $D(v, k, t)$ for $D_1(v, k, t)$.

Upper bounds for $D_\lambda(v, k, t)$ were given by Johnson cited from [1; page 409-410] as stated in the following theorems.

**Theorem 1.1.** $D_\lambda(v, k, t) \leq \lfloor \frac{v \cdot D_\lambda(v-1, k-1, t-1)}{k} \rfloor$. *Iterating this bound yields* $D_\lambda(v, k, t) \leq U_\lambda(v, k, t)$. *where* $U_\lambda(v, k, t) = \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \cdots \lfloor \frac{\lambda(v-t+1)}{k+t-1} \rfloor \rfloor \rfloor$.

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*Further, if* $\lambda(v-1) \equiv 0 (\mathrm{mod}\ (k-1))$ *and* $\lambda v(v-1) \equiv -1 (\mathrm{mod}\ k)$, *then* $D_\lambda(v,k,2) \leq U_\lambda(v,k,2) - 1$.

**Theorem 1.2.** *Suppose* $d = D_\lambda(v,k,t) = qv + r$, *where* $0 \leq r \leq v-1$. *Then* $q(q-1)v + 2qr \leq (t-1)d(d-1)$. *From this it follows that* $D_\lambda(v,k,t) \leq \lfloor \frac{v(k+1-t}{k^2-v(t-1)} \rfloor$.

This note is motivated by a remark in Stinson ([1]; page 410) that for $v \equiv 5 (\mathrm{mod}\ 6)$, $v \geq 17$, essentially nothing is known about the number $D(v,4,3)$, the packing number for a 3-$(v,4,1)$ packing and by the well known construction of SQS$(2v)$ from SQS$(v)$, see for example Lindner and Rodger [2].

For the most part, our notation and terminology follows that of Lindner and Rodger [2].

## 2 Construction from $v$ to $2v$

Observe that the well known $2v$ construction for SQS to produce SQS$(2v)$ from an SQS$(v)$, can be applied for packing and for any even integer $v$ where a packing exists.

**Theorem 2.1.** *A* 3-$(v,4,1)$ *packing* $(X, \mathcal{B})$ *for even* $v$ *implies the existence of a* 3-$(2v,4,1)$ *packing.*

**Proof** A 3-$(v,4,1)$ packing $(X, \mathcal{B})$ is given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_{v-1}\}$ be a 1-factorization of $K_v$ defined on $X$. Then the following blocks together produce a 3-$(2v,4,1)$ packing on the set of vertices $X \times \{1,2\}$ and set of blocks $\mathcal{B}'$, where $\mathcal{B}'$ is defined as follows.

1. For each block $B \in \mathcal{B}$, $B \times \{1\}, B \times \{2\} \in \mathcal{B}'$.
2. For each $F_j \in \mathcal{F}$, $\{(a,1),(b,1),(c,2),(d,2)\} \in \mathcal{B}'$ if and only if $\{a,b\} \in F_j$ and $\{c,d\} \in F_j$.

□

**Corollary 2.2.** *The construction produces optimal packing for* $v \equiv 0 (\mathrm{mod}\ 6)$

**Proof** Let $v = 6t$. Then $D(6t,4,3) = \lfloor \frac{18t^3-9t^2-3t}{2} \rfloor = \frac{18t^3-9t^2-3t}{2}$. On the other hand the construction produces $\mathcal{B}'$ with $|\mathcal{B}'| = 2 \cdot \frac{18t^3-9t^2-3t}{2} + (3t) \cdot (3t) \cdot (6t-1) = 72t^3 - 18t^2 - 3t$, the exact number needed for optimal packing for 3-$(v,4,1)$, i.e, $D(12t,4,3)$ is achieved. This means, the construction produces an optimal 3-$(2v,4,1)$ packing, just like in the case of SQS$(v)$. □

For example, when $v = 6$, the packing number $D(6,4,3)$ is 3. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_5\}$ be a set of 1-factorization of $K_6$. For each $F_j \in \mathcal{F}$, $F_i$ contains 3 independent edges. Therefore we get a packing for $2v = 12$ with $3+3+3 \cdot 3 \cdot 5 = 51$ blocks. Note that $D(12,4,3)$ is 51.

## 3 Construction from $v$ to $2v+1$

A slight modification of above construction is very useful as it can be used to get 3-$(v,4,1)$ packing for $v \equiv 5 (\mathrm{mod}\ 6)$.

**Theorem 3.1.** *Packings* 3-$(v,4,1)$ *and* 3-$(v+1,4,1)$ *implies the existence of a* 3-$(2v+1,4,1)$ *packing.*

**Proof** Let $v$ be an odd integer. Suppose a packing 3-$(v,4,1)$ and a packing 3-$(v+1,4,1)$ with sets of blocks $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, are given. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_v\}$ be a 1-factorization of $K_{v+1}$. Then the following blocks together produce a 3-$(2v+1,4,1)$ packing on the set of vertices $\{(i,1) : i = 1,2,\cdots,v\} \cup \{(i,2) : i = 1,2,\cdots,v+1\}$ with set of blocks $\mathcal{B}'$ according to the following rules.

1. For each block $B \in \mathcal{B}_1$, $B \times \{1\} \in \mathcal{B}'$.
2. For each block $B \in \mathcal{B}_2$, $B \times \{2\} \in \mathcal{B}'$.
3. For each $F_j \in \mathcal{F}$, $\{(a,1),(b,1),(c,2),(d,2)\} \in \mathcal{B}'$ if and only if $\{a,b\} \in F_j$ and $\{c,d\} \in F_j$.

□

The construction produces $|\mathcal{B}_1| + |\mathcal{B}_2| + \frac{v-1}{2} \cdot \frac{v+1}{2} \cdot v$ blocks. Moreover the construction gives blocks with more structure than the packing obtained from deleting an element from a 3-$(2v+2,4,1)$ and it can produce optimal packing. For example, when $v = 7$, we get $7 + 14 + 3 \cdot 4 \cdot 7 = 105$ blocks, which gives an optimal packing as $D(15,4,1) = 105$.

Note that if $v \equiv 5 (\mathrm{mod}\ 6)$, then $2v+1 \equiv 5 (\mathrm{mod}\ 6)$. Let us first consider in the case when $v = 5$. It is known that $D(5,4,3) = 1$ and $D(6,4,3) = 3$. Thus by the above construction, we get $D(11,4,3) \geq 1 + 3 + 2 \cdot 3 \cdot 5 = 34$. It is also known that $D(11,4,3) = 35$. So, the construction achieves one block less than an optimal packing in this case. Now consider when $v = 11$. It is known that $D(11,4,3) = 35$ and $D(12,4,3) = 51$. Thus by the above construction, we get $D(23,4,3) \geq 35 + 51 + 5 \cdot 6 \cdot 11 = 416$. It is known that $D(23,4,3) \leq \lfloor \frac{23}{4} \lfloor \frac{22}{3} \lfloor \frac{21}{2} \rfloor \rfloor \rfloor = 419$. Thus $416 \leq D(23,4,3) \leq 419$. For $v = 17$, we know that $D(35,4,3) \leq 1583$ and by the construction gives that $D(35,4,3) \geq 1579$. Finally, it is known that $D(27,4,3) = 702$, and the construction of 3-$(27,4,1)$ above produces $65 + 91 + 6 \cdot 7 \cdot 13 = 702$ blocks. In fact, we have the following corollary.

**Corollary 3.2.** *If* $v \equiv 1 (\mathrm{mod}\ 12)$, *say* $v = 12s+1$. $D(24s+3,4,3) = (24s+2)(24s+3)s = 576s^3 + 120s^2 + 6s$ *and the construction also produces the same number of blocks required to give optimal packing.*

# 4 Generalization of $v$ to $2v$ construction from SQS($v$) to $(2t-1)$-$(v, 2t, 1)$ packing for any $t$

Suppose large set exists for $(v, t)$, in other words assume that all $t$-subsets of $v$-set can be partitioned to BIBD$(v, t, 1)$. Note that the set of all $2t$-subsets of a $v$-set is a $(v, \binom{v}{t}, \binom{v}{t-1}, t, \binom{v}{t-1} \cdot (\frac{t-1}{v-1}))$. So there are $s = \binom{v}{t-1} \cdot (\frac{t-1}{v-1})$ $BIBD(v, t, (\lambda = 1)$-designs. Call them $F_1, F_2, \cdots, F_s$. We have the following construction.

**Theorem 4.1.** *Suppose there exists a* $(2t-1)$-$(v, 2t, 1)$ *packing with* $M$ *blocks, then there exists a* $(2t-1)$-$(2v, 2t, 1)$ *packing with* $2M + [\frac{t-1}{v-1} \cdot \binom{v}{t-1}]^2 \cdot [\frac{\binom{v}{t}}{\binom{v}{t-1} \cdot \frac{t-1}{v-1}}]$.

**Proof** As usual let $V = \{1, 2, \cdots, v\}$ and the blocks of $(2t-1)$-$(v, 2t, 1)$ packing be $\mathcal{B}$. The blocks of $(2t-1)$-$(2v, 2t, 1)$ are in $\mathcal{B}'$ on the set $V \times \{1, 2\}$, according to the following rules.

1. If $B \in \mathcal{B}$, then $B \times \{1\}$ and $B \times \{2\}$ are in $\mathcal{B}'$.
2. For each $F_j$, if $\{a_1, a_2, \cdots, a_t\}$ and $\{b_1, b_2, \cdots, b_t\}$ are any two not necessary distinct blocks in $F_j$, then $\{(a_1, 1), (a_2, 1), \cdots, (a_t, 1), (b_1, 2), (b_2, 2), \cdots, (b_t, 2)\}$ is in $\mathcal{B}'$.

Note that two blocks of size $t$ in $F_j$ intersect in only 1 point. Therefore the intersection between any two distinct blocks directly obtained from $\mathcal{B}$ and from $F_j$ is at most $t$ and the intersection between the blocks from $\mathcal{B}$ can be $(2t-1)$. Intersection of blocks obtained from the same $F_j$ can be at most $(t+1)$. Intersection of blocks obtained from different $F_j$ can be at most $2(t-1) = 2t-2$.

□

For example, let $v = 7$, we have $D(7, 6, 1) = 1$. Since $\binom{7}{3} = 35$, there is a partition of all subsets of size 3 into 5 2-$(7, 3, 1)$. So there exists 5-$(14, 6, 1)$ packing with $1 + 1 + 5 \times 7 \times 7 = 247$ blocks. The bound for 5-$(14, 6, 1)$ is $\lfloor \frac{14}{6} \lfloor \frac{13}{5} \lfloor \frac{12}{4} \lfloor \frac{11}{3} \lfloor \frac{10}{2} \rfloor \rfloor \rfloor \rfloor \rfloor = 326$. The construction does not provide close enough number of blocks but notice that block intersections are much lower than $2t-2$ for many pairs.

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# A Construction For Group Divisible Designs With Two Groups

Narong Punnim and Dinesh G. Sarvate

ABSTRACT. We construct new group divisible designs with two groups with any block size. As special cases block size six and with 2 groups and group divisible designs with block size 6 and 8 groups are constructed.

## 1. Introduction

A group divisible design, GDD$(m, n, k; \lambda_1, \lambda_2)$, is a collection of $k$- subsets of a $mn = v$-set $X$ called blocks and a partition of the $v = nm$ elements of $X$ into $m$ subsets (called groups) of size $n$ which satisfies the following properties: each point of $X$ appears in $r$ of the $b$ blocks; each pair of distinct points within the same group appear $\lambda_1$ times together in blocks. Any two points not in the same group appear together in $\lambda_2$ blocks. In general it is hard to construct GDDs where the number of groups is less than the block size. This note gives a new construction for GDD$(2k, t, k+2; \binom{k}{2}+1, 2(k-1))$ for any $t$ for which $k$ mutually orthogonal latin squares exist, which can be improved to a GDD construction with two groups of size $kt$ for any $t$ and with block size $k+2$ if a BIBD$(t, k+2, \binom{k}{2}+1)$. Observe it means we have a construction for new GDD family with two groups of size $4t$ for any $t$ for which there are four mutually orthogonal latin squares and with block size 6. For necessary definitions, the reader is referred to [**2**].

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*Key words and phrases.* BIBD, GDD, MOLS, PBIBD, group divisible design, latin square.

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## 2. The Construction

The following theorem generalizes a construction given in [3] for block size 5.

THEOREM 1. *If for any positive integer $t$, $k$ mutually orthogonal latin squares of order $t$ exist then there exists a GDD$(2k, t, k+2; \binom{k}{2}+1, 2(k-1))$ and further if a BIBD$(t, k+2, \lambda = \binom{k}{2}+1)$ exists then there exists a GDD$(2, n = kt, k+2; \lambda_1 = (\binom{k}{2}+1)s, \lambda_2 = 2(k-1)s)$ for any positive integer $s$.*

PROOF. Suppose the mutually orthogonal latin squares $L_i$, $i = 1, 2, \ldots, k$ of order $t$ are on symbols $1, 2, 3, \ldots, t$ and rows and columns are also labeled by $1, 2, 3, \ldots, t$. Arbitrarily partition the group $G_i$ for $i = 1, 2$ into subsets $G_{ij}$, $j = 1, 2, \ldots, k$. Identify elements of $G_{ij}$ with $1, 2, ..., t$ arbitrarily by a one-to-one correspondence. When we use the phrase replace the entries or labels of a latin square by elements of some $G_{ij}$, this correspondence is intended to be used. Replace labels of the rows of latin squares with $G_{11}$ and columns by $G_{12}$ and replace the entries of $L_j$ by the elements of $G_{2,j}$ for $j = 1, 2, \ldots, k$. Construct $t^2$ blocks

$$\{\text{label of } x^{th} \text{ row, label of } y^{th} \text{ column, } L_1(x,y),\ L_2(x,y),\ldots,\ L_k(x,y)\}$$

where $L_z(x,y)$ is $(x,y)^{th}$ entry of $L_z$ for $z = 1, 2, \ldots, k$. Repeat this construction for blocks by replacing the row and column labels by all remaining pairs $(G_{1i}, G_{1j})$ from $1 \le i < j \le k$ respectively.

At this time we have $\binom{k}{2}t^2$ blocks. Observe, for $i \neq j$, each element from $G_{1i}$ comes with every element of $G_{1j}$ exactly once and also each element from $G_{2i}$ comes with every element of $G_{2j}$ exactly $\binom{k}{2}$ times. Also, for all $i, j$, including $i = j$, each element from $G_{1i}$ comes with every element of $G_{2j}$ exactly $k-1$ times. Now interchange the roles of $G_{1i}$ with $G_{2i}$ and again construct $\binom{k}{2}t^2$ blocks. These $2\binom{k}{2}t^2$ blocks together satisfy the condition that an element from $G_1$ comes with every element of $G_2$ exactly $2(k-1)$ blocks and elements from $G_i x$ with $G_i y$ for $i = 1, 2$ and $x \neq y$ occur together in exactly $\binom{k}{2}+1$ blocks. Hence, we have a GDD$(2k, t, k+2; \binom{k}{2}+1, 2(k-1))$ for any $t$ for which $k$ mutually orthogonal latin squares exist. Now, assuming the existence of a BIBD$(t, k+2, \binom{k}{2}+1)$, include in the collection of blocks so far constructed, the blocks of BIBD$(t, k+2, \binom{k}{2}+1)$ on $G_{ij}$ for $i = 1, 2$ and $j = 1, 2, \ldots, k$. This completes the construction of GDD$(2, n = kt, k+2; \lambda_1 = (\binom{k}{2}+1), \lambda_2 = 2(k-1))$. Taking $s$ copies of the constructed GDD gives the required design for any $s$.

□

We would like to restate an useful version of the above result for block size 6 and the proof is repeated to serve as a concrete example.

COROLLARY 1. *If for any positive integer t, four mutually orthogonal latin squares of order t and a BIBD*$(t, 6, \lambda = 7)$ *exist then there exists a GDD*$(2, n = 4t, 6; \lambda_1 = 7s, \lambda_2 = 6s)$ *for any positive integer s.*

PROOF. Suppose the mutually orthogonal latin squares $L_1, L_2, L_3$, and $L_4$ of order $t$ are on symbols $1, 2, 3, ...t$ and rows and columns are also labeled by $1, 2, 3..., t$. Arbitrarily partition the group $G_i$ for $i = 1, 2$ into subsets $G_{ij}$, $j = 1, 2, 3, 4$. Identify elements of $G_{ij}$ with $1, 2, ..., t$ arbitrarily by a one-to-one correspondence. When we use the phrase replace the entries or labels of a latin square by elements of some $G_{ij}$, this correspondence is intended to be used. Replace labels of the rows of all four latin squares with $G_{11}$ and columns by $G_{12}$ and replace the entries of $L_x$ by the elements of $G_{2,x}$ for $x = 1, 2, 3, 4$. Construct $t^2$ blocks

$$\{\text{label of } x^{th} \text{ row, label of } y^{th} \text{ column, } L_1(x, y),\ L_2(x, y),\ L_3(x, y),\ L_4(x, y)\}$$

where $L_z(x, y)$ is $(x, y)^{th}$ entry of $L_z$ for $z = 1, 2, 3, 4$. Repeat this construction for blocks by replacing the row and column labels by all remaining five pairs $(G_{1i}, G_{1j})$ from $1 \leq i < j \leq 4$ respectively.

Note at this time we have $6t^2$ blocks. Observe, for $i \neq j$, each element from $G_{1i}$ comes with every element of $G_{1j}$ exactly once and also each element from $G_{2i}$ comes with every element of $G_{2j}$ exactly six times (in both cases ). Also, for all $i, j$, including $i = j$, each element from $G_{1i}$ comes with every element of $G_{2j}$ exactly three times. Now interchange the roles of $G_{1i}$ with $G_{2i}$ and again construct $6t^2$ blocks. These $12t^2$ blocks satisfy the condition that an element from $G_1$ comes with every element of $G_2$ exactly six blocks and elements from $G_i x$ with $G_i y$ for $i = 1, 2$ and $x \neq y$ occur together in exactly 7 blocks. Hence, we have a GDD$(8, t, 6; 7, 6)$ for any $t$ for which four mutually orthogonal latin squares exist. Now, assuming the existence of a BIBD$(t, 6, 7)$, include in the collection of blocks so far constructed, the blocks of BIBD$(t, 6, 7)$ on $G_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3, 4$. This completes the construction of a GDD$(2, 4t, 6; 7, 6)$. Taking $s$ copies of the constructed GDD gives the required design for any $s$. □

COROLLARY 2. *(a) For* $t \geq 5$ *except* $t = 14, 18, 22$, *a GDD*$(8, t, 6; 7, 6)$ *exists as according to Table 2.73 in* [1] *for all other values of t 4 MOLS exist.*

*(b) Except for finitely many t's, for all* $t \equiv 1, 6 (\text{mod } 15)$, *a BIBD*$(t, 6, 7)$ *exists and hence for infinitely many values of t's, there exists a GDD*$(2, t, 6; 7s, 6s)$.

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# The Hamiltonian Number of Cubic Graphs

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**Abstract.** A Hamiltonian walk in a connected graph $G$ of order $n$ is a closed spanning walk of minimum length in $G$. The Hamiltonian number $h(G)$ of a connected graph $G$ is the length of a Hamiltonian walk in $G$. Thus $h$ may be considered as a measure of how far a given graph is from being Hamiltonian. We prove that if $G$ runs over the set of connected cubic graphs of order $n$ and $n \neq 14$ then the values $h(G)$ completely cover a line segment $[a, b]$ of positive integers. For an even integer $n \geq 4$, let $\mathcal{C}(3^n)$ be the set of all connected cubic graphs of order $n$. We define $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{C}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{C}(3^n)\}$. Thus for an even integer $n \geq 4$, the two invariants $\min(h, 3^n)$ and $\max(h, 3^n)$ naturally arise. Evidently, $\min(h, 3^n) = n$. The exact values of $\max(h, 3^n)$ are obtained in all situations.

**Key Words**: Hamiltonian walk, Hamiltonian number, cubic graph.

**AMS Subject Classification**: 05C12, 05C45

## 1 Introduction

If $G$ is a connected graph of size $m$, there is always a closed spanning walk of length at most $2m$. Goodman and Hedetniemi [6] introduced the concept of a *Hamiltonian walk* in a connected graph $G$. It is defined as a closed spanning walk of minimum length in $G$. They denoted the length of a Hamiltonian walk of $G$ by $h(G)$. Therefore, for a connected graph $G$ of order $n \geq 3$, it follows that $h(G) = n$ if and only if $G$ is Hamiltonian. Thus $h$ may be considered as a measure of how far a given graph is from being Hamiltonian. Hamiltonian walks were studied further by T. Asano, T. Nishizeki, and T. Watanabe [2, 3], J. C. Bermond [4], and P. Vacek [11]. Since every connected 2-regular graph is Hamiltonian, it is reasonable to investigate the Hamiltonian number in the class of cubic graphs. In [10], it was shown that all cubic graphs of order $n$ for $n = 4, 6, 8$ are Hamiltonian. Let $P(k, m)$ be a generalized Petersen graph such that $V(P(k, m)) = \{u_i, v_i : i = 0, 1, \cdots, k-1\}$ and $E(P(k, m)) = \{u_i u_{i+1}, v_i v_{i+m}, u_i v_i : i = 0, 1, \cdots, k-1\}$ where addition

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is taken modulo $k$ and $m \leq \frac{k}{4}$. In [1] Alspach completed the determination of the parameters $k, m$ for which $P(k,m)$ is Hamiltonian. He proved that the generalized Petersen graph $P(k,m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5(\text{mod } 6)$. We showed in [8] that $h(P(5,2)) = 11$ and $h(k,m) = 2k+1$ for all integers $k \equiv 5(\text{mod } 6)$ and $m = 2$.

The following result is known [5] and will be applied throughout the paper.

**Theorem 1.** *Let $G$ be a connected graph and $B_1, B_2, \cdots, B_k$ be the blocks of $G$. Then $h(G) = \sum_{i=1}^{k} h(B_i)$.* □

## 2 Graph notations

For a graph $G = (V,E)$ and $X \subseteq E$, by $G - X$ we denote the graph obtained from $G$ by removing all edges of $X$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph $G$ and $X \subseteq V$, $G - X$ is the graph obtained by removing all vertices of $X$ and all edges incident with these vertices from $G$. If $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from $G$ by adding all edges from $X$. Similarly, if $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graph $G$ and $H$ we define their union, $G \cup H$, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Since the operation "$\cup$" is associative, we can extend this definition to a finite union of pairwise disjoint graphs. If $G_1 \cong G_2 \cong \cdots \cong G_k \cong G$ and $G_1, G_2, \cdots, G_k$ are pairwise disjoint graphs, then we will use $kG$ for $G_1 \cup G_2 \cup \cdots \cup G_k$. A *cubic graph* is a graph in which every vertex has degree 3. Let $\mathcal{C}(3^n)$ be the set of all connected cubic graphs of order $n$. It is well known that, for $n \geq 10$, $\mathcal{C}(3^n)$ can be partitioned as $\mathcal{C}_1(3^n) \cup \mathcal{C}_2(3^n) \cup \mathcal{C}_3(3^n)$, where $\mathcal{C}_1(3^n)$ is the set of all connected cubic graphs of order $n$ containing a cut edge, $\mathcal{C}_2(3^n)$ is the set of all 2-connected cubic graphs of order $n$ containing a set of two cut edges, and $\mathcal{C}_3(3^n)$ is the set of all 3-connected cubic graphs of order $n$. It is also well known that the edge-connectivity and the vertex-connectivity of a given cubic graph are equal.

The following graph constructions and notations will be used from now on.

1. Let $G$ be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from $G$ by replacing $v$ by a triangle, matching the vertices of the triangle to the former neighbors of $v$. Thus $G * v$ is also a cubic graph containing a triangle.

2. Let $G$ be a cubic graph. We denote $G^+$ for a graph obtained from $G$ by subdivision an edge of $G$. Thus $K_4^+$ is unique. When the subdivision vertex is specified, say $u$, we use $G(u)$ for a graph with $V(G(u)) = V(G) \cup \{u\}$ and $E(G(u)) = (E(G) - xy) \cup \{xu, uy\}$, where $xy \in E(G)$.

3. Let $G$ and $H$ be vertex disjoint cubic graphs. Let $u, v$ be new vertices. We denote $\langle G(u), H(v) \rangle$ for a connected cubic graph of minimum order containing

$G(u) \cup H(v)$ as its induced subgraph. Note that the graphs $G(u)$ and $H(v)$ are not unique but the graph $\langle G(u), H(v) \rangle$ is uniquely determined by $G(u)$ and $H(v)$.

4. Let $G$, $H$ and $K$ be pairwise vertex disjoint cubic graphs. Let $x, y, z$ be new vertices. A connected cubic graph of minimum order containing $G(x) \cup H(y) \cup K(z)$ as its induced subgraph is denoted by $\langle G(x), H(y), K(z) \rangle$. Note that the graph $\langle G(x), H(y), K(z) \rangle$ is uniquely determined by $G(x), H(y)$ and $K(z)$ and $|\langle G(x), H(y), K(z) \rangle| = |G(x)| + |H(y)| + |K(z)| + 1$. More generally, if $G_1, G_2, \cdots, G_k$ are pairwise disjoint graphs and for all $i = 1, 2. \cdots, k$, vertices of $G_i$ are of degree 2 or 3 with at least one vertex of degree 2, then $\langle G_1, G_2, \cdots, G_k \rangle$ denotes a connected cubic graph of minimum order containing $G_1, G_2, \cdots, G_k$.

5. We use $K_4^-$ for the graph obtained from $K_4$ by removing an edge.

## 3 Cycles in 2-connected cubic graphs

A *factor* of a graph $G$ is a spanning subgraph of $G$. A *$k$-factor* is a spanning $k$-regular subgraph. In particular, a 1-factor of a graph $G$ is a 1-regular spanning subgraph of $G$ and a 2-factor is a 2-regular spanning subgraph of $G$. Let $G$ be a connected cubic graph containing a 1-factor $F_1$ and $F$ be a graph with $V(F) = V(G)$ and $E(F) = E(G) - E(F_1)$. Then $F$ is a 2-factor of $G$.

By a well-known theorem of Petersen [7], every 2-connected cubic graph $G$ has a 2-factor. Thus if $G$ is a 2-connected cubic graph, then the edge set of $G$ can be partitioned into a 1-factor and a 2-factor. The following theorem due to Schönberger [9] and it is considered as an extension of the Petersen theorem.

**Theorem 2.** *Let $G$ be a 2-connected cubic graph and $e \in E(G)$. Then $G$ has a 1-factor containing $e$.* □

As a consequence of Theorem 2, if $G$ is a 2-connected cubic graph and $e, f$ are two incident edges of $G$, then $G$ has a 2-factor containing both $e$ and $f$.

In [10], it was shown that there are two 2-connected cubic graphs of order 6 and five 2-connected cubic graphs of order 8. All of those graphs are Hamiltonian. For an integer $m \geq 3$, let $C_m$ denote an $m$-cycle, a cycle of order $m$.

Let $G$ be a 2-connected cubic graph of order $n \geq 10$ and $F = \bigcup_{i=1}^{k} C_{p_i}$ be a 2-factor of $G$. If the girth of $G$ is at least 5, then for each $i = 1, 2, \cdots, k$, $p_i \geq 5$.

Let $T = \{x, y, z\}$ be a triangle of a cubic graph $G$. Then $T$ is called a *pure triangle* if $x, y, z$ have no other neighbor in common. If $G$ contains a pure triangle $T = \{x, y, z\}$, then we define $G'$ to be the graph obtained from $G$ by identifying the vertices $x, y, z$ to a new vertex $v$, and joining $v$ to the third neighbors of $x, y$ and $z$. Thus $G'$ is a 2-connected simple cubic graph of order $n - 2$.

**Theorem 3.** *Let $G$ be a 2-connected graph of order $n \geq 6$. Then $G$ has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of $G$ such that for all $i = 1, 2, \cdots, k$, $p_i \geq 4$. Moreover, if $n = 4q$, for some integer $q$, then $k < \frac{n}{4}$.*

*Proof.* Let $G$ be a 2-connected graph of order $n \geq 6$. The result of this theorem holds if the girth of $G$ is at least 5. Thus we can assume that the girth of $G$ is at most 4.

Suppose that $G$ contains a pure triangle $T = \{x, y, z\}$. Let $x', y', z'$ be the third neighbors of $x, y, z$, respectively. Then we define $G'$ to be the graph obtained from $G$ by identifying the vertices $x, y, z$ to a new vertex $v$, matching $v$ to the neighbors of $x, y, z$ not in $V(T)$. Thus $G'$ is 2-connected simple cubic graph of order $n-2$. By induction, there is a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of $G'$ such that for all $i = 1, 2, \cdots, m$, $q_i \geq 4$. Let $C'$ be the cycle in $F'$ containing $v$ and suppose without loss of generality that $C'$ contains $x', v, z'$ as its consecutive vertices. By replacing $v$ by $x, y, z$ yields a cycle $C$ of $G$. Thus we obtain a 2-factor $F = (F' - \{C'\}) \cup \{C\}$ of $G$ satisfying the desired property.

Since $K_4^-$ consists of two triangles with a common edge, any triangle of $K_4^-$ can not belong to any 2-factor of $G$. Therefore, if $G$ does not contain a pure triangle, then any 2-factor of $G$ is a union of cycles of length at least 4.

Suppose that $n = 4q$, for some integer $q$. Suppose further that $G$ does not contain a pure triangle and all 2-factors of $G$ is a union of 4-cycles. Let $F = \bigcup_{i=1}^{k} C_{q_i}$ be a 2-factor of $G$ such that $q_1 = q_2 = \cdots = q_k = 4$. Let $C$ be a 4-cycle in $F$, $V(C) = \{x, y, z, w\}$, $E(C) = \{xy, yz, zw, wx\}$ and $x', y', z', w'$ are the third neighbors of $x, y, z, w$, respectively. If $xz \notin E(G)$ and $yw \notin E(G)$, then $xx', yy', zz', ww'$ are independent and hence $x', y', z', w'$ are pairwise distinct. By Theorem 2, let $F'$ be a 2-factor of $G$ containing $x'x, xy$. Then, by assumption that all cycles in $F'$ are 4-cycles, either $\{x', x, y, y'\}$ or $\{x', x, y, z\}$ induces a 4-cycle in $G$. Since $x' \neq z'$, it follows that $\{x', x, y, y'\}$ induces a 4-cycle in $G$. Similarly, each of $\{z', z, w, w'\}, \{x', x, w, w'\}, \{y', y, z, z'\}$ also induces a 4-cycle in $G$. Thus $G$ is a Hamiltonian graph of order 8. Suppose that all 4-cycles in $F$ induce $K_4^-$ in $G$. Then $G$ is Hamiltonian. Therefore, $G$ has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of $G$ with $k < \frac{n}{4}$, as required. □

By using the result in Theorem 2, we obtain the following stronger result :

**Theorem 4.** *Let $G$ be a 2-connected graph of order $n \geq 6$ and $e \in E(G)$. Then $G$ has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ such that for all $i = 1, 2, \cdots, k$, $p_i \geq 4$ and $e \notin E(F)$. Moreover, if $n = 4q$, for some integer $q$, then $k < \frac{n}{4}$.*

*Proof.* By Theorem 2 and Theorem 3, the result of this theorem holds $G$ does not contain a pure triangle as its subgraph. The result also holds if $n = 6, 8$. Suppose that $G$ contains a pure triangle $T = \{x, y, z\}$, $n \geq 10$, and let $G'$ be the graph as described earlier. Thus $G'$ is of order $n-2$. By induction, for every edge $e \in G'$, $G'$ has a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ such that for all $i = 1, 2, \cdots, m$, $q_i \geq 4$ and $e \notin E(F)$. Let $C'$ be the cycle in $F'$ containing $v$. If $e \notin \{vx', vy', vz'\}$, then, without loss of generality, we may assume that $y', v, z'$ are as consecutive vertices in $C'$. Let $C$ be a cycle obtained from $C'$ by replacing $v$ by $y, x, z$. Thus $F = (F' - \{C'\}) \cup \{C\}$ is a 2-factor of $G$ with the desired property. If $e \in \{vx', vy', vz'\}$, say $e = vx'$, then $y', v, z'$ are as consecutive vertices on $C'$. Let $C$ be a cycle obtained from $C'$ by replacing $v$ by $y, x, z$. Thus $F = (F' - \{C'\}) \cup \{C\}$

is a 2-factor of $G$ with the desired property. If $e \in \{xy, yz, xz\}$, say $e = yz$, then there exists a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of $G'$ for all $i = 1, 2, \cdots, m$, $q_i \geq 4$, $vx' \notin E(F')$. Thus there exists a cycle $C'$ in $F'$ such that $C'$ contains $y', v, z'$ as its consecutive vertices. Thus we can extend $F'$ to a 2-factor $F$ of $G$ as described above. Thus the proof is complete. □

## 4 The range of Hamiltonian numbers

For an even integer $n \geq 4$, we have already denoted that $h(3^n) = \{h(G) : G \in \mathcal{C}(3^n)\}$. We put $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{C}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{C}(3^n)\}$. It is well-known that for any even integer $n \geq 4$, there exists a Hamiltonian cubic graph of order $n$. Thus $\min(h, 3^n) = n$. It is also well-known that $\max(h, 3^n) = \min(h, 3^n) = n$ if and only if $n = 4, 6, 8$. For $n = 10, 12$, we have $\max(h, 3^n) = n+2$. Observe that for $n = 10, 12$, a connected cubic graph $G$ having $h(G) = \max(h, 3^n)$ being a graph with a cut edge. The problem of determining $\max(h, 3^n)$ is more challenging. The following facts are useful.

1. If a connected graph $G$ contains an edge $e$ such that $G - e$ is connected, then $h(G) \leq h(G - e)$.

2. If $G$ is a 2-connected cubic graph of order $n$ and $F = \bigcup_{i=1}^{k} C_{p_i}$ is a 2-factor of $G$, then there exists a set $X = \{e_1, e_2, \ldots, e_{k-1}\} \subseteq E(G) - E(F)$ such that $F + X$ is connected. Thus $h(G) \leq h(F + X) = n + 2(k - 1)$. In particular, if $G$ is a 2-connected cubic graph of order $n$, then, by Theorem 3, $h(G) \leq n + 2(k - 1)$, where $k \leq \lfloor \frac{n-2}{4} \rfloor$.

3. For an integer $n \geq 10$, let $H = \langle (q-2)K_4^-, 2K_4^+ \rangle$ if $n = 4q+2$ and $H = \langle (q-2)K_4^-, K_4^+, K \rangle$ if $n = 4(q+1)$, where $K$ is a graph obtained from a cubic graph of order 6 and a subdivision of an edge. Then, by Theorem 1, $h(H) = n + 2(q - 1)$.

As a consequence of above observations, we obtain the following lemma.

**Lemma 1.** *Let $G$ be a 2-connected graph of order $n \geq 14$. Then there exists a graph $H \in \mathcal{C}_1(3^n)$ such that $h(G) \leq h(H)$.*

*Proof.* By using above observation and Theorem 3, we have $k \leq q$. Thus $h(G) \leq n + 2(k - 1) \leq n + 2(q - 1) = h(H)$. Note that $H \in \mathcal{C}_1(3^n)$. □

By Lemma 1 and Theorem 4, we obtain the following result.

**Corollary 1.** *Let $G$ be a 2-connected graph of order $n \geq 14$ and $e = xy \in E(G)$. Then there exist a closed spanning walk $W$ of $G$ such that $x, y$ appear as consecutive vertices on $W$ exactly once and a graph $H \in \mathcal{C}_1(3^n)$ such that $|W| \leq h(H)$.* □

Thus in order to find the value of $\max(h, 3^n)$, by Lemma 1, it is enough to search for a connected cubic graph $G \in \mathcal{C}_1(3^n)$ with $h(G) = \max(h, 3^n)$. Let

$G \in \mathcal{C}_1(3^n)$ with $h(G) = \max(h, 3^n)$. Let $B_1, B_2, \ldots, B_k$ be the blocks of $G$ with $|B_i| \geq 3$, for all $i = 1, 2, \ldots, k$. Thus there are $k-1$ blocks of $G$ of order 2. A block of $G$ of order 2 is called *trivial*, otherwise, it is called *nontrivial*. A nontrivial block $B$ is called a *leaf block* if $B$ has exactly one vertex of degree 2. It is clear that $G$ has at least two leaf blocks. Thus by a nontrivial block $B$, we mean in this paper, a 2-connected graph of order at least 3, $2 \leq d(x) \leq 3$, for all $x \in V(B)$, and $B$ contains at least one vertex of degree 2. Similarly, by a leaf block, we mean a nontrivial block with exactly one vertex of degree 2.

**Lemma 2.** *Let $B$ be a leaf block of order $b$ and $5 \leq b \leq 9$. Then $B$ is Hamiltonian.*

*Proof.* Since $B$ is a leaf block of order $b$, $b$ is odd and $b \geq 5$. Let $v$ be the vertex of degree 2 of $B$. Then $B$ can be pictured as in Figure 1. Let $B_1 = B_1' + u_r v_r$. Then $B_1$ is a 2-connected cubic graph of order at most 8 and hence $B_1$ Hamiltonian. By Theorem 2, $B_1$ has a Hamiltonian cycle $C'$ containing $u_r v_r$. The cycle $C'$ can be extended to a Hamiltonian cycle $C$ of $B$. Thus $B$ is Hamiltonian. □

Let $B$ be a nontrivial block. If $W : x_1, x_2, \ldots, x_{t+1} = x_1$ is a closed spanning walk of $B$ and $e = xy \in E(B)$, then we say that $x, y$ appear as consecutive vertices in $W$ if there exists $i$, $1 \leq i \leq t$, such that $\{x, y\} = \{x_i, x_{i+1}\}$. If $x, y, x$ appear in $W$ as consecutive vertices, then we say that $x, y$ appear twice in $W$.

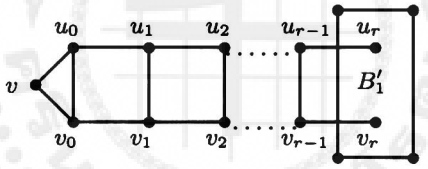


Figure 1

**Lemma 3.** *Let $B$ be a leaf block of order $b \geq 11$. Then there exist a closed spanning walk $W : x_1, x_2, \ldots, x_{t+1} = x_1$ of $B$ and a connected graph $H$ with a cut edge such that $B$ and $H$ are of the same degree sequence, and $|W| \leq h(H)$*

*Proof.* Let $B$ be a leaf block of $G$ of order $b \geq 11$. Thus $B$ can be pictured as shown in Figure 1. If $B$ is Hamiltonian of order $b \geq 11$, then we can choose a Hamiltonian walk $W$ of $B$ and a graph $H$ obtained from a connected cubic graph of order $b-1$ with a cut edge $e$ and a subdivision to the edge $e$. Thus $|W| \leq h(H)$ and $H$ and $B$ are of the same degree sequence. Suppose that $B$ is not Hamiltonian. Let $B_1 = B_1' + u_r v_r$. Thus $B_1$ is 2-connected cubic graph of order $b_1 \leq b-1$. If $b_1 \leq 8$, then $B$ is Hamiltonian and the result follows. Suppose that $b_1 \geq 10$. By Corollary 1, there exists a closed spanning walk $W_1$ of $B_1$ such that $u_r, v_r$ appear as consecutive vertices in $W_1$ by exactly once and a graph $H_1 \in \mathcal{C}_1(3^n)$ such that $|W_1| \leq h(H_1)$, where $B_1$ and $H_1$ are of the same degree sequence. Thus $W_1$ and $H_1$ can be easily extended to $W$ and $H$ with desired result. Thus the proof is complete. □

For a 2-connected cubic graph $G$, we have constructed a closed spanning walk $W$ of $G$ traveling along the cycles in a given 2-factor of $G$ and $k-1$ edges connecting between $k$ cycles in the 2-factor. It turns out that for every edge on the $k$ cycles appears exactly once in $W$ while every edge that connects to the cycles appears exactly twice in $W$. Lemma 2 showed similar result for a leaf block.

Let $B$ be a block of order $b \geq 3$. If $b = 3, 4$, then $B$ is Hamiltonian. If $b = 5$ and $B$ is not Hamiltonian, then $B \cong K_{2,3}$ and $h(B) = 6$. Note that the graph $K_{2,3}$ has a property that for every edge $e = xy$, there is a Hamiltonian walk containing $x, y$ as its consecutive vertices exactly once.

If $b \in \{6, 8\}$, $B$ contains exactly two vertices of degree 2, and the two vertices $x, y$ are not adjacent, then the graph $B' = B+xy$ is a cubic graph of order $b$. Thus $B'$ is Hamiltonian. By Theorem 4, $B'$ has a Hamiltonian cycle not containing $xy$. Thus $B$ is Hamiltonian. Suppose that the two vertices are adjacent. Let $B'$ be the graph obtained from $B$ by removing the two vertices of degree 2. Thus $B' \cong K_4^-$ if $b = 6$, and $B' \cong K$ if $b = 8$, where $K$ is obtained from a cubic graph of order 6 by removing one edge. Thus $B'$ is Hamiltonian and consequently $B$ is Hamiltonian.

If $b \in \{7, 9\}$ and $B$ contains exactly three vertices of degree 2, then the three vertices are not pairwise adjacent. Thus there are two vertices $x, y$ of degree 2 in $B$ that are not adjacent and $B' = B + xy$ is a leaf block of order $b$. By Lemma 2, $B'$ is Hamiltonian.

**Theorem 5.** *Let $B$ be a nontrivial block of order $b \geq 6$. Then $B$ is Hamiltonian or there exist a closed spanning walk $W : x_1, x_2, \ldots, x_{t+1} = x_1$ of $B$ and a connected graph $H$ with a cut edge satisfying the following conditions.*

1. *For every edge $xy$ of $B$, $x, y$ appear as its consecutive vertices in $W$ by at most twice,*

2. *$H$ and $B$ are of the same degree sequence, and*

3. $t \leq h(H)$.

*Proof.* Let $B$ be a nontrivial block of order $b \geq 6$. Suppose that $B$ is a non-Hamiltonian graph. If $b = 6$, then $B$ contains exactly four vertices of degree 2 and $h(B) = 7$. Let $H$ be a graph of order 6 obtained from two disjoint triangles and one edge connecting from one vertex of a triangle to one vertex of another triangle. Thus $H$ contains a cut edge and $h(H) = 8$. Thus the result follows if $b = 6$. Suppose $b \geq 7$, $B$ can be pictured as in Figure 1. Let $B'$ be the subgraph of $B$ induced by $\{v, u_0, v_0, u_1, v_1, \ldots, u_{r-1}, v_{r-1}\}$. Thus $V(B) = V(B') \cup V(B'_1)$ and $E(B) = E(B') \cup E(B'_1) \cup \{u_{r-1}u_r, v_{r-1}v_r\}$. Let $B_1 = B'_1 + u_r v_r$. Then $B_1$ is a block of order at most $b-1$. If $B_1$ is of order 5, then either $B_1$ is Hamiltonian or $B_1 \cong K_{2,3}$. Since every block of order 5 has a property that every edge is contained in its closed spanning walk of length 6, it follows that $h(B) \leq b+1$. It is easy to construct a graph $H$ with the desired property. Suppose that $B_1$ is order at least 6. By induction, there exist a closed spanning walk $W_1$ of $B_1$

in which for every edge $xy$ of $B_1$, $x, y$ appear as its consecutive vertices at most twice and a connected graph $H_1$ with an edge cut such that $|W_1| \leq h(H_1)$, where $B_1$ and $H_1$ are of the same degree sequence. We now construct a closed spanning walk $W$ of $B$ according to the following properties of $W_1$.

1. If $u_r, v_r$ appear exactly once as consecutive vertices on $W_1$, then we obtain a closed spanning walk $W$ of $B$ from $W_1$ by replacing $u_r, v_r$ by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r$. Thus $|W| = |W_1| - 1 + 2r + 2 = |W_1| + 2r + 1$.

2. If $u_r, v_r$ appear exactly twice as consecutive vertices on $W_1$, then we obtain a closed spanning walk $W$ of $B$ from $W_1$ by replacing $u_r, v_r$ by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r$ for the first pair and replace $u_r, v_r$ by $u_r, u_{r-1}, v_{r-1}, v_r$ for the second pair. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

3. If $u_r, v_r, u_r$ appear as consecutive vertices on $W_1$, then we obtain a closed spanning walk $W$ of $B$ from $W_1$ by replacing $u_r, v_r, u_r$ by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

4. If $u_r, v_r$ do not appear as consecutive vertices on $W_1$, then we obtain a closed spanning walk $W$ of $B$ from $W_1$ by replacing $u_r$ by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| - 1 + 2r + 3 = |W_1| + 2r + 2$.

Thus $W$ is a closed spanning walk of $B$ and $|W| \leq |W_1| + 2r + 3$ and for every edge $e = xy \in E(B)$, $x, y$ appear as consecutive vertices in $W$ at most twice. On the other hand, let $B'$ be a graph with $V(B') = V(B) - V(B_1)$ and $E(B') = E(B) - E(B_1)$. Since $H_1$ contains a cut edge $e = pq$, it follows that a graph $H$, where $V(H) = V(B') \cup V(H_1)$ and $E(H) = E(B') \cup E(H_1 - e) \cup \{u_{r-1}p, v_{r-1}q\}$, is a graph containing a cut edge. Thus $h(H) = h(B') + h(H_1) - 2 + 4 = h(H_1) + 2r + 1 + 2 = h(H_1) + 2r + 3 \geq |W|$. Thus the proof is complete. □

Let $G \in \mathcal{C}_1(3^n)$ such that $n \geq 14$ and $h(G) = \max(h, 3^n)$. Then, by Lemmas 2 and 3, there exists a leaf block of $G$ that is isomorphic to $K_4^+$, and by Theorem 5, all nontrivial blocks of $G$ are $K_{2,3}$ or Hamiltonian. The following lemma is easily obtained.

**Lemma 4.** $\max(h, 3^{14}) = 18$, $\max(h, 3^{16}) = 21$ *and* $\max(h, 3^{18}) = 24$.

*Proof.* We have constructed graphs $G_{14}, G_{16}$ and $G_{18}$ with $h(G_{14}) = 18$, $h(G_{16}) = 21$ and $h(G_{18}) = 24$. Let $G \in \mathcal{C}_1(3^{14})$ such that $h(G) = \max(h, 3^{14})$ and $G$ contains $K_4^+$ as a leaf block. Thus $G = \langle K_4^+, G_1 \rangle$, where $G_1$ is a connected graph of order 9 containing 8 vertices of degree 3 and a vertex of degree 2. If $G_1$ is 2-connected, then $G_1$ is Hamiltonian and hence $h(G) = 16 < 18 = h(G_{14})$. Thus $G_1$ contains a cut edge. Therefore $G \cong G_{14}$. Thus $\max(h, 3^{14}) = 18$. Let $G \in \mathcal{C}_1(3^{16})$ and $h(G) = \max(h, 3^{16})$. If $G$ has only two trivial blocks, then $h(G) \leq 16 + 4 = 20 < 21 = h(G_{16})$. Thus $G$ must have at least three trivial blocks. Since the order of $G$ is 16 and $G$ has at least two block leave of order 5, $G \cong G_{16}$. Let $G \in \mathcal{C}_1(3^n)$ and $h(G) = \max(h, 3^n)$. If $G$ contains a nontrivial block of order 3, then the block is a pure triangle and we can form a new graph $G'$ of order $n - 2$ by identifying the triangle to a new vertex, matching this

new vertex to the former neighbors of the triangle and $h(G) = h(G') + 3$. Let $G \in \mathcal{C}_1(3^{18})$ and $h(G) = \max(h, 3^{18})$. Thus if $G$ has a block of order 3, then $G \cong G_{18}$ and $h(G) = 24$. If $G$ does not have a nontrivial block of order 3, then $G \cong \langle 2K_4^+, 2K_4^- \rangle$. Thus $\max(h, 3^{18}) = 24$. □

Let $G \in \mathcal{C}_1(3^n)$ and $e = xy$ be a cut edge of $G$. A graph $G(K_4(v))$ is a graph obtained from $G$ and $K_4(v)$ by deleting $xy$ and adding edges $xz, zy, zv$, where $z$ is a new vertex. By Theorem 1, it follows that $h(G(K_4(v)) = h(G) + 9$. Note that the graph $G(K_4(v)) \in \mathcal{C}_1(3^{n+6})$. Let $G_{14} = \langle 2K_4^+, K_4^- \rangle, G_{16} = \langle 3K_4^+ \rangle$ and $G_{18} = \langle 3K_4^+, K_3 \rangle$. Thus $G_{14}, G_{16}$ and $G_{18}$ are connected cubic graphs of order 14, 16, and 18, respectively, and $h(G_{14}) = 18, h(G_{16}) = 21$ and $h(G_{18}) = 24$. Note that each of the graphs $G_{14}, G_{16}$ and $G_{18}$ contains a cut edge. Thus for an integer $n_i = 14 + 2i$, $i \geq 3$, a graph $G_{n_i} = G_{n_{i-3}}(K_4(v)) \in \mathcal{C}_1(3^{n_i})$ and $h(G_{n_i}) = 18 + 3i$. Thus $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer $i$. We show in the next theorem that the graph $G_{n_i}$ satisfies $h(G_{n_i}) = \max(h, 3^{n_i})$.

**Theorem 6.** *Let $i$ be a non-negative integer and $n_i = 14 + 2i$. Then* $\max(h, 3^{14+2i}) = 18 + 3i$.

*Proof.* We have already mentioned that $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer $i$. We have also obtained $\max(h, 3^{14+2i}) = 18 + 3i$, for $i = 0, 1, 2$. Suppose $i \geq 3$ and $G \in \mathcal{C}_1(3^{14+2i})$ with $h(G) = \max(h, 3^{14+2i})$. Let $B$ be a non-trivial block of $G$ of order $b$ which is not a leaf block and $B$ is of minimum order. The following five cases are considered.

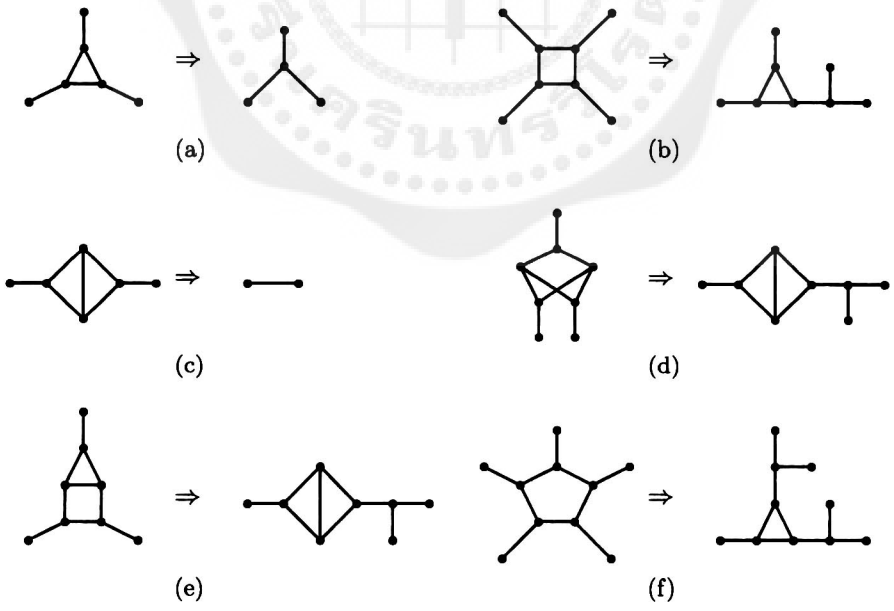


Figure 2

**Case 1.** If $b = 3$, then $B$ is a pure triangle. Let $G'$ be a graph obtained from $G$ by identifying $B$ to a new vertex $v$, matching $v$ to the former neighbors of $B$ (see Figure 2(a)). Clearly $G' \in \mathcal{C}_1(3^{14+2(i-1)})$ and by induction, $h(G) = h(G') + 3 \leq 18 + 3(i-1) + 3 = 18 + 3i$.

**Case 2.** If $b = 4$ and the induced subgraph of $B$ in $G$ is a 4-cycle, then we can replace $B$ by a graph in Figure 2(b) and the result follows by Case 1.

**Case 3.** If $b = 4$ and the induced subgraph of $B$ in $G$ is a $K_4^-$, then let $G'$ be a graph obtained from $G$ by deleting the $K_4^-$ and connecting the two neighbors of $K_4^-$ (see Figure 2(c)). Then $G' \in \mathcal{C}_1(3^{14+2(i-2)})$ and by induction, $h(G) = h(G') + 6 \leq 18 + 3(i-2) + 6 = 18 + 3i$.

**Case 4.** If $b = 5$, then $B$ is one of the graphs shown in Figure 2(d, e, f) and the result follows by Case 1, Case 2 or Case 3.

**Case 5.** If $b \geq 6$, then, by Theorem 5, $B$ is Hamiltonian. By an arrangement of blocks of $G$, we may assume that $B$ is adjacent to a leaf block $K_4^+$. Let $v$ be a vertex of degree 2 of $B$ such that $v$ is adjacent to the vertex of degree 2 of $K_4^+$. Since $b \geq 6$, we may assume that the neighbors $x, y$ of $v$ in $B$ are not adjacent. Let $G'$ be a obtained from $G$ by removing $V(K_4^+) \cup \{v\}$ and adding an edge $xy$. Thus $G' \in \mathcal{C}_1(3^{14+2(i-3)})$. Since $h(G) = h(G') + 1 + 7 = h(G') + 8$, by induction, $h(G) = h(G') + 8 \leq 18 + 3(i-3) + 8 < 18 + 3i$. Thus the proof is complete. □

Recall that the range of Hamiltonian numbers of connected cubic graphs of order $n$ is $h(3^n) = \{h(G) : G \in \mathcal{C}(3^n)\}$. It is easy to obtain $h(3^n)$ for small values of $n$. In other words, $h(3^n) = \{n\}$ if and only if $n = 4, 6, 8$ and $h(3^n) = \{n, n+1, n+2\}$ if and only if $n = 10, 12$. By using Theorems 1, 6 and Lemma 4, it is not difficult to show that $h(3^{14}) = \{14, 15, 16, 18\}$, $h(3^{16}) = \{16, 17, 18, 19, 20, 21\}$, $h(3^{18}) = \{18, 19, 20, 21, 22, 23, 24\}$ and $h(3^{20}) = \{20, 21, 22, 23, 24, 25, 26, 27\}$. Observe that for even integers $n$, $4 \leq n \leq 20$, $h(3^n)$ completely covers all integers from $\min(h, 3^n)$ to $\max(h, 3^n)$, except $n = 14$.

Let $j$ be a non-negative integer. A connected graph $G$ of order $n$ is called a *$j$-Hamiltonian* if $h(G) = n + j$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an *almost Hamiltonian graph*. We proved in [8] that for an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order $n$. Furthermore, we proved that an almost Hamiltonian cubic graph is 2-connected. Thus for an even integer $n \geq 10$, $\{n, n+1\} \subseteq h(3^n)$.

Let $n$ be an even integer and $n \geq 10$ and $H$ be a Hamiltonian cubic graph of order $n - 6$. Then $G = \langle K_4(u), H(v) \rangle$ exists and $h(G) = n + 2$. Thus for an even integer $n \geq 10$, $\{n, n+1, n+2\} \subseteq h(3^n)$.

Let $n$ be an even integer with $n \geq 16$. Consider the following elementary facts.

1. A graph $G = \langle P(u), K(v) \rangle$ satisfies $h(G) = n + 3$, where $P$ is the Petersen graph of order 10 and $K(v)$ is a Hamiltonian graph of order $n-11$ obtained from a Hamiltonian cubic graph $K$ of order $n-12$. Thus $\{n, n+1, n+2, n+3\} \subseteq h(3^n)$.

2. A graph $G = \langle 2K_4^+, K - e \rangle$ satisfies $h(G) = n + 4$, where $K$ is a Hamiltonian

cubic graph of order $n-10$ and $e \in E(K)$. Thus $\{n, n+1, n+2, n+3, n+4\} \subseteq h(3^n)$.

3. A graph $G = \langle 2K_4^+, K(v) \rangle$ satisfies $h(G) = n+5$, where $K(v)$ is a Hamiltonian graph of order $n-11$ obtained from a Hamiltonian cubic graph $K$ of order $n-12$. Thus $\{n, n+1, n+2, n+3, n+4, n+5\} \subseteq h(3^n)$.

4. Since $\max(h, 3^{14+2i}) = 18+3i$, the maximum value of $j$ in which there exists a $j$-Hamiltonian graph in $\mathcal{C}(3^{14+2i})$ is $4+i$.

5. If $j = 4+i$, then there exists $G \in \mathcal{C}_1(3^{14+2i})$ such that $G$ is a $j$-Hamiltonian graph in which all nontrivial blocks are Hamiltonian.

6. If $6 \leq j \leq 4+i$ and $G \in \mathcal{C}_1(3^{14+2i})$ such that $G$ is a $j$-Hamiltonian graph containing a Hamiltonian leaf block $B$ and $v \in V(B)$, then $G * v \in \mathcal{C}_1(3^{14+2(i+1)})$ and $G * v$ is a $j$-Hamiltonian graph with a Hamiltonian leaf block.

We have the following lemma.

**Lemma 5.** *If $n \geq 10$ and $\mathcal{C}(3^n)$ contains a $j$-Hamiltonian graph, then $\mathcal{C}(3^{n+2})$ contains a $j$-Hamiltonian graph.* □

By Lemma 5, if $h(3^n)+2 = \{h(G)+2 : G \in \mathcal{C}(3^n)\}$, then $h(3^n)+2 \subseteq h(3^{n+2})$. Thus by Theorem 6, we have the following theorem.

**Theorem 7.** *For an even integer $n \geq 4$ and $n \neq 14$. There exists an integer $b$ such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer $b$ is given in the following cases.*

1. *$b = n$ if and only if $n = 4, 6, 8$.*

2. *$b = n+2$ if and only if $n = 10, 12$.*

3. *If $n = 14+2i$ and $i \geq 1$, then $b = 18+3i$.* □

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