INTERPOLATION AND EXTREMAL THEOREMS: THE FOREST NUMBER OF GRAPHS

A DISSERTATION

BY

AVAPA CHANTASARTRASSMEE

Presented in Partial Fulfillment of the Requirements for the

Doctor of Philosophy in Mathematics

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การปรากฏค่าทุกค่าและค่าสุดขีด:

จำนวนฟอเรสต์ของกราฟ

บทคัดย่อ

ของ

อวภาส์ ฉันทศาสตร์รัศมี

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา

ตามหลักสูตรปริญญาปรัชญาดุษฎีบัณฑิต สาขาวิชาคณิตศาสตร์

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ให้ G เป็นกราฟ และ $F \subseteq V(G)$ เรียก F ว่า อินดิวส์ฟอเรสต์ ของ G ก็ต่อเมื่อ $<F>$ เป็นกราฟไม่มีวัฏจักร จำนวนฟอเรสต์ของ G (เขียนแทนด้วย $f(G)$) ซึ่งกำหนดโดย

$$f(G) := \max\{|F| : F \text{ เป็นอินดิวส์ฟอเรสต์ของ} G\}$$

เป็นที่รู้กันโดยทั่วไปว่าการหาจำนวนฟอเรสต์ของกราฟในคลาสของกราฟต่างๆ อาทิเช่น คลาสของกราฟเชิงระนาบ, คลาสของกราฟสองส่วน นั้นเป็น NP-hard

ให้ $\mathcal{J}$ เป็นคลาสของกราฟ เราศึกษาปัญหาของการหาเรนจ์ของจำนวนฟอเรสต์

$$f(\mathcal{J}) = \{f(G) : G \in \mathcal{J}\}$$

เราสามารถหาเรนจ์ของจำนวนฟอเรสต์ในบางคลาสของกราฟ ซึ่งเราเรียกว่า ปัญหาค่าสุดขีดสำหรับกราฟพารามิเตอร์ f นั่นเอง

จำนวนจุด (เส้น)อาร์บอริซิตี้ ของกราฟ G เขียนแทนด้วย $a(G)$ $(a_1(G))$ คือ จำนวนของผลแบ่งกั้นเซตของจุด (เส้น) ของ G ที่น้อยที่สุดที่ทำให้แต่ละกราฟย่อยที่อินดิวส์โดยผลแบ่งกั้นเป็นกราฟที่ไม่มีวัฏจักร ให้ $\mathbf{d}$ คือลำดับดีกรีเชิงกราฟ เราสามารถพิสูจน์ได้ว่า ถ้า $\pi \in \{a, a_1\}$ แล้วจะมีจำนวนเต็ม $x(\pi)$ และ $y(\pi)$ ซึ่งมี G เป็น realization ของ $\mathbf{d}$ ที่ทำให้ $\pi(G) = z$ ก็ต่อเมื่อ z เป็นจำนวนเต็มบวกที่สอดคล้อง $x(\pi) \leq z \leq y(\pi)$ เราสามารถหาค่าสุดขีดสำหรับกราฟพารามิเตอร์ a และ a_1

INTERPOLATION AND EXTREMAL THEOREMS:
THE FOREST NUMBER OF GRAPHS

AN ABSTRACT
BY
AVAPA CHANTASARTRASSMEE

Presented in Partial Fulfillment of the Requirements for the
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Avapa Chantasartrassmee. (2009). *Interpolation and Extremal Theorems: The Forest Number of Graphs.* Dissertation, Ph.D. (Mathematics). Bangkok: Graduate School, Srinakharinwirot University. Advisor Committee: Prof. Dr.Narong Punnim, Prof. Dr.Saad El-Zanati, Asst. Prof. Dr.Varaporn Saenpholphat.

Let G be a graph and $F \subseteq V(G)$. Then F is called an *induced forest* of G if the induced subgraph $\langle F \rangle$ of G contains no cycle. The *forest number* of G, denoted by $f(G)$, is defined by

$$f(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

The computation of forest numbers of the following families of graphs is shown to be NP-hard: planar graphs, bipartite graphs, perfect graphs.

Let $\mathcal{J}$ be a class of graphs. We consider the problem of determining the range of forest numbers of $\mathcal{J}$, denoted by $f(\mathcal{J})$, where

$$f(\mathcal{J}) = \{f(G) : G \in \mathcal{J}\}.$$

We are able to obtain some significant results on the range of forest numbers for several classes of graphs and this problem is, in fact, the extremal problem for the graph parameter f.

Let G be a graph. The *vertex (edge) arboricity* of G denoted by $a(G)$ ($a_1(G)$) is the minimum number of subsets into which the vertex (edge) set of G can be partitioned so that each subset induces an acyclic subgraph. Let $\mathbf{d}$ be a graphical sequence. We prove that if $\pi \in \{a, a_1\}$, then there exist integers $x(\pi)$ and $y(\pi)$ such that $\mathbf{d}$ has a realization G with $\pi(G) = z$ if and only if z is an integer satisfying $x(\pi) \leq z \leq y(\pi)$. We consider the corresponding extremal problem on the arboricities on the class of regular graphs.

The dissertation titled

"Interpolation and Extremal Theorems:

The Forest Number of Graphs"

by

Avapa Chantasartrassmee

has been approved by the Graduate School as partial fulfillment of the requirements for the Doctor of Philosophy in Mathematics of Srinakharinwirot University.

.... Dean of Graduate School

(Assoc. Prof. Dr.Somchai Santiwatanakul)

December , 2009

Dissertation Committee	Oral Defense Committee
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Avapa Chantasartrassmee

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

LIST OF FIGURES (Continue)

Figure Page

CHAPTER 1

INTRODUCTION

1.1 Prologue

The minimum number of edges whose removal eliminates all cycles in a given graph is known as the *cycle rank* of the graph, and this parameter has a simple expression, namely, $b(G) = |E(G)| - |V(G)| + \omega(G)$, where $\omega(G)$ is the number of components of G. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back to at least to the work of Kirchhoff [24] on spanning trees. This problem does not have a simple solution. This latter question is difficult even for some simply defined graphs. Thus the *decycling number* of a graph G, denoted by $\phi(G)$, is the minimum cardinality of a subset S of $V(G)$ whose removal eliminates all cycles in G. Clearly, $\phi(G) = 0$ if and only if G is a forest and $\phi(G) = 1$ if and only if G contains at least one cycle and there exists a vertex v of G that lies on all cycles of G. It is not easy to characterize the graphs G for which $\phi(G) = 2$.

There are several classes of graphs for which the decycling number is known. For example, $\phi(K_n) = n - 2$, where K_n is the complete graph of order $n \geq 3$ and $\phi(K_{r,s}) = \min\{r, s\}$, where $K_{r,s}$ is the complete bipartite graph with bipartitioning sets of size r and s. The decycling number is the same as the feedback number which was first used by Speckenmeyer [38]. Let G be a graph of order n and S be a minimum decycling set of G. Then $G - S$ contains no cycles. It is reasonable to define a counterpart graph parameter to ϕ as follows.

Let G be a graph and $F \subseteq V(G)$. Then F is called an *induced forest* of G if the induced subgraph $\langle F \rangle$ of G contains no cycle. The *forest number* of G, denoted by $f(G)$, is defined by

$$f(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

Clearly, for a graph G of order n, we have $\phi(G) + f(G) = n$. The graph parameter $f(G)$ was first defined by Staton [37] and Erdős, Sak and Sós [13].

It was shown by Karp [23] that determining the forest number of an arbitrary graph is NP-complete. In fact, the computation of forest numbers of the following families of graphs is shown to be NP-hard: planar graphs, bipartite graphs, and perfect graphs. On the other hand, the problem is known to be polynomial for various other families, including cubic graphs, permutation graphs, and interval graphs. These results suggest further investigations as to good bounds on the parameter and exact results when possible.

In 1984, Staton [37] studied some properties of induced forests in cubic graphs. More precisely, he obtained some upper bounds for the number of vertices in an induced forest in a cubic graph with a given girth. In particular, he proved that if the girth increases without bound, then the ratio of the number of vertices in a largest forest to the number of vertices in the whole graph approaches $\frac{3}{4}$. In 1988, Speckenmeyer [38] obtained a formula for the maximum size of an induced forest in a cubic graph in terms of the order of the graph and the cardinality of a maximum non-separating independent set.

In 2001, Alon, Mubayi and Thomas [1] proved that if G is a triangle-free graph of order n, then $f(G) \geq n - e/4$. In the same paper they obtained a formula for $f(G)$ in terms of the independence number.

Punnim [28, 29, 30, 31, 32, 33, 34, 35, 36] found several results on $f(G)$ for a variety of classes of graphs. In particular, by using the *probabilistic method* [28], he obtained a lower bound for the forest number of the the class of graphs with a fixed degree sequence as follows. Let G be a graph with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots d_n \geq 1$. Then

$$f(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

Moreover, this bound is sharp. The problem of finding a good upper bound for the class of graphs with a fixed degree sequence is more difficult. Punnim obtained a good upper bound for the class of regular graphs [31] and for the class of connected regular graphs [34]. This suggests the problem of finding the *range of* *forest numbers* in $\mathcal{J}$, denoted by $f(\mathcal{J})$, where

$$f(\mathcal{J}) = \{f(G) : G \in \mathcal{J}\}.$$

The purpose of this dissertation is to obtain some significant results on the range of forest numbers for several classes of graphs and this problem is, in fact, the extremal problem for the graph parameter f.

1.2 Notation and Terminology

In this dissertation, we consider only finite simple graphs. For the most part, our notation and terminology follows that of Chartrand and Lesniak [10].

A *graph* G is a finite nonempty set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges.* The *vertex set* of G is denoted by $V(G)$, and the *edge set* is denoted by $E(G)$.

If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent* vertices, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges.* It is convenient to henceforth denote an edge by uv or vu rather than by $\{u, v\}$.

The cardinality of the vertex set of a graph G is called the *order* of G and is commonly denoted by $\nu(G)$; while the cardinality of its edge set is the *size* of G and is often denoted by $\varepsilon(G)$. We use $|S|$ to denote the cardinality of a set S.

It is customary to define or describe a graph G by means of a diagram in which each vertex of G is represented by a point (which we draw as a small circle) and each edge $e = uv$ of G is represented by a line segment or curve joining the points corresponding to u and v. We then refer to this diagram as the graph G itself.

The *degree of a vertex* v in a graph G is the number of edges of G incident with v, which is denoted by $\deg_G v$ or simply by $\deg v$ if G is clear from the context. A vertex of degree 0 in G is also called an *isolated* vertex, while a vertex of degree 1 is also referred to as an *end-vertex* of G. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* is defined similarly and is denoted by $\Delta(G)$.

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called a *degree sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg_G v_i = d_i$ for all i. Given a graph G, a degree sequence of G can be easily determined. On the other hand, if a sequence $\mathbf{d} : d_1, d_2, \ldots, d_n$ of nonnegative integers is a degree sequence of some graph G, then $\mathbf{d}$ is called a *graphical sequence*. In this case, G is called a *realization* of $\mathbf{d}$.

Two graphs often have the same structure, differing only in the way their vertices and edges are labeled. To make this idea more precise, the concept of isomorphism is introduced as follows: a graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency and nonadjacency; that is, $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. It is easy to see that "is isomorphic to" is an equivalence relation on graphs; hence, this relation partitions the collection of all graphs into equivalence classes, two graphs being *nonisomorphic* if they belong to different equivalence classes.

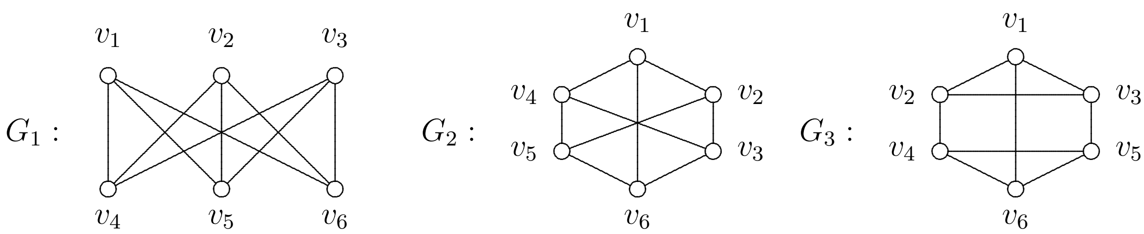


Figure 1: Isomorphic and nonisomorphic graphs.

If G_1 is isomorphic to G_2, then we say G_1 and G_2 are *isomorphic* and denote this by writing $G_1 \cong G_2$. If G_1 is not isomorphic to G_2, then we write $G_1 \ncong G_2$. Consider the graphs $G_i, i = 1, 2, 3,$ of Figure 1. Each G_i is a graph of order 6 and size 9 and the degree of every vertex of each graph is 3. Here, $G_1 \cong G_2$. On the other hand, G_3 contains three pairwise adjacent vertices whereas G_1 does not; so there is no isomorphism from G_1 to G_3 and therefore $G_1 \not\cong G_3$. Of course, $G_2 \not\cong G_3$.

If G is a graph of order n and size m, then $n \geq 1$ and $0 \leq m \leq \binom{n}{2} = n(n-1)/2$. Although every graph must have at least one vertex, this is not the case with the number of edges. A graph with no edges is called an *empty graph.* There is only one graph of order 1 (up to ismorphism), and this is referred to as the *trivial graph.* A *nontrivial graph* then has $n \geq 2$. Thus far, whenever we have considered a graph G that is defined by a diagram, each point of this diagram (that is, each vertex of G) has been labeled. Therefore, the set of all vertex labels is $V(G)$. Here we are dealing with a *labeled graph.* There are occasions, however, when we are only interested in the structure of a graph defined by a diagram and the vertex set of the graph is irrelevant. In this case, we have an *unlabeled graph.*

A labeled graph H is a *subgraph* of a labeled graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in such a case, we also say that G contains H as a subgraph. For unlabeled graphs H and G, we say that H is a subgraph of G if the vertices of H and G can be labeled so that as labeled graphs, H is a subgraph of G. So the graphs H_1 and H_2 of Figure 2 are subgraphs of the graph G. Whenever a subgraph H of a graph G has the same order as G, then H is called a *spanning subgraph* of G.

If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G *induced* by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. In Figure 2, the graph H_1 is not an induced subgraph of G. On the other hand, the graph H_2 of Figure 2 is an induced subgraph of G.

Let U is a nonempty set of vertices of a graph G. The *neighborhood* $N_G(U)$ of U is the set of all vertices of G adjacent with at least one element of U.

The simplest type of a subgraph of a graph G is that obtained by deleting vertices or edges. If $S \subseteq V(G)$, then $G - S$ denotes the subgraph with vertex set $V(G) - S$ and whose edges are all those of G not incident with a vertex of S. If $S = \{v\}$, we simply write $G - v$ for $G - \{v\}$. If $F \subseteq E(G)$, we can analogously define a graph $G - F$ as the subgraph having vertex set $V(G)$ and edge set $E(G) - F$. Also we simply write $G-e$ for $G-\{e\}$. Furthermore, if u and v are nonadjacent vertices of a graph G, then $G+e$, where $e=uv$, denotes the graph with vertex set $V(G)$ and the edge set $E(G)\cup\{e\}$. Clearly, G is a subgraph of $G+e$. These concepts are illustrated in Figure 2.

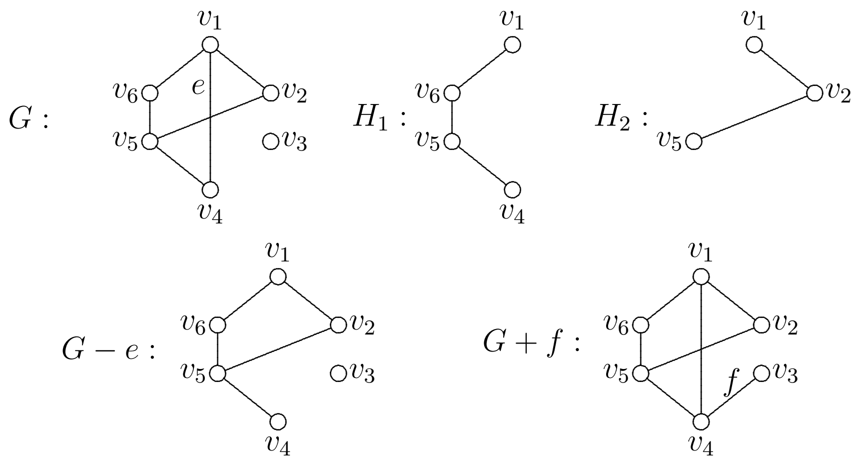


Figure 2: Subgraphs of a graph G.

A graph G is *regular of degree* r if deg $v=r$ for each vertex v of G. Such graphs are called r*-regular*. A 3-regular graph is also called a *cubic graph*. A 4-regular graph is called a *quartic graph*. A graph is *complete* if every two of its vertices are adjacent. A complete graph of order n and size m is therefore a regular graph of degree $n-1$ having $m=\binom{n}{2}$; we denote this graph by K_n. In Figure 3, we show all (nonisomorphic) regular graphs with $n=4$, including the complete graph $G_4\cong K_4$.

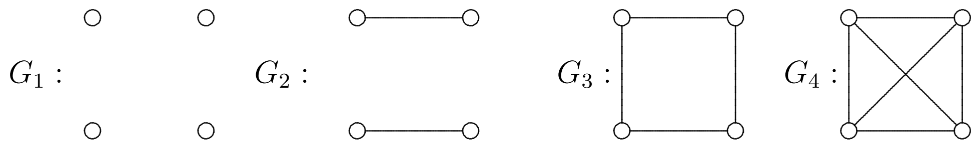


Figure 3: The regular graphs of order 4.

Any spanning subgraph of a graph G is referred to as a *factor* of G. A k-regular factor is called k*-factor*. Of course, if a graph H has a 1-factor, then H has even order.

The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Hence, if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. In Figure 3, the graphs G_1 and G_4 are complementary, as are G_2 and G_3. Thus the complement $\overline{K}_n$ of the complete graph K_n is the empty graph of order n.

A graph is *k-partite*, $k \geq 1$, if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \ldots, V_k$ (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of $V_j, i \neq j$. Every graph is k-partite for some k; indeed if n is the order of G then G is n-partite. If G is a 1-partite graph of order n, then $G = \overline{K}_n$. For $k = 2$, such graphs are called *bipartite* graphs; this class of graphs is particularly important and will be encountered many times. If G is an r-regular bipartite graph, $r \geq 1$, with partite sets V_1 and V_2, then $|V_1| = |V_2| = r$. This follows from the fact that its size is $m = r|V_1| = r|V_2|$. A quartic bipartite graph is also called *biquartic* graph.

A *complete k-partite graph* G is a k-partite graph with partite sets $V_1, V_2, \ldots,$ V_k having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $uv \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \ldots, n_k)$ or $K_{n_1,n_2,\ldots,n_k}$. (The order in which the numbers $n_1, n_2, \ldots, n_k$ are written is not important.) Note that a complete k-partite graph is complete if and only if $n_i = 1$ for all i, in which case it is K_k. A *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$, is then denoted by $K(r, s)$ or more commonly $K_{r,s}$. The graph $K_{1,s}$ is called a *star*.

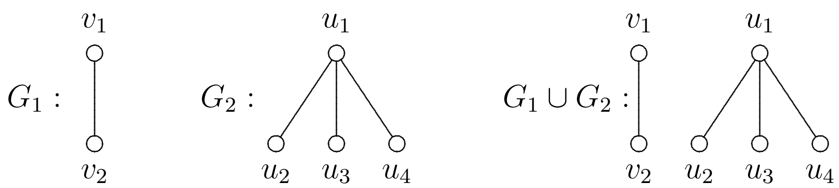


Figure 4: The union of graphs.

We next describe some binary operations defined on graphs. Let G_1 and G_2 be two disjoint graphs. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$. Let G and H be any two graphs, not necessarily disjoint; then $G*H$ is the graph with $V(G*H) = V(G) \cup V(H)$ and $E(G*H) = E(G) \cup E(H)$. It is clear that the binary operation "$*$" is associative. These concepts are illustrated in Figures 4 and 5.

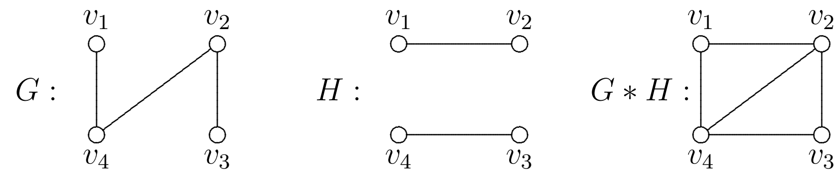


Figure 5: The graph $G * H$.

The *n-cube* Q_n can be considered as the graph whose vertices are labeled by the binary n-tuples $(a_1, a_2, \ldots, a_n)$ (that is, a_i is 0 or 1 for $1 \leq i \leq n$) and such that two vertices are adjacent if and only if their corresponding n-tuples differ at precisely one coordinate. The graph Q_n is an n-regular graph of order 2^n. The n-cubes, $n = 1, 2$ and 3, are shown in Figure 6 with appropriate labelings. The graphs Q_n are often called *hypercubes*.

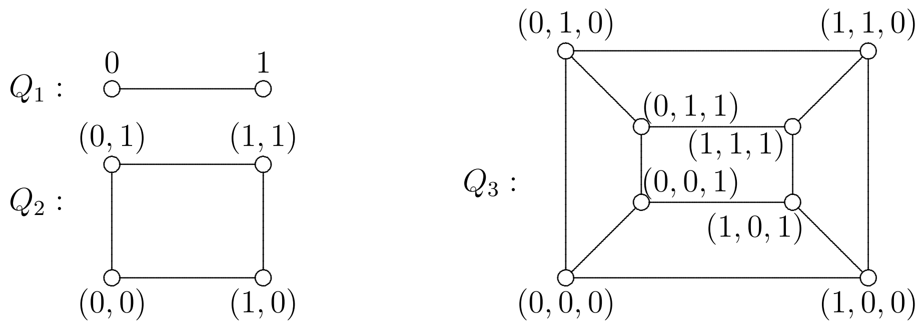


Figure 6: Cubes.

Let u and v be (not necessarily distinct) vertices of a graph G. A *$u - v$ walk* W of G is a finite, alternating sequence

$$W : u = u_0, e_1, u_1, e_2, u_2, \ldots, u_{k-1}, e_k, u_k = v$$

of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for $i = 1, 2, 3, \ldots, k$. The number k (the number of occurrences of edges) is called the *length* of W. A *trivial walk* contains no edges; that is, $k = 0$. We note that there may be repetition of vertices and edges in a walk. Often only the vertices of a walk are indicated since the edges present are then evident.

A $u - v$ walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A $u - v$ *path* is a $u - v$ walk in which no vertex is repeated. A nontrivial closed path $v_1, v_2, \ldots, v_n, v_1$ $(n \geq 3)$ of a graph G whose n vertices v_i are distinct is called a *cycle.* An *acyclic graph* has no cycles. The subgraph of a graph G induced by the edges of a path or cycle is also referred to as a *path* or *cycle* of G. A cycle is *even* if its length is even; otherwise it is *odd.* A cycle of length n is an n-cycle; a 3-cycle is also called *a triangle.* A graph of order n that is a path or a cycle is denoted by P_n or C_n, respectively.

We now introduce additional basic concepts in graph theory, namely connected and disconnected graphs. A vertex u is said to be *connected* to a vertex v in a graph G if there exists a $u - v$ path in G. A graph G is *connected* if every two of its vertices are connected. A graph that is not connected is *disconnected.* The relation "is connected to" is an equivalence relation on the vertex set of every graph G. Each subgraph induced by the vertices in a resulting equivalence class is called a *connected component* or simply a *component* of G. Equivalently, a component of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G; that is, a component of G is a subgraph that is maximal with respect to the property of being connected. The number of components of G is denoted by $k(G)$.

A vertex v of a graph G is called a *cut-vertex* of G if $k(G - v) > k(G)$. Thus, a vertex of a connected graph is a cut-vertex if its removal produces a disconnected graph. A nontrivial connected graph with no cut-vertices is called a *nonseparable graph.* A *block* of a graph G is a maximal nonseparable subgraph of G. If a connected graph G contains a single block, then G is nonseparable. Every two blocks have at most one vertex in common, namely a cut-vertex.

A *vertex-cut* in a graph G is a set U of vertices of G such that $G - U$ is disconnected. The *vertex-connectivity* or simply *connectivity* $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n-1$ if $G = K_n$ for some positive integer n. Hence $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is said to be *k-connected*, $k \geq 1$, if $\kappa(G) \geq k$. Thus G is 1-connected if any only if G is nontrivial and connected. In general, G is k-connected if and only if the removal of fewer than k vertices results in neither a disconnected nor a trivial graph.

A graph G is defined to be *hamiltonian* if it has a cycle containing all the vertices of G. A cycle of a graph G containing every vertex of G is called a *hamiltonian cycle* of G; thus, a hamiltonian graph is one that possesses a hamiltonian cycle. If G is a hamiltonian graph, then G is necessarily connected and G contains no cut-vertices, and, has order at least 3; thus G is 2-connected.

Among the connected graphs, the simplest yet most important are trees. A *tree* is an acyclic connected graph; while a *forest* is an acyclic graph. Thus every component of a forest is a tree. There is one tree of each of the orders $1, 2$ and 3; while there are two trees of order 4, three trees of order 5, and six trees of order 6.

If u and v are any two nonadjacent vertices of a tree T, then $T + uv$ contains precisely one cycle C. If, in turn, e is any edge of C in $T + uv$, then the graph $T + uv - e$ is once again a tree.

Let G be a graph and $F \subseteq V(G)$. F is called an *induced forest* of G if the induced subgraph $\langle F \rangle$ of G contains no cycle. The graph H_2 in Figure 2 is an induced forest of G while H_1 is not.

Let F be a graph. A graph G is *F-free* if G contains no induced subgraph isomorphic to F. Thus a K_2-free graph is empty.

1.3 Basic Concepts in Graph Theory

In this section we introduce some fundamental concepts of graph theory most of which are used in proving our results. Our first theorem is sometimes called *The First Theorem of Graph Theory.*

Theorem 1.3.1 *Let G be a graph of order n and size m. Then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

This result has an interesting consequence.

Theorem 1.3.2 *In any graph, there is an even number of odd vertices*

A necessary and sufficient condition for a sequence to be graphical was found by Havel [22] and later rediscovered by Hakimi [15].

Theorem 1.3.3 *A sequence $\mathbf{d} : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \ldots \geq d_n$, $n \geq 2$, $d_1 \geq 1$, is graphical if and only if the sequence $d' : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphical.*

By the definition of a graphical sequence, $\mathbf{d} = r^n$ is graphical if and only if $rn \equiv 0 \pmod 2$ and $n \geq r + 1$.

We are now prepared to present a useful characterization of bipartite graphs.

Theorem 1.3.4 *A nontrivial graph is bipartite if and only if it contains no odd cycles.*

The following theorem characterizes cut-vertices.

Theorem 1.3.5 *A vertex v of a graph G is a cut-vertex of G if and only if there exist vertices u and w $(u, w \neq v)$ such that v is on every $u - w$ path of G.*

The next result is a basic property of trees that is often useful when using mathematical induction to prove theorems dealing with trees.

Theorem 1.3.6 *Every nontrivial tree has at least two end-vertices.*

There is a number of ways to characterize trees. Three of these are particularly useful.

Theorem 1.3.7 *A graph G of order n and size m is a tree if and only if G is acyclic and $n = m + 1$.*

Theorem 1.3.8 *A graph G of order n and size m is a tree if and only if G is* *connected and* $m = n - 1$.

Theorem 1.3.9 *A graph* G *is a tree if and only if every two distinct vertices of* G *are connected by a unique path.*

Petersen [27] proved the following theorem.

Theorem 1.3.10 *If* G *is a 2-connected cubic graph, then* G *contains a 2-factor.*

1.4 Graph Transformations

We introduced in this section some important concepts of graph transformations.

Let $\mathcal{J}$ be a class of graphs or subgraphs of a graph. If there is a fixed type of transformation of a member of $\mathcal{J}$ into another, then this will provide a definition of a *graph transformation on* $\mathcal{J}$. Equivalently, a graph transformation may be defined as a relation ρ from $\mathcal{J}$ to $\mathcal{J}$. In other words $\rho \subseteq \mathcal{J} \times \mathcal{J}$. With this definition we can see that there are infinitely many such graph transformations. Let ρ be a graph transformation on $\mathcal{J}$. We define the *ρ-graph* as having $\mathcal{J}$ as its vertex set and there is an edge between two vertices G and H if and only if $(G, H) \in \rho$. If ρ is symmetric, it yields an undirected graph and a directed graph otherwise. Thus a ρ-graph arises from graphs in at least two different ways:

1. from a class of graphs, or
2. from a class of subgraphs of a fixed graph.

Harary [21] introduced a graph transformation and called it a *fundamental exchange* or an *edge exchange* as follows: Let G be a graph of order $n \geq 3$. The *tree graph*, $\mathcal{T}(G)$, of G is defined by specifying $V(\mathcal{T}(G))$ to be the set of all spanning trees of G, and two vertices $T_1, T_2 \in V(\mathcal{T}(G))$ are adjacent in $\mathcal{T}(G)$ if and only if T_1 and T_2 have exactly $n - 2$ edges in common. This is an example of an undirected ρ-graph. It was proved in [21] that the tree graph $\mathcal{T}(G)$ is connected. The spanning subgraphs T_1 and T_3 in Figure 7 are adjacent in $\mathcal{T}(G)$, but T_2 and T_3 are not.

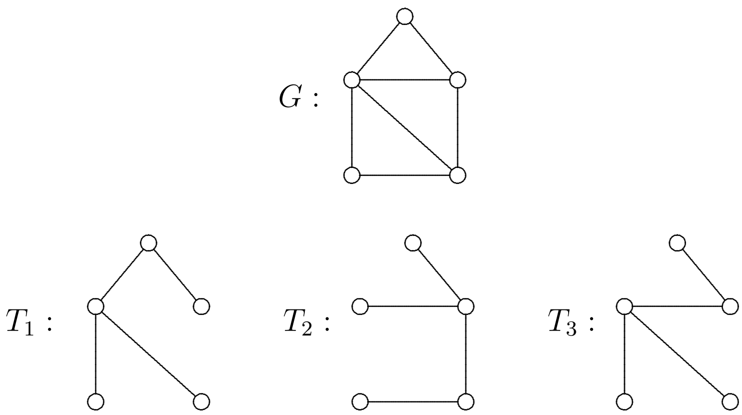


Figure 7: Some spanning subgraphs of G.

Let G be a graph. For distinct vertices a, b, c, and d in G, let $ab, cd \in E(G)$ be independent where $ac, bd \notin E(G)$. Define $G^{\sigma(a,b;c,d)}$, simply written G^{σ}, to be the graph obtained from G by deleting the edges ab and cd and adding the edges ac and bd (see the Figure 8). The operation $\sigma(a, b; c, d)$ is called a *switching operation.* Note that G^{σ} has the same degree sequence as G.

Let $\mathcal{G}$ be a class of graphs and $G \in \mathcal{J}$ where $\mathcal{J} \subseteq \mathcal{G}$. A switching σ is called a *$\mathcal{J}$-switching with respect to G* if $G^{\sigma} \in \mathcal{J}$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a *sequence of $\mathcal{J}$-switchings with respect to G* if $G^{\sigma_1\sigma_2\ldots\sigma_i} \in \mathcal{J}$ for all $i = 1, 2, \ldots, t$.

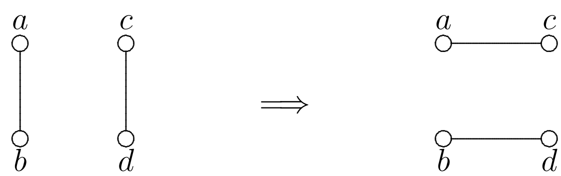


Figure 8: The switching operation.

The following theorem was shown by Havel [22] and Hakimi [15].

Theorem 1.4.1 *Let* $\mathbf{d}$ *be a graphical sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$, *then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

Let $\mathbf{d}$ be a graphical sequence. Let $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$ be the sets of realizations and connected realizations of $\mathbf{d}$, respectively. Since σ is a degree preserving transformation, it is reasonable to define $\Sigma(\mathbf{d})$ as a relation on $\mathcal{R}(\mathbf{d})$ as $(G,H) \in \Sigma(\mathbf{d})$ if $G \not\cong H$ and there is a switching σ on G such that $H \cong G^{\sigma}$. It is easy to see that $\sigma(a,b;c,d)$ is well-defined on G if and only if $\sigma(a,c;b,d)$ is well-defined on $G^{\sigma(a,b;c,d)}$. Thus the $\Sigma(\mathbf{d})$-graph is simple.

Figure 9 shows the $\Sigma(\mathbf{d})$-graphs when $\mathbf{d} = 2^{10}$ and $\mathbf{d} = 2^{11}$. It is easy to see that the $\Sigma(2^{10})$-graph and the $\Sigma(2^{11})$-graph are connected.

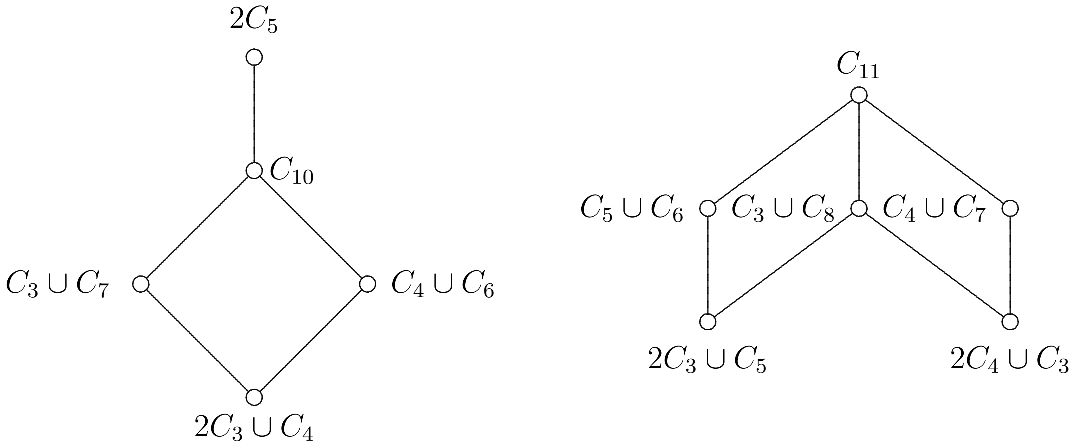


Figure 9: The $\Sigma(2^{10})$-graph and $\Sigma(2^{11})$-graph.

The following theorems were proved by Taylor [40, 41].

Theorem 1.4.2 *Let* $\mathbf{d}$ *be a graphical sequence. Then the* $\Sigma(\mathbf{d})$*-graph is connected.*

Theorem 1.4.3 *Let* $\mathbf{d}$ *be a graphical sequence. Then subgraph of the* $\Sigma(\mathbf{d})$*-graph induced by* $\mathcal{CR}(\mathbf{d})$ *is connected.*

For positive integers n and m with $0 \leq m \leq \binom{n}{2}$, let $\mathcal{G}(n,m)$ and $\mathcal{CG}(n,m)$ be the sets of all non-isomorphic graphs and connected graphs, respectively, of order n and size m. Let $G \in \mathcal{G}(n,m)$ with $e \in E(G)$ and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G - e + f)$. A transformation $t(e,f)$ is called a *jumping transformation.* Now let $\mathcal{T}(n,m)$ be a relation on $\mathcal{G}(n,m)$ defined by $(G,H) \in \mathcal{T}(n,m)$ if $G \not\cong H$ and H can be obtained from G by a jumping transformation. Since $\mathcal{T}(n,m)$ is symmetric, it follows that the $\mathcal{T}(n,m)$-graph is simple.

The concept of jumping transformation is not new. In the early 1980's, a series of papers by Harary and various co-authors [16, 17, 18, 19, 20, 21] introduced it and called it an *edge exchange*.

1.5 Interpolation Graph Parameters

In this section, we give the definition of interpolation theorems which leads to extremal problems. Then we will detail the interpolation graph parameter.

As discussed when the concept of graph isomorphism was introduced, the relation " $\cong$ " is an equivalence relation on the set of all simple graphs. Consequently, this relation partitions the set of all graphs into equivalence classes each of which consists of all graphs having the same algebraic properties. The following are examples of the algebraic properties preserved under the relation " $\cong$ ": being a bipartite graph, being a cubic graph, being a connected graph, being a planar graph, being a Hamiltonian graph, etc.

A *graph invariant* is a function π from the set of all graphs to any range X of values (numerical, vectorial or any other) such that π takes the same value on isomorphic graphs. In particular if the range of values is numerical (real, rational or integers), then the graph invariant is called a *graph parameter*.

If X is the set of all integer sequences, then $\pi(G) = \mathbf{d}(G)$, where $\mathbf{d}(G)$ is the degree sequence of G, is an example of an invariant. If X is the category of all graphs, then the function $\pi(G) = Aut(G)$, where $Aut(G)$ is the group of automorphisms of G, is a group invariant of graphs.

Let $\mathcal{G}$ be the class of graphs and $\mathcal{J} \subseteq \mathcal{G}$. A graph parameter π is called an *interpolation graph parameter over* $\mathcal{J}$ if there exist integers a and b such that

$$\{\pi(G) : G \in \mathcal{J}\} := [a, b] := \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

Interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter π and a subset $\mathcal{J}$ of $\mathcal{G}$, does π interpolate over $\mathcal{J}$? If π interpolates over $\mathcal{J}$, then $\{\pi(G) : G \in \mathcal{J}\}$

is uniquely determined by $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$. The second part of the interpolation theorems for a graph parameter is to find $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$ for the corresponding interpolation graph parameters and this part is the extremal problems in graph theory. In the case where $\mathcal{J} \subseteq \mathcal{G}$ consists of all connected graphs, we simply write $\mathrm{Min}(\pi, \mathcal{J})$ and $\mathrm{Max}(\pi, \mathcal{J})$ for $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$, respectively.

An "extremal problem" asks for the maximum or minimum values of a function over a class of objects. In graph theory, we use the term extremal problem for finding an extreme value of some graph parameters over some class of graphs. Extremal graph theory is a huge area. The archetypal example is the Turán problem: find the maximum number of edges in a graph not containing H as a subgraph.

Proving that a is the maximum of $\pi(G)$ for graphs in a class $\mathcal{J}$ requires showing two things:

1. $\pi(G) \leq a$ for all $G \in \mathcal{J}$ and
2. $\pi(G) = a$ for some $G \in \mathcal{J}$.

The proof of the bound must apply to every $G \in \mathcal{J}$. We can prove that the bound is achievable by constructing an example. Changing " $\leq$ " to " $\geq$ " yields the criteria for a minimum.

The *forest number* $f(G)$ of a graph G is the maximum cardinality of an induced forest in G. That is

$$f(G) := \max\{|F| : \text{ F is an induced forest of } G\}.$$

The class of those graphs G for which $f(G) = n$ consists of all forests, and $f(G) = n - 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is easy to see that $f(K_n) = 2$.

A subset U of the vertex set of a graph G is said to be an *independent set* of G if the induced subgraph $\langle U \rangle$ of G is an empty graph. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$.

CHAPTER 2

REVIEW OF THE LITERATURE

In this chapter we review some relevant works on interpolation theorems and extremal problems on the forest number of some classes of graphs.

The interest in the interpolation properties of graph parameters was motivated by a question posed by Chartrand during a conference held at Kalamozoo, Michigan in 1980. He posed the following question: If a graph G contains spanning trees having m and n end-vertices, with $m < n$, does G contain a spanning tree with k end-vertices for every integer k with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of graph parameters over the set of all spanning trees of a given graph.

In 1963, Erdős and Gallai [12] proved that any regular graph on n vertices has chromatic number $k \leq \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communicaiton, Erdős wrote, "probably such a graph exists for every $k \leq \frac{3n}{5}$, except possibly for trivial exceptional cases."

In 1990, Caccetta and Pullman [4] determined the minimum number of edges in a regular connected graph on n vertices, containing a complete subgraph of order $k \leq \frac{n}{2}$. This enabled them to confirm and strengthen the conjecture of Erdős by showing that if $k \geq 1$, then for every $n \geq \frac{5k}{3}$, there exists a connected regular k-chromatic graph on n vertices. This is an example of the interpolation properties of the graph parameter χ of the class of all connected regular graphs of order n.

In 1984, Staton [37] studied some properties of induced forests in cubic graphs. More precisely, he obtained some upper bounds for the number of vertices in an induced forest in a cubic graph with a given girth. In particular, he proved that if the girth increases without bound, then the ratio of the number of vertices in a largest forest to the number of vertices in the whole graph approaches $\frac{3}{4}$. In 1988, the maximum induced forest in cubic graphs was also studied by Speckenmeyer [38].

In 1986, Erdős, Saks and Sós [13] first defined a graph parameter called the *forest number* which is denoted by $f(G)$ of a graph G as the maximum cardinality of an induced forest of G.

Let G be a connected graph on n vertices. A subset F of $V(G)$ is a *feedback* vertex set if $G-F$ is a forest and a subset J of $V(G)$ is a nonseparating independent set if no two vertices of J are adjacent and $G-J$ is connected. In a graph G, $f'(G)$ and $z(G)$ denote the cardinality of a minimum feedback set and that of a maximum nonseparating independent set, respectively. In 1988, Speckenmeyer [38] derived the equation $f'(G) = \frac{n}{2} - z(G) + 1$ for cubic graphs G. It is easy to see that $f(G) = n - f'(G)$.

2.1 Interpolation Theorems

In this section we review some significant results on interpolation theorems on the graph parameter f.

Let $\mathcal{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ. We define the (Δ, n)-*graph* as the graph whose vertex set is $\mathcal{G}(\Delta, n)$ and two graphs in $\mathcal{G}(\Delta, n)$ are adjacent if one can be obtained from the other by adding or deleting an edge. Note that the (Δ, n)-graph exists if and only if $n \geq \Delta + 1$. In 2005, Punnim [31] proved the following results.

Theorem 2.1.1 *The (Δ, n)-graph is connected.*

Theorem 2.1.2 *If G_1 and G_2 are adjacent in the (Δ, n)-graph, then $|f(G_1) - f(G_2)| \leq 1$.*

Theorem 2.1.3 *The forest number f is an interpolation graph parameter over $\mathcal{G}(\Delta, n)$.*

Corollary 2.1.4 *For any class of graphs $\mathcal{G}(\Delta, n)$, there exist integers a and b such that there is a graph $G \in \mathcal{G}(\Delta, n)$ with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Let $\mathbf{d}$ be a graphical sequence. It was shown in [40, 41] that the $\Sigma(\mathbf{d})$-graph is connected and the subgraph of $\Sigma(\mathbf{d})$-graph induced by $\mathcal{CR}(\mathbf{d})$ is connected. It follows that for a graph G of degree sequence $\mathbf{d}$ and a switching σ, if $|\pi(G) - \pi(G^\sigma)| \leq 1$, then π is an interpolation graph parameter over $\mathcal{R}(\mathbf{d})$.

In 2003, Punnim [29] proved the following result.

Theorem 2.1.5 *If σ is a switching on a graph G, then $|f(G) - f(G^\sigma)| \leq 1$.*

The following theorems are immediate consequences of the previous result.

Theorem 2.1.6 *Let $\mathbf{d}$ be a graphical sequence. Then f is an interpolation graph parameter over $\mathcal{R}(d)$.*

Theorem 2.1.7 *Let $\mathbf{d}$ be a graphical sequence. Then f is an interpolation graph parameter over $\mathcal{CR}(d)$.*

Let π be an interpolation graph parameter over $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. We use the following notation.

- $\min(\pi, \mathbf{d}) = \min\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$,
- $\max(\pi, \mathbf{d}) = \max\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$,
- $\mathrm{Min}(\pi, \mathbf{d}) = \min\{\pi(G) : G \in \mathcal{CR}(\mathbf{d})\}$,
- $\mathrm{Max}(\pi, \mathbf{d}) = \max\{\pi(G) : G \in \mathcal{CR}(\mathbf{d})\}$.

Theorem 2.1.8 *For any graphical sequence $\mathbf{d}$, there exist integers $a := \min(f, \mathbf{d})$ and $b := \max(f, \mathbf{d})$ such that $\mathbf{d}$ has a realization G with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Theorem 2.1.9 *For any graphical sequence $\mathbf{d}$, there exist integers $a := \mathrm{Min}(f, \mathbf{d})$ and $b := \mathrm{Max}(f, \mathbf{d})$ such that $\mathbf{d}$ has a realization G with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

We now review the interpolation results on the graph parameter f over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$. The idea of investigating the interpolation property of a graph parameter π over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$ is as follows:

1. By using the facts that the $\mathcal{T}(n, m)$-graph and the subgraph induced by $\mathcal{CG}(n, m)$ are connected, and

2. for any $G \in \mathcal{G}(n,m)$ and a jumping transformation $t(e,f)$ on G, $|\pi(G) - \pi(G^{t(e,f)})| \leq 1$.

Punnim [36] recently proved the following results.

Theorem 2.1.10 *The $\mathcal{T}(n,m)$-graph is connected.*

Theorem 2.1.11 *The subgraph of $\mathcal{T}(n,m)$ induced by $\mathcal{CG}(n,m)$ is connected.*

Theorem 2.1.12 *For any $G \in \mathcal{G}(n,m)$ and a jumping transformation $t(e,f)$ on G, $|f(G) - f(G^{t(e,f)})| \leq 1$.*

Theorem 2.1.13 *The forest number f is an interpolation graph parameter over $\mathcal{G}(n,m)$.*

Theorem 2.1.14 *The forest number f is an interpolation graph parameter over $\mathcal{CG}(n,m)$.*

Let π be an interpolation graph parameter over $\mathcal{G}(n,m)$ and $\mathcal{CG}(n,m)$. We use the following notation.

- $\min(\pi; n,m) = \min\{\pi(G) : G \in \mathcal{G}(n,m)\}$,
- $\max(\pi; n,m) = \max\{\pi(G) : G \in \mathcal{G}(n,m)\}$,
- $\mathrm{Min}(\pi; n,m) = \min\{\pi(G) : G \in \mathcal{CG}(n,m)\}$,
- $\mathrm{Max}(\pi; n,m) = \max\{\pi(G) : G \in \mathcal{CG}(n,m)\}$.

Theorem 2.1.15 *Let n and m be integers satisfying $0 \leq m \leq \binom{n}{2}$. Then there exist integers $a := \min(f; n,m)$ and $b := \max(f; n,m)$ such that there exists $G \in \mathcal{G}(n,m)$ with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Theorem 2.1.16 *Let n and m be integers satisfying $0 \leq m \leq \binom{n}{2}$. Then there exist integers $a := \mathrm{Min}(f; n,m)$ and $b := \mathrm{Max}(f; n,m)$ such that there exists $G \in \mathcal{CG}(n,m)$ with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

2.2 Extremal Problems

The second part of interpolation theorems for the graph parameter f is reviewed in this section. We consider four subsections.

2.2.1 Large Induced Forests in Sparse Graphs

In 1987, the minimum possible value of $f(G)$, where G ranges over all graphs on n vertices and m edges, was determined by Alon, Kahn and Seymour [2], for every n and m. In particular, the results imply that if the average degree of G is at most $d \geq 2$, then $f(G) \geq \frac{2n}{d+1}$. This is sharp whenever $d+1$ divides n as shown by a disjoint union of cliques of order $d+1$. For bipartite graphs, one can do better, since trivially $f(G) \geq n/2$. In 2001, Alon, Mubayi and Thomas [1] proved the following result.

Theorem 2.2.1 *If G is a triangle-free graph of order n with the minimum degree $\Delta(G) = 3$, then $f(G) \geq 5n/8$ and this is sharp whenever n is divisible by 8.*

In the same paper [1], they also obtained a bound for $f(G)$ in terms of the *independence number* $\alpha(G)$ of G.

Theorem 2.2.2 *Let G be a connected graph of order n with maximum degree $\Delta(G) = \Delta \geq 3$. Then*

$$f(G) \geq \alpha(G) + \frac{n - \alpha(G)}{(\Delta - 1)^2}.$$

Moreover, they proved a special case of Theorem 2.2.2 that is independently interesting.

Theorem 2.2.3 *If G is a triangle-free graph of order n and size m, then $f(G) \geq n - m/4$.*

As mentioned earlier, Theorem 2.2.3 is sharp for $m \leq 3n/2$ and $m \equiv 0 \pmod{12}$, as shown by disjoint copies of Q_3. For 4-regular graphs it gives $f(G) \geq n/2$, but the best example they can find has $f(G) = 4n/7$. For 5-regular graphs, Theorem 2.2.3 gives $f(G) \geq 3n/8$, but the best example they can find has $f(G) = n/2$.

The following corollary follows immediately from Theorem 2.2.2.

Corollary 2.2.4 *If G is bipartite graph of order n with maximum degree $\Delta(G) = \Delta \geq 3$, then*

$$f(G) \geq \left(\frac{1}{2} + \frac{1}{2(\Delta-1)^2}\right) n$$

and this is sharp for $\Delta = 3$, and $n \equiv 0 \pmod 8$.

The cube Q_3 shows that the bound in Corollary 2.2.4 is sharp for $\Delta = 3$.

2.2.2 The Forest Number of Regular Graphs.

Let $\mathbf{d}$ be a graphical sequence. We have already defined the class $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ and the range of forest numbers. By the definition of degree sequence, $\mathbf{d} = r^n$ is graphical if and only if $rn \equiv 0 \pmod 2$ and $n \geq r+1$. Moreover, $\mathcal{R}(r^n)$ contains a disconnected graph if and only if $n \geq 2r+2$. It is easy to see that $f(\mathcal{R}(0^n)) = f(\mathcal{R}(1^n)) = \{n\}$. When $r = 2$, we have $n-1, \frac{2n}{3} \in f(\mathcal{R}(2^n))$. Because of this fact, from now on we will consider the cases when $r \geq 3$ and $n \geq r+1$.

It was remarked in [3] that if G is a connected graph with maximum degree $\Delta(G) = \Delta$, then $f(G) \leq \frac{|V(G)|\Delta - |E(G)| - 1}{\Delta - 1}$. Thus if G is a connected r-regular graph of order n, then $f(G) \leq \frac{nr-2}{2(r-1)}$.

In 2005, Punnim [31] obatined the values of $\max(f, r^n)$ for all r and n as stated in the following theorem.

Theorem 2.2.5

$$\max(f, r^n) = \begin{cases} n - r + 1 & if\ r+1 \leq n \leq 2r-1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & if\ n \geq 2r. \end{cases}$$

The problem of finding $\min(f, r^n)$ is difficult if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the class $\mathcal{G}(\Delta, n)$ of all graphs of order n and of maximum degree Δ. It is easy to see that

$\min(f, \mathcal{G}(\Delta, n)) = n$. In order to investigate the exact values of $\max(f, \mathcal{G}(\Delta, n))$, Punnim [31] first gave its upper bound and obtained the following results.

Lemma 2.2.6 *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq 1$, then $f(G) \geq \frac{2n}{\Delta+1}$.*

Theorem 2.2.7 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphical sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then*

1. $\min(f, \mathbf{d}) = 2$ *if and only if* $\mathcal{R}(\mathbf{d}) = \{K_n\}$ *and*
2. *if $K_n \notin \mathcal{R}(\mathbf{d})$, then $\min(f, \mathbf{d}) = 3$ if and only if there exists a union of stars as a realization of $\overline{\mathbf{d}}$, where $\overline{\mathbf{d}} = (n - d_n, n - d_{n-1}, \ldots, n - d_1)$.*

Theorem 2.2.8 *Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then*

1. $\min(f, \mathcal{G}(\Delta, n)) = 2q$, *if $t = 0$,*
2. $\min(f, \mathcal{G}(\Delta, n)) = 2q + 1$, *if $t = 1$, and*
3. $\min(f, \mathcal{G}(\Delta, n)) = 2q + 2$, *if $2 \leq t \leq \Delta$.*

Theorem 2.2.9 *For $r \geq 3$, and $r + 1 \leq n \leq 2r + 1$,*

1. $\min(f, r^n) = 2$, *if and only if $n = r + 1$,*
2. $\min(f, r^n) = 3$, *if and only if $n = r + 2$,*
3. $\min(f, r^n) = 4$, *for all even integers n, $r + 3 \leq n$,*
4. $\min(f, r^n) = 4$, *for all odd integers n, $r + 3 \leq n$ and $n \geq f(j)$, and*
5. $\min(f, r^n) = 5$, *for all odd integers n, $r + 3 \leq n$ and $n < f(j)$,*

where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod 4$.

Theorem 2.2.10 *For $n \geq 2r + 2$ and $r \geq 3$, write $n = (r + 1)q + t, q \geq 2$ and $0 \leq t \leq r$. Then*

1. $\min(f, r^n) = 2q$ *if $t = 0$,*

2. $\min(f, r^n) = 2q + 1$ *if* $t = 1$,

3. $\min(f, r^n) = 2q + 2$ *if* $2 \leq t \leq r - 1$, *and*

4. $\min(f, r^n) = 2q + 3$ *if* $t = r$.

Punnim [31] proved the following results.

Theorem 2.2.11 *For any integer* $n \geq 4$, *we have*

$$\max(f, 3^n) = \lfloor \frac{3n-2}{4} \rfloor, \quad \textit{and}$$

$$\min(f, 3^n) = \begin{cases} \frac{n}{2} & \textit{if } \frac{n}{2} \textit{ is even,} \\ \frac{n}{2} + 1 & \textit{if } \frac{n}{2} \textit{ is odd.} \end{cases}$$

For each $c \in f(\mathbf{d})$, let $\mathcal{R}(\mathbf{d}; c)$ denote the subgraph of the $\Sigma(\mathbf{d})$-graph induced by the vertices corresponding to graphs with the forest number c. Punnim considered the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d}; c)$. In general, what is the structure of the $\mathcal{R}(\mathbf{d}; c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d}; c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with forest number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with forest number c. Punnim [31] proved the following theorem.

Theorem 2.2.12 *The subgraph of* $\Sigma(3^n)$*-graph induced by* $\mathcal{R}(3^n; \lfloor \frac{3n-2}{4} \rfloor)$ *is connected.*

2.2.3 The Forest Number of Connected Regular Graphs.

In this section we review some results of Punnim [34]. It is easy to observe that the values of $\mathrm{Min}(f, r^n)$ and $\mathrm{Max}(f, r^n)$ are easily obtained for all $r \in \{0, 1, 2\}$. The problem of finding $\mathrm{Min}(f, 3^n)$ and $\mathrm{Max}(f, 3^n)$ is more difficult.

A *cubic tree* is a tree consisting of vertices of degree 1 or 3 only. It is easy to see that if T is a cubic tree of order n, then $n = 2k + 2$, where k is the number of veritces of T of degree 3. Let $\mathcal{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of K_4 with one subdivided edge (see the graph K_4' in Figure 10) at every vertex of degree 1. It is easy to see that $\mathrm{Min}(f, 3^4) = 2, \mathrm{Min}(f, 3^6) = 4, \mathrm{Min}(f, 3^8) = 5$ and $\mathrm{Min}(f, 3^{10}) = 6$.

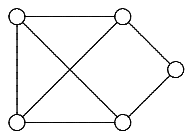


Figure 1: The graph K_4'.

In 1996, a lower bound for the forest number in connected cubic graphs was obtained by Liu and Zhao [25] as stated in the following theorem.

Theorem 2.2.13 *Let G be a connected cubic graph of order $n \geq 12$. Then $f(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathcal{T}$ and $f(G) \geq \frac{5}{8}n$ if $G \notin \mathcal{T}$.*

In [34] Punnim obtained the following results.

Theorem 2.2.14 *Let n be an integer with $n \geq 12$.*

$$\mathrm{Min}(f, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2 \pmod 8, \\ \lceil \frac{5}{8}n \rceil & \text{otherwise.} \end{cases}$$

Corollary 2.2.15 *Let $r \geq 3$ and nr be even. Then*

$$\mathrm{Max}(f, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & \text{if } n \geq 2r. \end{cases}$$

Thus the values of $\mathrm{Max}(f, r^n)$ for all r and n are already obtained. In particular the values of $\mathrm{Max}(f, 3^n)$ and $\mathrm{Min}(f, 3^n)$ are found for all n.

Note that $\mathcal{R}(r^n) = \mathcal{CR}(r^n)$ if and only if $r + 1 \leq n \leq 2r + 1$. Thus $\mathrm{Min}(f, r^n) = \min(f, r^n)$ for all $n \in \{r + 1, r + 2, \ldots, 2r + 1\}$. In this case, the results were already obtained in the previous section. We close this section with the following theorem and conjecture by Punnim [34].

Theorem 2.2.16 *If G is a connected r-regular graph of order $n \geq 2r+2$, then $f(G) \geq \frac{2n}{r}$ for all $r \geq 4$.*

Conjecture 2.2.17 *Let G be a connected r-regular graph of order $n = rq + t, 0 \leq t \leq r-1, r \geq 4$ and $q \geq 1$. Then $\mathrm{Min}(f, r^n) = 2q+2$ if $3 \leq t \leq \frac{r}{2}$, for all $r \geq 6$.*

2.2.4 The Forest Number of Cubic Planar Graphs.

In 2002, Bau and Beineke [3] posed the following problems:

Problem 2.2.18 *Which cubic graphs G of order n satisfy $f(G) = \lfloor \frac{3n-2}{4} \rfloor$?*

Problem 2.2.19 *Which cubic planar graphs G of order n satisfy $f(G) = \lfloor \frac{3n-2}{4} \rfloor$?*

In 2005, Punnim [33] proved that if $\mathcal{R}(3^n)$ is the class of all cubic graphs of order n, then $\mathrm{Max}(f, 3^n) = \lfloor \frac{3n-2}{4} \rfloor$. In this section, we answer the second question by reviewing some results of Punnim [35].

Let $\mathcal{P}(3^n)$ be the class of all connected cubic planar graphs of order n. For small n, the difference between $\mathrm{Min}(f, 3^n)$ and $\mathrm{Max}(f, 3^n)$ is relatively small and it is easy to show by construction that $f(\mathcal{P}(3^n)) = f(\mathcal{CR}(3^n))$, for all $n \leq 26$. We now consider the interpolation property of f over $\mathcal{P}(3^{34})$. Since $\mathrm{Min}(f, 3^{34}) = 21$, and $\mathrm{Max}(f, 3^{34}) = 25, \mathcal{CR}(3^{34})$ can be partitioned into five classes $C_i, 21 \leq i \leq 25$, where $C_i = \{G \in \mathcal{CR}(3^{34}) : f(G) = i\}$ and $C_i \neq \emptyset$, for all $i \in \{21, 22, 23, 24, 25\}$. Consider the graphs $G_i; 1 \leq i \leq 5$ in Figure 11. Thus for each $i = 21, 22, 23, 24, 25, C_i \cap \mathcal{P}(3^{34}) \neq \emptyset$. Therefore, f interpolates over $\mathcal{P}(3^{34})$. The same result can be obtained in $\mathcal{P}(3^{2n})$ for $n \in \{14, 15, 16\}$.

Punnim [35] proved the following theorem and corollary.

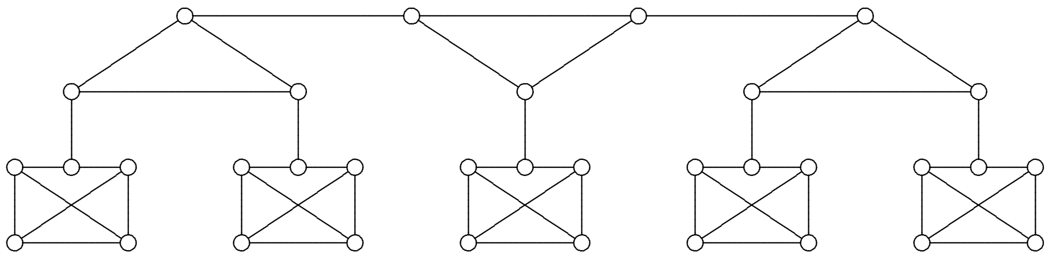
Theorem 2.2.20 *The forest number f is an interpolation graph parameter over $\mathcal{P}(3^n)$ with*

$$\min\{f(G) : G \in \mathcal{P}(3^n)\} = \mathrm{Min}(f, 3^n), \text{ and}$$

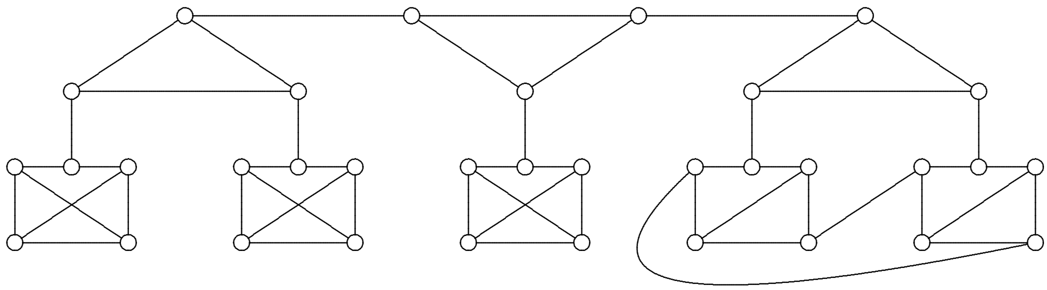
$$\max\{f(G) : G \in \mathcal{P}(3^n)\} = \mathrm{Max}(f, 3^n).$$

Let $\mathcal{P}(3^n; f = \lfloor\frac{3n-2}{4}\rfloor)$ be the class of cubic planar graphs G of order n and $f(G) = \lfloor\frac{3n-2}{4}\rfloor$.

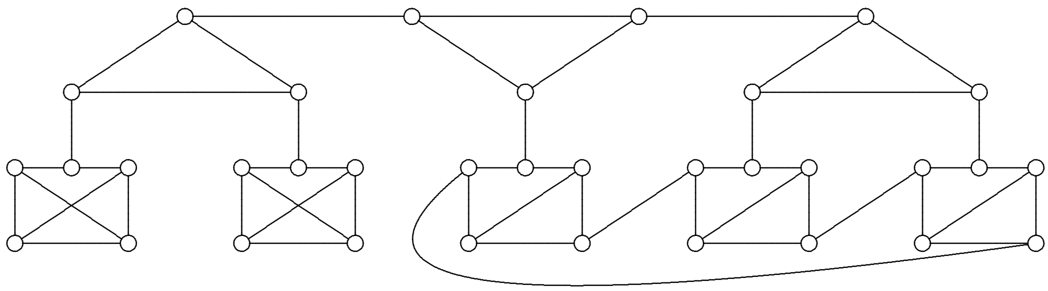
Corollary 2.2.21 *The subgraph of $\Sigma(3^n)$-graph induced by $\mathcal{P}(3^n; f = \lfloor\frac{3n-2}{4}\rfloor)$ is connected.*



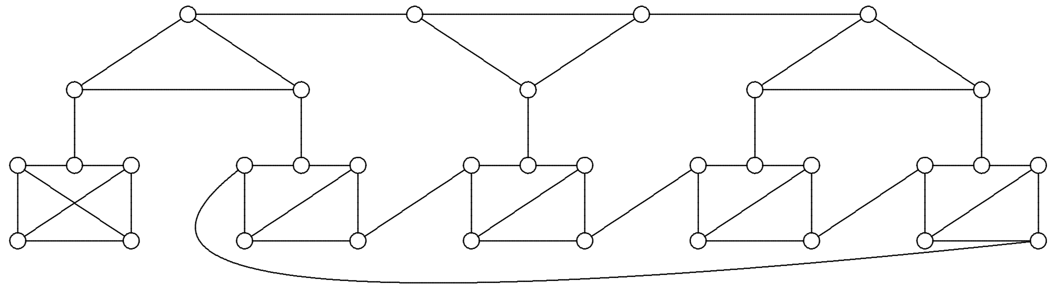
G_1 of order 34 and $f(G_1) = 21 = \text{Min}(f, 3^{34})$.



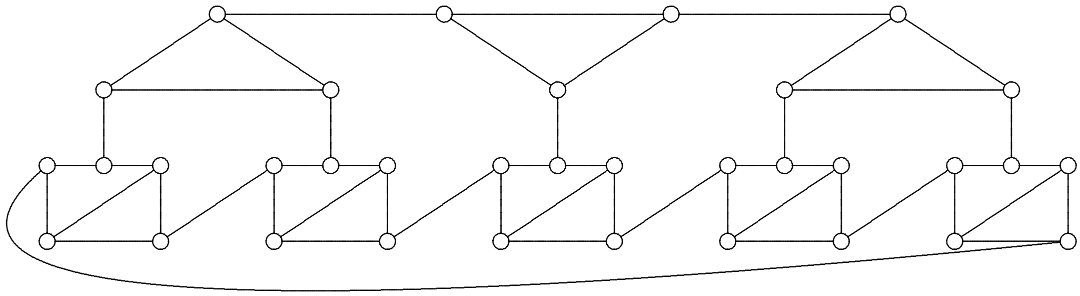
G_2 of order 34 and $f(G_2) = 22 = \text{Min}(f, 3^{34}) + 1$.



G_3 of order 34 and $f(G_3) = 23 = \text{Min}(f, 3^{34}) + 2$.



G_4 of order 34 and $f(G_4) = 24 = \mathrm{Min}(f, 3^{34}) + 3$.



G_5 of order 34 and $f(G_5) = 25 = \mathrm{Min}(f, 3^{34}) + 4$.

Figure 2: Cubic planar graphs of order 34.

CHAPTER 3

INTERPOLATION AND EXTREMAL THEOREMS: THE FOREST NUMBER OF GRAPHS

We have already reviewed some interpolation theorems and extremal problems for the graph parameter f in some classes of graphs. In this chapter, we present our comprehensive work concerning interpolation and extremal results for the graph parameter f. The chapter contains four sections.

We first state results of our work on the range of forest numbers in some classes of cubic graphs of order $2n$. Punnim [28, 29, 32] had obtained interpolation results on the graph parameter f in the classes $\mathcal{R}(\mathbf{d})$ and $\mathcal{CR}(\mathbf{d})$. In particular, he proved that f interpolates over $\mathcal{R}(3^n)$ and $\mathcal{CR}(3^n)$. He also proved that f interpolates over $\mathcal{P}(3^n)$, where $\mathcal{P}(3^n)$ is the class of all cubic planar graphs of order $2n$. Furthermore, the corresponding extremal problem on respective classes of cubic graphs was investigated in [31, 34, 35]. The results prompt us to consider the problem of determining the range of some classes of cubic graphs. Let $\mathcal{J}_1 = \mathcal{CR}(3^n)$ and $\mathcal{J}_2$ be the class of all connected cubic K_4'-free graphs of order n, where K_4' is the graph obtained from K_4 and a subdivision of an edge. Finally, let $\mathcal{J}_3$ be the class of all connected cubic triangle-free graphs of order n. It is clear that $\mathcal{J}_3 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_1$. Let $X_n = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$. By using the concept of graph transformation, we can prove that f interpolates over $\mathcal{J}$ for all $\mathcal{J} \in X_n$. Furthermore, the extremal results are also obtained in all classes in X_n. Details can be found in Section 3.1.

A continuation of our work on interpolation and extremal problems on f is extended to another classes of graphs. For positive integers n and m with $0 \le m \le \binom{n}{2}$, let $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$ be the sets of all non-isomorphic graphs and connected graphs, respectively, of order n and size m. It was shown in [36] by means of graph transformation that f interpolates over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$. The extremal results on f over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$ are obtained in all situations. Details can be found in Section 3.2.

The problem of determining the range of forest numbers in the classes of r-regular bipartite graphs of order $2n$ seems difficult. Alon [1] obtained the following result: if G is a 4-regular bipartite graph of order $2n$, then $f(G) \geq \frac{10}{9} \cdot n$. This result prompts us to consider the problem of determining the range of forest numbers in the class of biquartic graphs of order $2n$. By using concepts from symmetric design theory, we are able to improve the bound from $\frac{10}{9} \cdot n$ to $\frac{8}{7} \cdot n$. Furthermore, we prove that if G runs over the set $\mathcal{B}(4^{2n})$ of all 4-regular bipartite graphs of order $2n$, then the values of $f(G)$ completely cover a line segment $[a, b]$. We are able to obtain the extremal results for the forest number in the class $\mathcal{B}(4^{2n})$ in this section. Details can be found in Section 3.3.

A closely related graph parameter to f is that of the arboricity of graph which is defined as the concept of decomposing a graph into the minimum number of trees or forests. This concept was first introduced by Nash-Williams and Tutte. In 1961, Tutte [43] and Nash-Williams [26] independently gave a condition for when a graph could be decomposed into k forests. More precisely, there are two types of decomposing a graph G into forests, namely, vertices or edges, as we can see in the following definition.

For a graph G, it is always possible to partition $V(G)$ into subsets $V_i, 1 \leq i \leq k$, such that each induced subgraph $\langle V_i \rangle$ contains no cycle. The *vertex arboricity* $a(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. Thus $a(G) = 1$ if and only if G is a forest. For a few classes of graphs, the vertex arboricity is easily determined. For example, $a(C_n) = 2$. If n is even, then $a(K_n) = n/2$; while if n is odd, then $a(K_n) = (n+1)/2$. Also $a(K_{r,s}) = 1$ if $r = 1$ or $s = 1$ and $a(K_{r,s}) = 2$ otherwise. The *edge arboricity* or simply the *arboricity* $a_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic subgraph of G. As with vertex arboricity, a nonempty graph has arboricity 1 if and only if it is a forest.

We prove that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, then the values $a(G)$ and $a_1(G)$ completely cover a line segment $[x(\pi), y(\pi)]$ of positive integers where $\pi \in \{a, a_1\}$. Thus for an arbitrary graphical sequence $\mathbf{d}$ and $\pi \in \{a, a_1\}$, two invariants $x(\pi) := \min(\pi, \mathbf{d})$ and $y(\pi) := \max(\pi, \mathbf{d})$ naturally arise. We are able to obtain the extremal results for the arboricity in the class of $\mathcal{R}(r^n)$. The exact values of $\min(a, r^n)$ are found in all situations and $\max(a, r^n)$ for all $n \geq 2r + 2$. Details can be found in section 3.4.

3.1 Constrained Switchings in Cubic Graphs

3.1.1 Connected Subgraphs of $\Sigma(3^n)$-graph

In this section, we put $\mathcal{J}_1 = \mathcal{CR}(3^n)$, the class of connected cubic graphs of order n. Let $\mathcal{J}_2$ be the class of all connected cubic K_4'-free graphs of order n.

Finally, let $\mathcal{J}_3$ be the class of all connected cubic triangle-free graphs of order n. It is clear that $\mathcal{J}_3 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_1$. Let $X_n = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$.

A 2-connected cubic graph G may contain several 2-factors. If we choose one such 2-factor F of G, then $F' = (V(G), E(G) - E(F))$ is a 1-factor of G.

A connected cubic graph G which contains K_4' as a subgraph is not 2-connected. We first consider the class $\mathcal{J}_1 - \mathcal{J}_2$ separately. Observe further that $\mathcal{J}_1 - \mathcal{J}_2 \neq \emptyset$ if and only if n is even and $n \geq 10$.

Theorem 3.1.1 *Let $G_1, G_2 \in \mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2$. Then $G_1 = G_2$ or there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G_1 such that $G_1^{\sigma_1\sigma_2\ldots\sigma_t} = G_2$.*

Proof. Let $\mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2$ and $G_1, G_2 \in \mathcal{J}$. Since G_1 and G_2 contain K_4' as an induced subgraph, there exist $e_1 \in G_1$ and $e_2 \in G_2$ such that $G_1 - e_1 = G_1' \cup K_4'$ and $G_2 - e_2 = G_2' \cup K_4'$. The structure of graphs G_1 and G_2 are indicated in Figure 12.

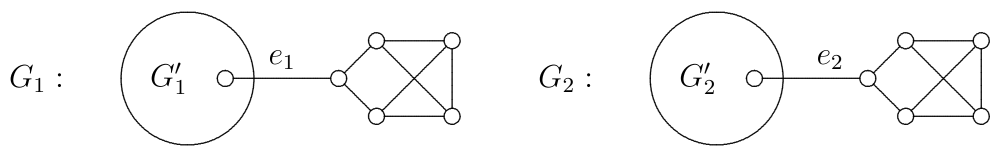


Figure 1: The graphs G_1 and G_2.

It turns out that the graphs G_1' and G_2' are connected and have the same degree sequence. Thus $G_1' = G_2'$ or by Theorem 1.4.3, there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $(G_1')^{\sigma_1\sigma_2\ldots\sigma_t} = G_2'$. The sequence of switchings can be considered as a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G_1 and $G_1^{\sigma_1\sigma_2\ldots\sigma_t} = G_2$. Thus the proof is complete. $\square$

Since the class of cubic graphs of order $n \leq 10$ is well-known [39], we will consider from now on only the connected cubic graphs of order n where $n \geq 12$. Let $\mathcal{J} \in X_n$ and $\mathcal{J} \neq \mathcal{J}_1 - \mathcal{J}_2$. We have the following theorems.

Theorem 3.1.2 *Let $\mathcal{J} \in X_n$, $\mathcal{J} \neq \mathcal{J}_1 - \mathcal{J}_2$, and $G \in \mathcal{J}$. If G contains a cut-vertex, then there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\ldots\sigma_t}$ is 2-connected.*

Proof. Let $G \in \mathcal{J}$ such that G is not 2-connected. Since G is cubic, G contains at least three blocks. Let S be the set of all cut-vertices of G and $v \in S$. Since $\deg_G v = 3$, there exists a component G_i of $G - v$ such that v is adjacent to exactly one vertex u of G_i and this vertex is also a cut-vertex of G. This situation is shown in Figure 13.

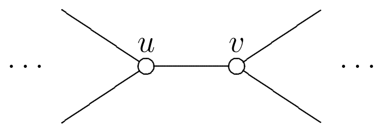


Figure 2: A part of a graph G.

Thus $|S| \geq 2$ and $|S| = 2$ if and only if G contains exactly three blocks. We will proceed by induction on the number of blocks, $b(G)$, of G. If $b(G) = 3$, then there exists a vertex-cut $S = \{u, v\}$ of G. Thus $uv \in E(G)$ and the blocks of G are B_1, B_2, and $B_3 = \langle\{u, v\}\rangle$ such that B_1 and B_2 contain u and v respectively. It is clear that B_1 and B_2 have order at least 5.

If $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$, then choose $e_1 = ab \in E(B_1)$ and $e_2 = cd \in E(B_2)$ such that e_1 is not adjacent to u and e_2 is not adjacent to v. Thus $\sigma(a, b; c, d)$ is a $\mathcal{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is 2-connected.

If $\mathcal{J} \in \{\mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$, then at least one B_i must contain a triangle as its subgraph. Suppose B_1 contains a triangle T. Choose $e_1 = ab \in E(B_1)$ and $e_2 = cd \in E(B_2)$ such that $e_1 \notin E(T)$. Thus $\sigma(a,b;c,d)$ is a $\mathcal{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is a 2-connected graph containing a triangle.

Let k be an integer greater than 2 and suppose that the theorem is true for all connected cubic graphs H with $b(H) < k$. Let G be a connected cubic graph with $b(G) = k$. Thus there exist two vertex-disjoint blocks B_1 and B_2 of G, each of which has order at least 5. We can analogously define the switching $\sigma_1 = \sigma(a,b;c,d)$ as described above and the resulting graph $G_1 = G^{\sigma_1}$ reduces the number of blocks of G by one. By induction, there exists a sequence of $\mathcal{J}$-switchings $\sigma_2, \sigma_3, \ldots, \sigma_t$ with respect to G_1 such that $G_1^{\sigma_2\sigma_3\ldots\sigma_t} = G^{\sigma_1\sigma_2\ldots\sigma_t}$ is 2-connected. This completes the proof. $\square$

Theorem 3.1.3 *Let $\mathcal{J} \in X_n$, $\mathcal{J} \neq \mathcal{J}_1 - \mathcal{J}_2$, and $G \in \mathcal{J}$. If G is 2-connected, then G is hamiltonian or there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\ldots\sigma_t}$ is hamiltonian.*

Proof. Let G be a 2-connected cubic graph. By Theorem 1.3.10, there is a 2-factor F of G. Let $c(F)$ be the number of cycles in F. Thus we may choose F of G with minimum $c(F)$. If $c(F) = 1$, then G is hamiltonian. Suppose $c(F) = 2$ and $F = C_1 \cup C_2$.

If $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$, then choose $e = ab \in E(C_1)$ and $f = cd \in E(C_2)$ such that $ac, bd \in E(G)$. Thus $\sigma(a,b;c,d)$ is a $\mathcal{J}$-switching with respect to G and $G^{\sigma(a,b;c,d)}$ is a hamiltonian graph.

If $\mathcal{J} \in \{\mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$, then at least one of the C_i's must contain three vertices which induce a triangle in G. Suppose C_1 contains three vertices which induce a triangle T and C_1 has order at least 4. Choose $e = ab \in E(C_1)$ and $f = cd \in E(C_2)$ such that $e \notin E(T)$. Thus $G^{\sigma(a,b;c,d)}$ is hamiltonian and contains a triangle. Suppose C_1 has order 3 and $V(C_1) = \{x, y, z\}$. Then C_2 has order at least 9. Let $\{x', y', z'\} \in V(C_2)$ such that $xx', yy', zz' \in E(G)$. Since G is not hamiltonian, $x'y', y'z', x'z' \notin E(C_2)$. Let $u \in V(C_2)$ and $ux' \in E(C_2)$. Thus there is a vertex $v \in V(C_2)$ such that $uv \in E(G)$. Since $x'y', x'z' \notin E(C_2)$, $u \notin \{y', z'\}$.

Case 1. Suppose that $y'v \notin E(G)$ or $z'v \notin E(G)$. Without loss of generality, we may assume that $y'v \notin E(G)$. Then $G^{\sigma(y,y';u,v)}$ is a hamiltonian graph containing a triangle. We have the situation shown in Figure 14.

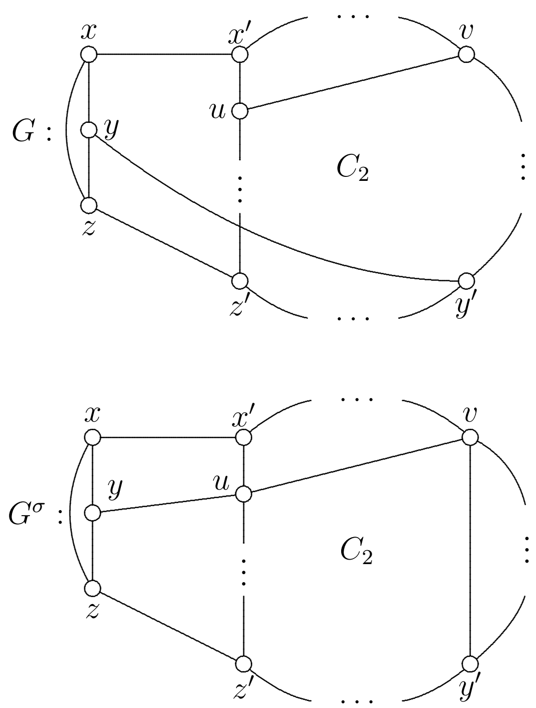
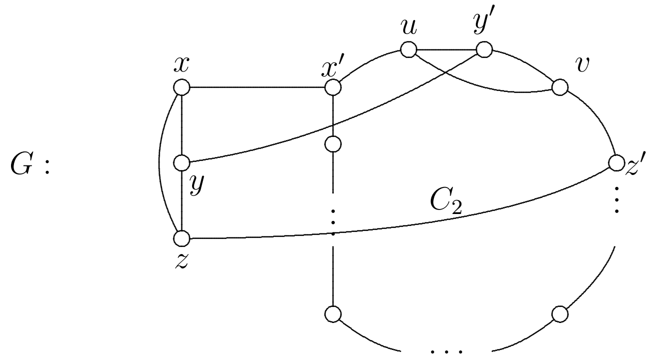


Figure 3: A part of the graphs G and G^σ in Case 1.

Case 2. Suppose that $y'v \in E(G)$ and $z'v \in E(G)$. Since $N(v) = \{u, y', z'\}$, there is at least one edge $e \in \{y'v, z'v\}$ satisfying $e \in E(C_2)$. Without loss of generality, we may assume that $z'v \in E(C_2)$. Therefore if $y'u \in E(G)$, then $G^{\sigma(x,x';u,v)}$ is a hamiltonian graph containing a triangle and if $y'u \notin E(G)$, then $G^{\sigma(y,y';v,u)}$ is a hamiltonian graph containing a triangle. We have the situation shown in Figures 15 and 16.

Suppose that $c(F) = k \geq 3$, we can analogously apply an appropriate $\mathcal{J}$-switching or a sequence of $\mathcal{J}$-switchings with respect to G to obtain a 2-connected cubic graph G' containing a 2-factor F' with $c(F') < c(F)$. This completes the proof. $\square$



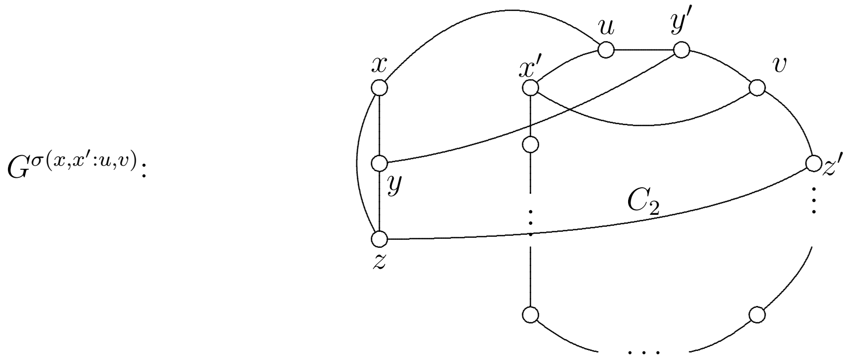


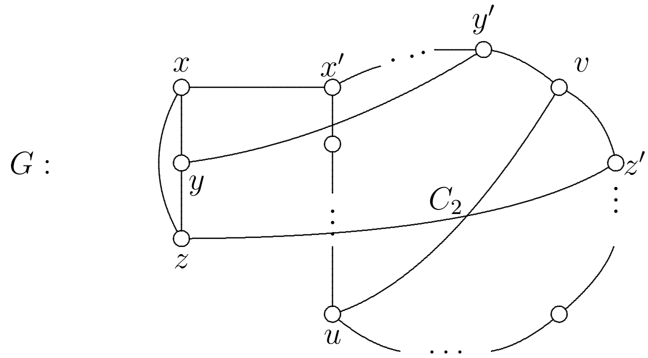
Figure 4: A part of the graphs G and G^σ in Case 2 when $y'u \in E(G)$.

Let $n \geq 6$ be an integer. Let $G(2n)$ be a graph of order $2n$ with $V(G(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(G(2n)) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\} \cup \{v_0v_n, v_1v_{n+1}\} \cup \{v_jv_{2n-j+1} : j = 2, 3, \ldots, n-1\}$. The structure of the graph $G(2n)$ is shown in Figure 17.

It is clear that $G(2n)$ is a cubic triangle-free graph containing a hamiltonian cycle C with $E(C) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\}$.

Let $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ and let $G \in \mathcal{J}$. Suppose further that G is hamiltonian. Up to isomorphism, we may assume that G contains C as its hamiltonian cycle. Thus the following sets are uniquely determined.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \ldots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and



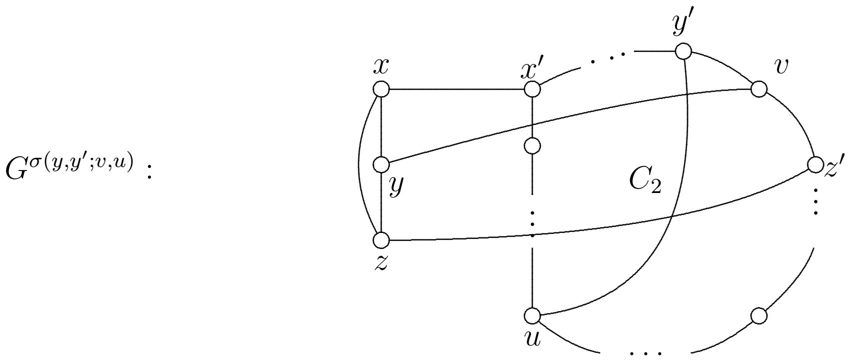


Figure 5: A part of the graphs G and G^σ in Case 2 when $y'u \notin E(G)$.

$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

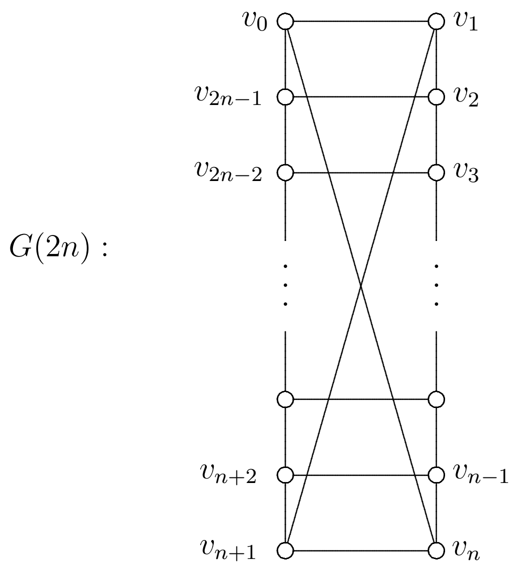


Figure 6: The graph $G(2n), n \geq 6$.

Since $e \in [L, R](G)$ if and only if $e \in [R, L](G)$, we have $[L, R](G) =$

G :

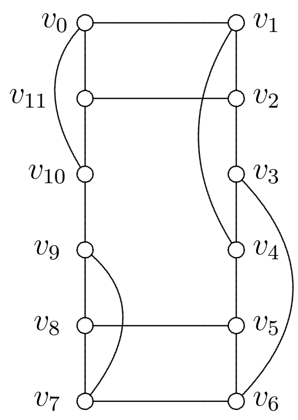


Figure 7: The cubic graph G of order 12.

$[R,L](G)$ and $|[L,L](G)| = |[R,R](G)|$. Let $e = v_iv_j \in [L,L](G)$. Then there exists $f = v_sv_t \in [R,R](G)$. If $i \in \{0, 2n-1\}$ and $s \in \{1,2\}$, then $\sigma_1 = \sigma_1(v_i, v_j; v_t, v_s)$ is a $\mathcal{J}$-switching with respect to G (See Figure 19). Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)|-1$. If $i \in \{n+1, n+2\}$ and $s \in \{n-1, n\}$, then $\sigma_1 = \sigma_1(v_i, v_j; v_t, v_s)$ is a $\mathcal{J}$-switching with respect to G. Thus $|[L,L](G^{\sigma_1})| = |[R,R](G^{\sigma_1})| = |[L,L](G)|-1$. In the other case, a switching $\sigma_1 = \sigma_1(v_i, v_j; v_s, v_t)$ is a $\mathcal{J}$-switching with respect to G. Thus the number of $[L,L](G^{\sigma_1})$ can be decreased by 2 from the number of $[L,L](G)$. With this observation, there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that $[R,R](G^{\sigma_1\sigma_2\ldots\sigma_t}) = [L,L](G^{\sigma_1\sigma_2\ldots\sigma_t}) = \emptyset$. For Figure 20, there exist two $\mathcal{J}$-switching $\sigma_1 = \sigma(v_0, v_8; v_6, v_1)$ and $\sigma_2 = \sigma(v_{12}, v_{10}; v_3, v_5)$ such that $[R,R](G^{\sigma_1\sigma_2}) = [L,L](G^{\sigma_1\sigma_2}) = \emptyset$.

Theorem 3.1.4 *Let $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$. If $G \in \mathcal{J}$ and $[R,R](G) = \emptyset$, then $G \cong G(2n)$ or there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\ldots\sigma_t} \cong G(2n)$.*

Proof. If $v_0v_n \notin E(G)$, then there exist $i \in \{3, 4, \ldots, n-1\}$ and $j \in \{n+3, n+4, \ldots, 2n-1\}$ such that $v_0v_i, v_nv_j \in E(G)$. Let $\sigma_1 = \sigma(v_0, v_i; v_n, v_j)$. Then σ_1 is a $\mathcal{J}$-switching with respect to G and $v_0v_n \in [L,R](G^{\sigma_1})$. Similarly if $v_1v_{n+1} \notin E(G^{\sigma_1})$, then there exists a $\mathcal{J}$-switching σ_2 with respect to $G_1 = G_1^{\sigma_1}$ such that $G_2 = G^{\sigma_1\sigma_2}$ and $v_0v_n, v_1v_{n+1} \in G_2$ (See Figure 21.).

If $v_2v_i \in E(G_2)$ and $i \neq 2n-1$, then there exists j, $3 \leq j \leq n-1$,

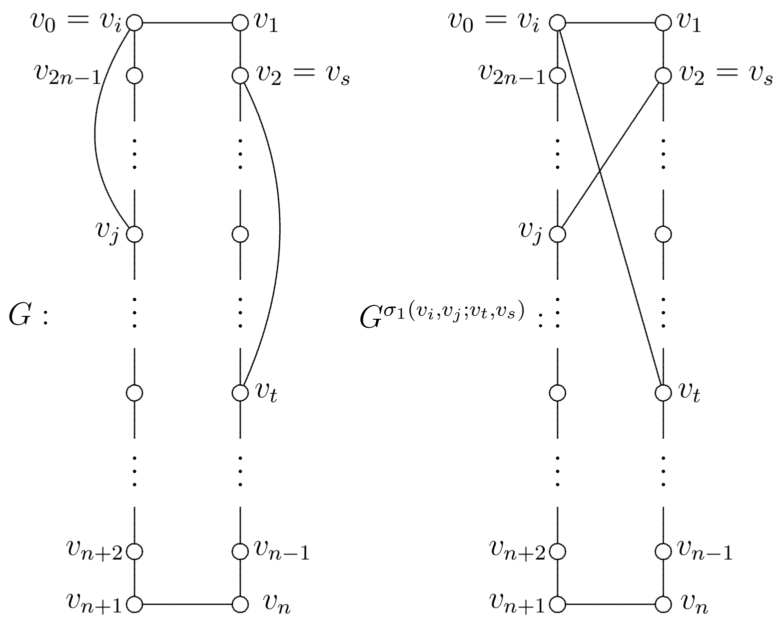


Figure 8: The graphs G and G^{σ} when $i = 0$ and $s = 2$.

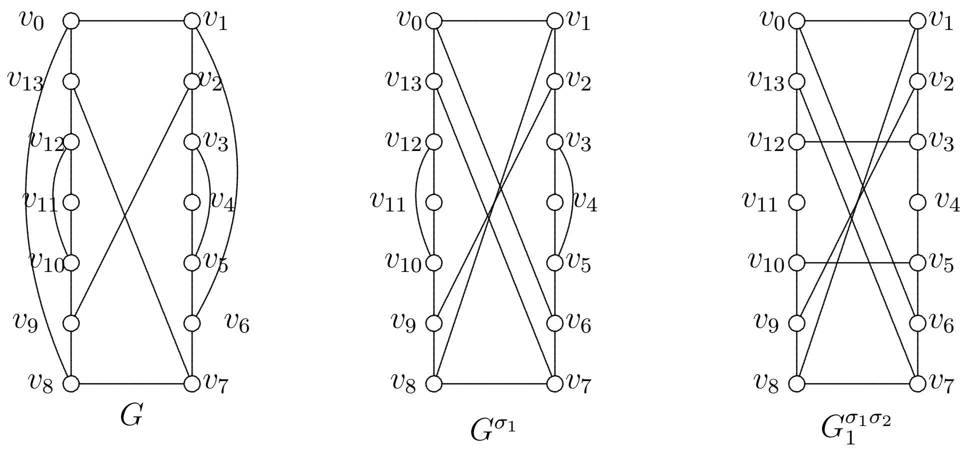


Figure 9: The graph G with $[R, R] \neq \emptyset$.

such that $v_{2n-1}v_j \in E(G_2)$. Thus there exists a $\mathcal{J}$-switching $\sigma_3 = \sigma(v_2, v_i; v_{2n-1}, v_j)$ with respect to G_2 such that $G_3 = G_2^{\sigma_3} = G_1^{\sigma_1\sigma_2\sigma_3}$ and $v_2v_{2n-1} \in E(G_3)$. Let j be the smallest integer in $\{3, 4, \ldots, n-1\}$ such that $v_jv_{2n-j+1} \notin E(G_3)$. Then there exist i and k such that $j < k < n$, $n+1 < i < 2n-j+1$ and $v_jv_i, v_kv_{2n-j+1} \in E(G_3)$. Thus $v_jv_{2n-j+1} \in E(G_4 = G_3^{\sigma_4})$, where $\sigma_4 = \sigma(v_j, v_i; v_{2n-j+1}, v_k)$. Similarly there exists a sequence of $\mathcal{J}$-switchings $\sigma_5, \sigma_6, \ldots, \sigma_t$ with respect to G_4 such that $G(2n) = G_4^{\sigma_5\sigma_6\ldots\sigma_t} = G^{\sigma_1\sigma_2\ldots\sigma_t}$. This completes the proof. $\square$

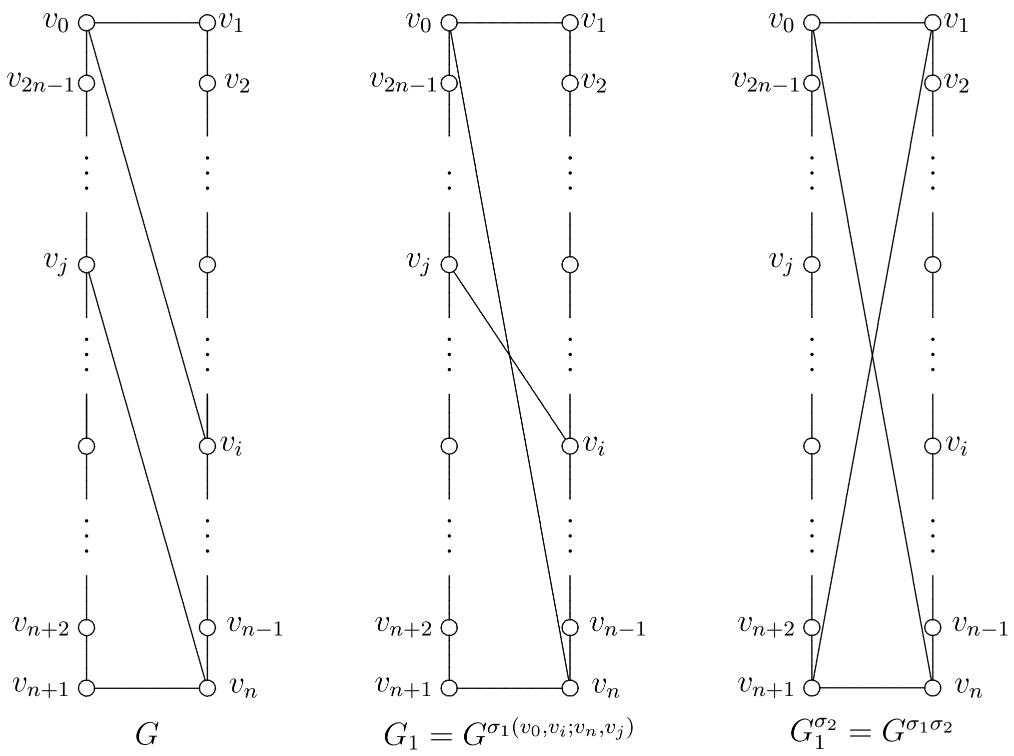


Figure 10: A step in the proof of Theorem 3.1.4.

By Theorem 3.1.4, we have the following theorem.

Theorem 3.1.5 *Let $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$. Then the subgraph of the $\Sigma(3^n)$-graph induced by $\mathcal{J}$ is connected.*

Let $n \geq 6$ be an integer. Let $H(2n)$ be a graph of order $2n$ with $V(H(2n)) = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $E(H(2n)) = \{v_i v_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\} \cup \{v_0v_2, v_1v_{2n-1}\} \cup \{v_{n-1}v_{n+1}, v_nv_{n+2}\} \cup \{v_jv_{2n-j+1} : j = 3, 4, \ldots, n-2\}$. The structure of a graph $H(2n)$ is shown in Figure 22.

It is clear that $H(2n)$ is a cubic hamiltonian graph of order $2n$ containing a triangle. Again put C as its hamiltonian cycle with $E(C) = \{v_iv_{i+1} : i = 0, 1, \ldots, 2n-1 \pmod{2n}\}$.

Let $\mathcal{J} \in \{\mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$ and let $G \in \mathcal{J}$. Suppose further that G is hamiltonian. Up to isomorphism, we may suppose that G contains C as its hamiltonian cycle. Thus the following sets are uniquely determined.

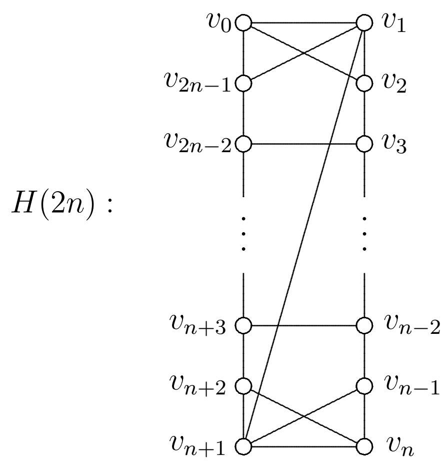


Figure 11: The graph $H(2n), n \geq 6$.

$RV(G) = \{v_i \in V(G) : i = 1, 2, 3, \dots, n\}$,
$LV(G) = V(G) - RV(G)$,
$[L, R](G) = \{uv \in E(G) : u \in LV(G) \text{ and } v \in RV(G)\} - E(C)$,
$[R, L](G) = \{uv \in E(G) : u \in RV(G) \text{ and } v \in LV(G)\} - E(C)$,
$[L, L](G) = \{uv \in E(G) : u, v \in L(G)\} - E(C)$, and
$[R, R](G) = \{uv \in E(G) : u, v \in RV(G)\} - E(C)$.

Since $e \in [L, R](G)$ if and only if $e \in [R, L](G)$, we have $[L, R](G) = [R, L](G)$ and $|[L, L](G)| = |[R, R](G)|$. Suppose that $v_0v_2 \notin E(G)$. Thus there exist i, j such that $v_0v_i, v_2v_j \in E(G)$ and $G_1 = G^{\sigma(v_0, v_i; v_2, v_j)}$ contains v_0v_2 as its edge. By using the same argument as described previously, there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \dots, \sigma_t$ with respect to G such that $G_t = G^{\sigma_1\sigma_2\dots\sigma_t}$ and $[L, L](G_t) = \emptyset$. Again by using the same argument as described in the proof of Theorem 3.1.4, we have the following theorem.

Theorem 3.1.6 *Let $\mathcal{J} \in \{\mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$. If $G \in \mathcal{J}$ and $[R, R](G) = \emptyset$, then $G \cong H(2n)$ or there exists a sequence of $\mathcal{J}$-switchings $\sigma_1, \sigma_2, \dots, \sigma_t$ with respect to G such that $G^{\sigma_1\sigma_2\dots\sigma_t} \cong H(2n)$.*

By Theorem 3.1.6, we have the following theorem.

Theorem 3.1.7 *Let $\mathcal{J} \in \{\mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$. Then the subgraph of the $\Sigma(3^n)$-graph induced by $\mathcal{J}$ is connected.*

Combining results in this subsection we can conclude the following theorem.

Theorem 3.1.8 *Let $\mathcal{J} \in X_n$. Then the subgraph of the $\Sigma(3^n)$-graph induced by $\mathcal{J}$ is connected.*

3.1.2 Interpolation and Extremal Results

If G is a graph and σ is a switching on G, then by Theorem 2.1.5, $|f(G) - f(G^\sigma)| \leq 1$. With this result and the results in Subsection 3.1.1 we have the following theorem.

Theorem 3.1.9 *Let $\mathcal{J} \in X_n$. Then f is an interpolation graph parameter over $\mathcal{J}$.*

We now answer the second part of the interpolation theorem. In other words, we will find $\min(f, \mathcal{J})$ and $\max(f, \mathcal{J})$ for all $\mathcal{J} \in X_n$.

Let G be a graph and X, Y be disjoint nonempty subsets of $V(G)$. Denote by $e(X)$ the number of edges in the subgraph $\langle X \rangle$ of G and by $e(X, Y)$ the number of edges in G connecting vertices in X to vertices in Y.

Let G be a cubic graph of order n and F be a maximum induced forest of G. Let $|F| = a$ and therefore $G - F$ has order $n - a$. An upper bound of $\max(f, \mathcal{J}_1)$ can be obtained by the following obvious identities.

$$\frac{3n}{2} = |E(G)| = e(F) + e(F, V(G - F)) + e(V(G - F)) \text{ and}$$

$$3(n - f) = e(F, V(G - F)) + 2e(V(G - F)).$$

It follows that $\frac{3n}{2} = e(F) + 3(n - a) - e(V(G - F)) \leq a - 1 + 3(n - a)$. Therefore $a \leq \frac{3n-2}{4}$.

Since $\mathcal{J}_3 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_1$, $\max(f, \mathcal{J}_3) \leq \max(f, \mathcal{J}_2) \leq \max(f, \mathcal{J}_1) \leq \frac{3n-2}{4}$.

We will show in the next theorem that $\max(f, \mathcal{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$ by constructing a connected cubic triangle-free graph G of order n with $f(G) = \lfloor \frac{3n-2}{4} \rfloor$. Note that several classes of graphs can be constructed to reach the bound even if we restrict the graph to be planar and $n \geq 8$. We now turn to consider the graph $G(2n)$ as we have constructed in Subsection 3.1.1. It is clear that $G(2n)$ is a connected cubic triangle-free graph of order $2n$ if $n \geq 3$. We have the following theorem.

Theorem 3.1.10 *If $n = 2m \geq 6$, then $f(G(2m)) = \lfloor \frac{3n-2}{4} \rfloor$.*

Proof. Let $G = G(2m) - \{v_0v_m, v_1v_{m+1}\}$ and $S = \{v_1, v_3, v_5, \ldots, v_m\}$ if m is odd, and $S = \{v_1, v_3, v_5, \ldots, v_{m-1}, v_m\}$ if m is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. Since $v_1, v_m \in S$, adding edges v_0v_m, v_1v_{m+1} to G will not create any cycles in F. Therefore $f(G(2m)) = \lfloor \frac{3m-1}{2} \rfloor = \lfloor \frac{3n-2}{4} \rfloor$. $\square$

Corollary 3.1.11 $\max(f, \mathcal{J}_3) = \max(f, \mathcal{J}_2) = \max(f, \mathcal{J}_1) = \lfloor \frac{3n-2}{4} \rfloor$.

Let G be a connected cubic graph of order n containing K_4' as a subgraph. Put $G' = G - K_4'$. Thus G' has order $n - 5$ and size $\frac{3n}{2} - 8$. Moreover $f(G) = f(G') + f(K_4') = f(G') + 3$. Let F' be a maximum induced forest of G' and $|F'| = a'$. We have $\frac{3n}{2} - 8 = e(V(G')) \leq (a' - 1) + 3(n - 5 - a') = 3n - 16 - 2a'$.

It follows that $a' \leq \frac{3n-16}{4}$ and $f(G) \leq \frac{3n-16}{4} + 3 = \frac{3n-4}{4}$.

Theorem 3.1.12 $\max(f, \mathcal{J}_1 - \mathcal{J}_2) = \lfloor \frac{3n-4}{4} \rfloor$.

Proof. It is clear that $\mathcal{J}_1 - \mathcal{J}_2 \neq \emptyset$ if and only if n is even and $n \geq 10$. If $n = 10$, we can take G as the graph obtained from two copies of K_4' and joining the two vertices of degree 2 as shown in Figure 23.

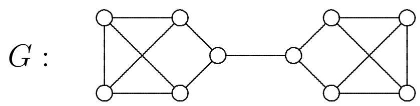


Figure 12: The graph G obtained from two copies of K_4'.

Let $n - 6 = 2m \geq 6$ and $H = G(2m)$. Thus by Theorem 3.1.10, we have $f(H) = \lfloor \frac{3(n-6)-2}{4} \rfloor$. In the proof of Theorem 3.1.10, let $e = v_0v_1 \in E(H)$ and define a graph K with $V(K) = V(H) \cup \{x\}$ and $E(K) = (E(H) - e) \cup \{v_0x, xv_1\}$ and finally let G be the graph obtained from K and a copy of K_4' by joining the two vertices of degree 2. The structure of such a graph G is shown in Figure 24. Thus $f(G) = \lfloor \frac{3(n-6)-2}{4} \rfloor + 1 + 3 = \lfloor \frac{3(n-6)-2}{4} \rfloor + 4 = \lfloor \frac{3n-4}{4} \rfloor$. $\square$

$G:$

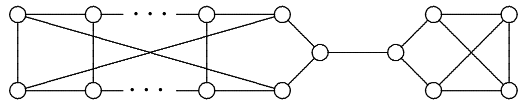


Figure 13: A step in the proof of Theorem 3.1.12.

Since $\mathcal{J}_2 - \mathcal{J}_3 \subseteq \mathcal{J}_1 - \mathcal{J}_3$, $\max(f, \mathcal{J}_2 - \mathcal{J}_3) \leq \max(f, \mathcal{J}_1 - \mathcal{J}_3) \leq \lfloor \frac{3n-2}{4} \rfloor$.

Theorem 3.1.13 $\max(f, \mathcal{J}_2 - \mathcal{J}_3) = \max(f, \mathcal{J}_1 - \mathcal{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$.

Proof. Let m be an integer such that $n = 2m$. The graph $H(2m)$ contains a triangle. Let $S = \{v_1, v_3, \ldots, v_m\}$ if m is odd and $S = \{v_1, v_3, \ldots, v_{m-1}, v_m\}$ if m is even. Thus $F = G - S$ is a tree of order $\lfloor \frac{3n-2}{4} \rfloor$. Thus $\max(f, \mathcal{J}_2 - \mathcal{J}_3) = \max(f, \mathcal{J}_1 - \mathcal{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$. $\square$

Thus the values of $\max(f, \mathcal{J})$ are obtained for all $\mathcal{J} \in X_n$.

The problem of finding lower bounds of $f(G)$, where G runs over a class of cubic graphs, have been investigated in the literature. First observe that if G is a cubic graph of order n and F is a maximum induced forest of G, then $G - F$ is a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if n is a multiple of 4. By Theorem 2.2.10, if $n = 4q + t,\ t = 0, 2$, then $\min(f, \mathcal{R}(3^n)) = 2q + t$. Liu and Zhao [25] proved that $f(G) \geq \frac{5}{8}n - \frac{1}{4}$ for any connected cubic graph G of order n. This means that for any connected cubic graph of order n, the bound on the order of a maximum induced forest can be improved from 50 percent of the number of vertices to 60 percent.

By Theorem 2.2.14, we know that for an even integer $n \geq 12$,

$$\min(f, \mathcal{J}_1) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 \pmod 8, \\ \lceil \frac{5}{8}n \rceil & otherwise. \end{cases}$$

Note that for a graph G that has been constructed in such a way that $f(G) = \min(f, \mathcal{J}_1)$, G must contain K_4' as a subgraph. Thus we have the following corollary.

Corollary 3.1.14 *Let n be an even integer with $n \geq 12$. If $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3\}$, then*

$$\min(f, \mathcal{J}) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 \pmod 8), \\ \lceil \frac{5}{8}n \rceil & otherwise. \end{cases}$$

Zheng and Lu [45] proved that $f(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without any triangles, except for two cubic graphs with $n = 8$ and $f(G) = 5$. This means that if we look at the class of connected cubic graphs of order n containing no triangle, then the bound can be improved from 60 percent of the number of vertices to 66 percent.

It is easy to see that there exists a cubic graph G of order n containing triangles with $f(G) \geq \frac{2n}{3}$.

We are able to extend the result of Zheng and Lu [45] by proving that $f(G) \geq \frac{2n}{3}$ for any connected cubic K_4'-free graph G of order $n \geq 10$.

Five connected cubic graphs of order 8 are given in [39], all of which have maximum induced forests of order 5. Alon, Mubayi and Thomas [1] proved that if G is a $\{K_4, K_4'\}$-free graph of order n and size m with maximum degree 3, then $f(G) \geq n - \frac{m}{4}$. Consequently, if G is a connected cubic K_4'-free graph of order $n \geq 10$, then $f(G) \geq \frac{5n}{8}$.

Observe that if G is a connected K_4'-free graph of order 8 and $\Delta(G) = 3$, then $f(G) \geq 5$, and $f(G) = 5$ if and only if G is a cubic graph.

Theorem 3.1.15 *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n with $\Delta(G) = 3$. Then $f(G) \geq \frac{2n}{3}$.*

Proof. Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n. By calculation we found that $\lceil \frac{5n}{8} \rceil = \lceil \frac{2n}{3} \rceil$ for all n with $4 \leq n \leq 10$ and $n \neq 8$. For $n = 8$ we also have $f(G) \geq \frac{2n}{3}$. Thus the theorem holds for all $4 \leq n \leq 10$. Now suppose $n \geq 11$ and G contains a triangle T with $V(T) = \{x, y, z\}$. If there exists a vertex in $V(T)$, say x, such that $\deg_G x = 2$, then by induction on n there exists a maximum induced forest F_1 of $G - T$ with $|F_1| \geq \frac{2(n-3)}{3}$. Hence $F = F_1 \cup \{x, y\}$ is an induced forest of G and $|F| \geq \frac{2n}{3}$. Suppose that for all triangles $T = \{x, y, z\}$ of G, $\deg_G x = \deg_G y = \deg_G z = 3$. Since G is a K_4-free graph, $|N(T)| \geq 2$.

Case 1. Suppose x and y have a common neighbor u and v is a neighbor of z. Since G is a K_4'-free graph, u and v are not adjacent in G. Thus by induction on n, $G-T$ contains an induced forest of order at least $\frac{2(n-3)}{3}$. Since $\deg_{G-T} u = 1$, any maximum induced forest of $G-T$ must contain u. If there is a maximum induced forest F_1 of $G-T$ which does not contain a $u-v$ path, then $F_1 \cup \{y,z\}$ is an induced forest of G of order at least $\frac{2n}{3}$. Suppose for any maximum induced forest F_1 of $G-T$, F_1 contains a $u-v$ path. Since $G' = G-T+uv$ satisfies the conditions of the theorem, there is a maximum induced forest F' of G' of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{y,z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F')$, then $F = (F'-uv) \cup \{y,z\}$ is a maximum induced forest of G of order at least $\frac{2n}{3}$.

Case 2. Suppose x, y, z have different neighbors u, v and w respectively. Since $n \geq 11$, the subgraph $\langle \{u,v,w\} \rangle$ of G is not a triangle. Thus there exist two vertices in $\{u,v,w\}$, say u, v, such that $uv \notin E(G)$. We consider the following subcases.

Subcase 2.1 Suppose that $G-T+uv$ contains a K_4. Let H be the copy of K_4 in $G-T+uv$. Thus $uv \in E(H)$ and $vw \notin E(G)$. Since $G' = G-T-V(H)$ satisfies the conditions of the theorem, $f(G') \geq \frac{2(n-7)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 3$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, $f(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-7)}{3} + 2 + 3 = \frac{2n+1}{3} \geq \frac{2n}{3}$.

Subcase 2.2 Suppose that $G-T+uv$ contains K_4'. Let H be the copy of K_4' in $G-T+uv$. Thus $uv \in E(H)$. Since $G' = G-T-V(H)-\{w\}$ satisfies the conditions of the theorem, $f(G') \geq \frac{2(n-9)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 4$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, we have $f(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-9)}{3} + 2 + 4 = \frac{2n}{3}$.

Subcase 2.3 Suppose that $G-T+uv$ contains a cubic graph H of order 8. Thus $uv \in E(H)$. Since $G' = G-T-V(H)$ satisfies the conditions of the theorem, $f(G') \geq \frac{2(n-11)}{3}$. Let F' be a maximum induced forest of G' and F_1 be a maximum induced forest of $H-uv$. Since $|F_1| = 6$ and there exists an induced forest $F = F' \cup \{y,z\} \cup F_1$ of G, $f(G) \geq |F| = |F' \cup \{y,z\} \cup F_1| \geq \frac{2(n-11)}{3} + 2 + 6 = \frac{2n+2}{3} > \frac{2n}{3}$.

Subcase 2.4 Suppose that $G - T + uv$ satisfies the conditions of the theorem. Thus there exists a maximum induced forest F' of $G - T$ of order at least $\frac{2(n-3)}{3}$. If $uv \notin E(F')$, then $F = F' \cup \{x,y\}$ is an induced forest of G of order at least $\frac{2n}{3}$. If $uv \in E(F)$, then $F = (F' - uv) \cup \{x,y\}$ is an induced forest of G of order at least $\frac{2n}{3}$. Thus the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.1.15.

Corollary 3.1.16 *Let G be a connected cubic K_4'-free graph of order $n \neq 8$. Then $f(G) \geq \frac{2n}{3}$.*

According to the results of Zheng and Lu [45], $f(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$. They claim that the lower bound is best possible but no proof of this was given in their paper.

We now construct a class of graphs to show that $\min(f, \mathcal{J}_3) = \min(f, \mathcal{J}_2) = \lceil \frac{2n}{3} \rceil$.

First observe the following facts:

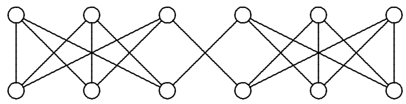
1. $f(K_{3,3}) = 4$.
2. $f(Q_3) = 5$, where Q_3 is the 3-cube.
3. There is a switching σ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $f((2K_4')^\sigma) = 7$. Let $K = (2K_4')^\sigma$.
4. If $e_1 \in E(K_{3,3})$ and $e_2 \in E(Q_3)$, then $f(K_{3,3} - e_1) = 4$ and $f(Q_3 - e_2) = 6$. Let $P = K_{3,3} - e_1$ and $Q = Q_3 - e_2$.
5. Let n be an even integer with $n \geq 12$. Write $n = 6q + t$, $t = 0, 2, 4$ and construct a connected cubic triangle-free graph according to the value of t:
   (a) If $t = 0$, construct a graph G of order $6q$ by taking q copies of P and joining q appropriate edges between the q copies of P.

(b) If $t = 2$, construct a graph G of order $6q + 2$ by taking $q - 1$ copies of P and a copy of Q and then joining q appropriate edges between them.

(c) If $t = 4$, construct a graph G of order $6q + 4$ by taking $q - 1$ copies of P and a copy of K and then joining q appropriate edges between them.

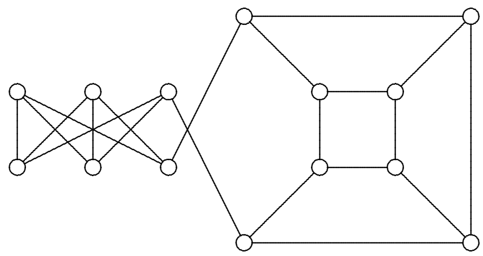
It is easy to check that the graphs G constructed above satisfy $f(G) = \lceil \frac{2n}{3} \rceil$. Thus we have the following theorem. Figure 25 shows the cubic graphs of order $n = 12, 14, 16$, respectively, in which $f(G) = \lceil \frac{2n}{3} \rceil$.

G_1 :



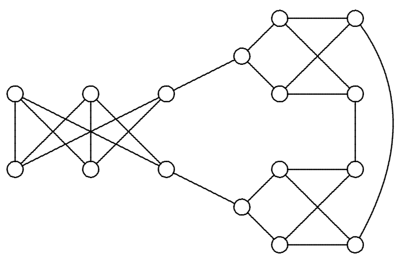
G_1 of order 12 and $f(G_1) = 8$.

G_2 :



G_2 of order 14 and $f(G_2) = 10$.

G_3 :



G_3 of order 16 and $f(G_3) = 11$.

Figure 14: The cubic graphs of order $n = 12, 14, 16$ in which $f(G) = \lceil \frac{2n}{3} \rceil$.

Theorem 3.1.17 $\min(f, \mathcal{J}_2) = \min(f, \mathcal{J}_3) = \min(f, \mathcal{J}_2 - \mathcal{J}_3) = \lceil \frac{2n}{3} \rceil$.

3.2 The Forest Number of (n, m)-Graphs

We now find for the extremal values of f for (n, m)-graphs. Note that f is an interpolation graph parameter over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$. We now answer the second part of the interpolation theorem.

Let $G \in \mathcal{G}(n, m)$ and F be a maximum induced forest of G. Let $|F| = a$. Therefore $G - F$ has order $n - a$. An upper bound of m can be obtained by the following inequality:

$$m = e(G - F) + e(G - F, F) + e(F) \leq \binom{n-a}{2} + a(n - a) + (a - 1).$$

Let $a = n - i$ for any $i \in \{1, 2, \ldots, n - 2\}$. We get $m \leq (i + 1)n - \frac{i^2+3i+2}{2}$.

For an integer $i = 1, 2, \ldots, n - 2$, let $\mathbf{M}_n(n - i) = (i + 1)n - \frac{i^2+3i+2}{2}$. It is clear that $\mathbf{M}_n(n - i)$ is an integer. We can show that $\max(f; n, m) = n - i$ if and only if $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$ by constructing a graph $G \in \mathcal{G}(n, m)$ with $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$ and $f(G) = n - i$ as follows.

Let G be a graph with $V(G) = X \cup Y$ where $X = \{v_1, v_2, \ldots, v_{n-i}\}$, $Y = \{u_1, u_2, \ldots, u_i\}$ and $E(G) = \{v_j v_{j+1} : 1 \leq j \leq n - i - 1\} \cup \{uv : u, v \in Y\} \cup \{u_j v_k : 1 \leq j \leq i - 1, 1 \leq k \leq n - i\} \cup \{u_i v_k : 1 \leq k \leq m - \mathbf{M}_n(n - i + 1) + 1\}$. The structure of a graph G when $n = 12$ and $i = 3$ is shown in Figure 26. It is easy to see that the induced subgraph $\langle X \rangle$ of G is a maximum induced forest.

It is easy to check that $f(G) = n - i$ and $G \in \mathcal{CG}(n, m)$. Thus we have the following theorem.

Theorem 3.2.1 *Let n and m be integers satisfying $0 \leq m \leq \binom{n}{2}$. Then $\max(f; n, m) = n - i$ if and only if $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$, and $\mathrm{Max}(f; n, m) = n - i$ if and only if $m \geq n - 1$ and $\mathbf{M}_n(n - i + 1) < m \leq \mathbf{M}_n(n - i)$.*

The problem of finding $\min(f; n, m)$ is difficult and finding $\mathrm{Min}(f; n, m)$ is even more difficult. In order to obtain the values of $\min(f; n, m)$, we first find the minimum number of edges of a graph order n having the forest number a. Let $\mathcal{G}(n; f = a)$ be the set of graphs of order n having the forest number a. It is clear

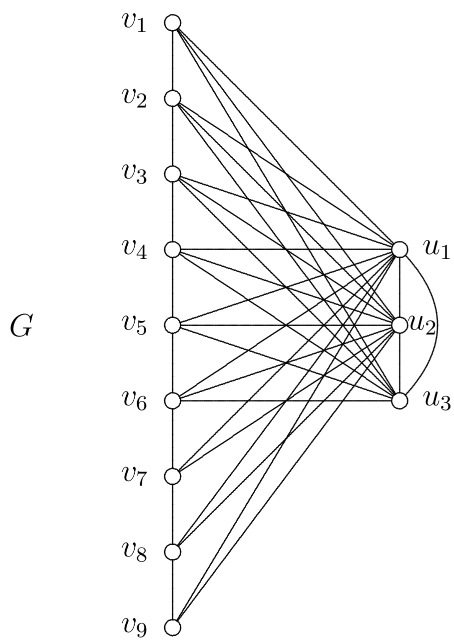


Figure 15: The graph G of order 12, size 35 and $f(G) = 9$.

that $\mathcal{G}(n; f = a) \neq \emptyset$ if and only if $2 \leq a \leq n$. For integers n and a, let

$$\mathbf{m}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{G}(n; f = a)\}.$$

Further, $\mathbf{m}_n(n) = 0,\ \mathbf{m}_n(n-1) = 3$ and $\mathbf{m}_n(2) = \binom{n}{2}$. In Figure 27, G_1, G_2 and G_3 are the graphs with $\varepsilon(G_1) = \mathbf{m}_5(5), \varepsilon(G_2) = \mathbf{m}_5(4)$ and $\varepsilon(G_1) = \mathbf{m}_5(2)$. It is easy to see that for a graph G of order $n \geq 2$, $f(G) = 2$ if and only if $G \cong K_n$.

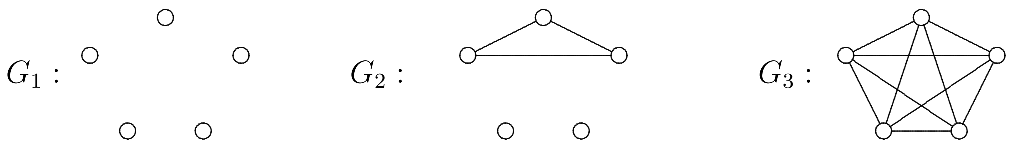


Figure 16: The graphs of order 5 and size $\mathbf{m}_5(5), \mathbf{m}_5(4)$ and $\mathbf{m}_5(2)$.

We now find $\mathbf{m}_n(a)$ for $2 < a < n$. The following lemma which was proved in [28] provides a characterization of graphs with forest number 3.

Lemma 3.2.2 *Let G be a graph of order $n \geq 3$ and $G \ncong K_n$. Then $f(G) = 3$ if and only if $\overline{G}$ is a union of stars.*

Now we have $\mathbf{m}_n(3) = \binom{n}{2} - n + 1$, for all $n \geq 4$. In 2005, Punnim [33] prove the following lemma.

Lemma 3.2.3 *If G is a graph of order n with maximum degree $\Delta(G) = \Delta$, then $f(G) \geq \frac{2n}{\Delta+1}$.*

Lemma 3.2.4 *If G is a graph of order n with $\Delta(G) = \Delta$ and $f(G) = 2q+1$ for some integer q, then $n \leq (\Delta+1)q+1$.*

Proof. We will proceed by induction on q. Suppose that $n = (\Delta+1)q+t$ for some integer $t \geq 2$. Since $f(G) = 2q+1$, it follows by Lemma 3.2.3 that $2 \leq t \leq \Delta$. Suppose that $q = 1$, that is $n = \Delta+1+t$. Thus $\delta(\overline{G}) = t \geq 2$ and hence $\overline{G}$ is not a union of stars. By Lemma 3.2.2, $f(G) \geq 4$. Therefore the result holds for $q = 1$. Suppose that there exists a graph G of order $n = (\Delta+1)q+t$ with $f(G) = 2q+1, q \geq 2$ as small as possible. Let v be a vertex of G of degree Δ and let H be the graph obtained from G by deleting v and its neighbors. Thus H has order $(\Delta+1)(q-1)+t$ and by minimality of q, there exists a maximum induced forest F in H of order at least $2(q-1)+2 = 2q$. Since $F_v = F \cup \{v\}$ is an induced forest of G and $|F_v| = |F|+1$, it follows that $|F| = 2q$ and F_v is a maximum induced forest in G of order $2q+1$. Note that if T is a maximum induced forest in a graph G and $v \in V(G-T)$, then there exists a nontrivial connected component C of T such that v is adjacent to at least 2 vertices in C. Since F_v contains at most $2q$ vertices of degree at least 1 and $G - F_v$ has order $(\Delta-1)q+t-1$, there are at least $2(\Delta-1)q+2(t-1)$ edges from $G - F_v$ to the nontrivial components of F_v. Since F_v has at most $2q$ vertices of degree at least 1 and $\frac{2(\Delta-1)q+2(t-1)}{2q} > \Delta-1$, it follows that there exists a vertex in F_v of degree at least $\Delta+1$. Thus we get a contradiction. □

By Lemma 3.2.3, we have a lower bound on the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n, then $\Delta(G) \geq \lceil \frac{2n}{f(G)} \rceil - 1$. In particular, if $f(G) = 2q$ for some integer q, then $\Delta(G) \geq \lceil \frac{n}{q} \rceil - 1$. By Lemma 3.2.4 the lower bound of $\Delta(G)$ can be improved if $f(G)$ is odd. That is, if $f(G) = 2q+1$ for some integer q, then $n \leq (\Delta(G)+1)q+1$, which is equivalent to $\Delta(G) \geq \lceil \frac{n-1}{q} \rceil - 1$. We have the following corollary.

Corollary 3.2.5 *Let G be a graph of order n and q be a positive integer. If $f(G) = 2q$, then $\Delta(G) \geq \lceil \frac{n}{q} \rceil - 1$, and if $f(G) = 2q+1$, then $\Delta(G) \geq \lceil \frac{n-1}{q} \rceil - 1$.*

Let $\mathcal{G}^*(n; f = a) = \{G \in \mathcal{G}(n; f = a) : G$ is a union of $\lceil \frac{a}{2} \rceil$ cliques$\}$. It is clear that $\mathcal{G}^*(n; f = a) \subset \mathcal{G}(n; f = a)$. In Figure 28, $G = 3K_3 \cup K_1$ is a union of 4 cliques and $f(G) = 7$. We have the following theorem.

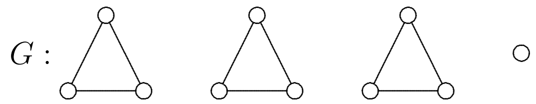


Figure 17: The graph G of order 10, size 9, and $f(G) = 7$.

Theorem 3.2.6 *Let G be a graph of order n with $f(G) = a$. Then there exists a graph $H \in \mathcal{G}^*(n; f = a)$ such that $\varepsilon(H) \leq \varepsilon(G)$.*

Proof. We will proceed by induction on n. The result holds for $n = 1$. Suppose that $n \geq 2$. If there exists a vertex v of G such that $f(G - v) = f(G) - 1 = a - 1$, then, by induction, there exists a graph $H' \in \mathcal{G}^*(n - 1; f = a - 1)$ such that $\varepsilon(H') \leq \varepsilon(G - v)$. Let $H = H' \cup \{v\}$. Thus $H \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) = \varepsilon(H') \leq \varepsilon(G - v) \leq \varepsilon(G)$. Suppose that for every vertex v of G, $f(G - v) = f(G) = a$. Let v be a vertex of G of degree $\Delta(G)$. Thus, by induction, there exists a graph $H' \in \mathcal{G}^*(n - 1; f = a)$ such that $\varepsilon(H') \leq \varepsilon(G - v)$. We consider two cases.

Case 1. Suppose that $f = 2q$ for some integer q. Thus H' is a union of at least q cliques. Therefore there exists a component C' of H' such that C' is a clique of order at most $\lfloor \frac{n-1}{q} \rfloor$. By Corollary 3.2.5, $d_G(v) \geq \lceil \frac{n}{q} \rceil - 1$. Note that $\lceil \frac{n}{q} \rceil - 1 = \lfloor \frac{n-1}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-1}{q} \rfloor$ edges from v to C'. Then $H = H' * C \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) \leq \varepsilon(H') + \lfloor \frac{n-1}{q} \rfloor \leq \varepsilon(G - v) + d_G(v) \leq \varepsilon(G)$, as required.

Case 2. Suppose that $n \geq 4$ and $f = 2q + 1$ for some integer q. Thus H' is a union of at least $q + 1$ cliques one of which is an isolated vertex. Therefore there exists a component C' of H' such that C' is a nontrivial clique of order at most $\lfloor \frac{n-2}{q} \rfloor$. By Corollary 3.2.5, $d_G(v) \geq \lceil \frac{n-1}{q} \rceil - 1$. Note that $\lceil \frac{n-1}{q} \rceil - 1 = \lfloor \frac{n-2}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-2}{q} \rfloor$ edges form v to C'. Then $H = H' * C \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) \leq \varepsilon(H') + \lfloor \frac{n-2}{q} \rfloor \leq \varepsilon(G - v) + d_G(v) \leq \varepsilon(G)$, as required. $\square$

By Theorem 3.2.6, we know the structure of graphs of order n with prescribed forest numbers. In general, for a graph $G \in \mathcal{G}(n; f = a)$, there may be many such graphs $H \in \mathcal{G}^*(n; f = a)$. We now look for such a graph H with a minimum number of edges.

It is easy to see that for any integer $n \geq 6$, $\mathbf{m}_n(4) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ and for $n \geq 7$, $\mathbf{m}_n(5) = \binom{\lceil \frac{n-1}{2} \rceil}{2} + \binom{\lfloor \frac{n-1}{2} \rfloor}{2}$.

Theorem 3.2.7 is the famous result of Turán [42] which is viewed as the origin of extremal graph theory.

The *Turán graph* $T_{n,r}$ is the complete r-partite graph of order n whose partite sets differ in size by at most 1. It is clear that the complement of the Turán graph $T_{n,r}$ is a union of r cliques.

Theorem 3.2.7 *Among the graphs of order n containing no complete subgraph of order $r+1$, $T_{n,r}$ has the maximum number of edges.*

By Theorem 3.2.7, we know that among the graphs of order n which is a union of r cliques, the complement of the Turán graph $T_{n,r}$ has the minimum number of edges.

Turán's Theorem can be applied for finding $\mathbf{m}_n(a)$ for $4 \leq a \leq n-1$. We first establish certain facts and notation.

1. Let $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \ldots \cup p_kK_k$. Then the order of G is $p_1 + 3p_3 + 4p_4 + \ldots + kp_k$ and $f(G) = p_1 + 2(p_3 + p_4 + \ldots + p_k)$. Suppose that $p_1 \geq 2$, $p_k \geq 1$ and $k \geq 4$. Then by replacing $2K_1 \cup K_k$ by $K_3 \cup K_{k-1}$ we obtain a graph H with $\varepsilon(H) \leq \varepsilon(G)$. Furthermore, $\varepsilon(H) = \varepsilon(G)$ if and only if $k = 4$.

2. $\mathbf{m}_n(n-1) = 3$ if $n \geq 4$. Let $G \in \mathcal{G}(n; f = n-1)$. Then $\varepsilon(G) = 3$ if and only if $n \geq 4$ and $G = (n-3)K_1 \cup K_3$.

3. Let a be an integer satisfying $\frac{2n}{3} \leq a \leq n-1$. If (p, q) is the solution of $p + 3q = n$ and $p + 2q = a$, then $G = pK_1 \cup qK_3$ satisfies $f(G) = a$.

4. Let a be an integer satisfying $\frac{2n}{3} \leq a \leq n-2$ and $G \in \mathcal{G}(n; f = a)$ such that $\varepsilon(G) = \mathbf{m}_n(a)$. Then by Theorem 3.2.6, we can choose $G = p_1K_1 \cup p_3K_3 \cup$

$p_4K_4 \cup \ldots \cup p_kK_k \in \mathcal{G}^*(n; f = a)$ and $k \le 4$. If $k = 4$, then $p_1 \ge 2$. Thus by the first fact above, there exists a graph $H = pK_1 \cup qK_3$ such that $p + 3q = n$, $p + 2q = a$ and $\varepsilon(H) = \varepsilon(G) = \mathbf{m}_n(a)$.

5. Let a be an integer with $a < \frac{2n}{3}$ and $G \in \mathcal{G}(n; f = a)$. Then, by Lemma 3.2.3, $\Delta(G) \ge 3$. Thus if $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \ldots \cup p_kK_k$, $f(G) = a < \frac{2n}{3}$ and $\varepsilon(G) = \mathbf{m}_n(a)$, then $p_1 \le 1$ and $k \ge 4$.

6. If $n = rq + t$, $0 \le t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.

7. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

8. Let $t(n, r) = \varepsilon(T_{n,r})$. Then for a fixed n, by using elementary arithmetic, we get $t(n, r-1) < t(n, r)$ for all r, $2 \le r \le n$. In fact $t(n, r) - t(n, r-1) \ge \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

Let $\bar{t}(n, r) = \binom{n}{2} - t(n, r)$. Summarizing the results, we have the following theorems.

Theorem 3.2.8 *Let n and a be integers satisfying $2 \le a \le n - 1$. Then*

1. $\mathbf{m}_n(n) = 0$,

2. $\mathbf{m}_n(n-1) = 3$ *if* $n \ge 3$ *and* $G = (n-3)K_1 \cup K_3$ *is the only graph of order* n *satisfying* $f(G) = n - 1$ *and* $\varepsilon(G) = 3$,

3. $\mathbf{m}_n(n-i) = 3i$ *if* $1 \le i \le \lceil \frac{n}{3} \rceil$,

4. *furthermore, if* $4 \le a < \frac{2n}{3}$, *then* $\mathbf{m}_n(a) = \bar{t}(n, q)$ *if* $a = 2q$, *and* $\mathbf{m}_n(a) = \bar{t}(n-1, q)$ *if* $a = 2q + 1$, *for some integer* q, *and*

5. *moreover,* $\mathbf{m}_n(3) = \binom{n-1}{2}$ *if* $n \ge 3$, *and* $\mathbf{m}_n(2) = \binom{n}{2}$ *if* $n \ge 2$.

Theorem 3.2.9 *Let n, m and a be integers satisfying $0 \le m \le \binom{n}{2}$. Then*

1. $\min(f; n, m) = \max(f; n, m) = n$ *if and only if* $m \in \{0, 1, 2\}$,

2. $\min(f; n, m) = \max(f; n, m) = 2$ *if and only if* $m = \binom{n}{2}$, *and*

3. if $3 \leq a \leq n-1$, *then* $\min(f; n, m) = a$ *if and only if* $\mathbf{m}_n(a) \leq m < \mathbf{m}_n(a-1)$.

We now find the minimum number of edges of a connected graph of order n having the forest number a. Let $\mathcal{CG}(n; f = a)$ be the set of connected graphs of order n having the forest number a. For integers n and a, let

$$\mathbf{cm}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{CG}(n; f = a)\}.$$

Further, $\mathbf{cm}_n(n) = n-1$, $\mathbf{cm}_n(2) = \binom{n}{2}$. It is easy to see that C_5 is a connected graph of order 5 and size 5 such that $f(C_5) = 4$ and K_5 is a connected graph of order 5 and size 10 such that $f(K_5) = 2$. We now find $\mathbf{cm}_n(a)$ for $2 < a < n$.

Let $\mathcal{CG}^*(n; f = a) = \{G \in \mathcal{CG}(n; f = a) : G$ is obtained from $\lceil \frac{a}{2} \rceil$ disjoint cliques and $\lceil \frac{a}{2} \rceil - 1$ edges$\}$. In Figure 29, G is a connected graph of order 10 obtained from 4 disjoint cliques and 3 edges such that $f(G) = 7$. We have the following theorem.

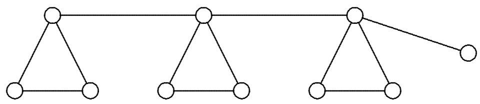


Figure 18: The connected graph G of order 10, size 12, and $f(G) = 7$.

Theorem 3.2.10 *Let* G *be a connected graph of order* n *with* $f(G) = a$. *Then there exists a graph* $H \in \mathcal{CG}^*(n; f = a)$ *such that* $\varepsilon(H) \leq \varepsilon(G)$.

Proof. We will proceed by induction on n. The result holds for $n = 1$. Let $n \geq 2$. Suppose that there exists a vertex v of G such that $f(G - v) = f(G) - 1 = a - 1$. Let $G_1, G_2, \ldots, G_k$ be the k components of $G - v$ having $n_1, n_2, \ldots, n_k$ vertices, respectively. Then $\deg_G(v) \geq k$. The structure of such a graph G is shown in Figure 30. Thus for each $i \in \{1, 2, \ldots, k\}$, there exists a graph $H_i \in \mathcal{CG}^*(n_i; f = f(G_i))$ such that $\varepsilon(H_i) \leq \varepsilon(G_i)$.

Let $w_i \in V(H_i)$ and H be a graph with $V(H) = \bigcup_{i=1}^{k} H_i \cup \{v\}$ and $E(H) = \bigcup_{i=1}^{k} E(H_i) \cup \{vw_i : i = 1, 2, \ldots, k\}$. Then $H \in \mathcal{CG}^*(n; f = a)$ and $\varepsilon(H) = k + \sum_{i=1}^{k} \varepsilon(H_i) \leq \deg_G(v) + \sum_{i=1}^{k} \varepsilon(G_i) \leq \varepsilon(G)$. This situation is shown in Figure 31.

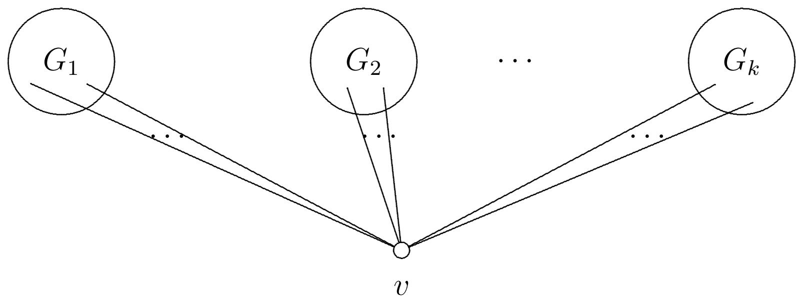


Figure 19: The graph G in the proof of Theorem 3.2.10.

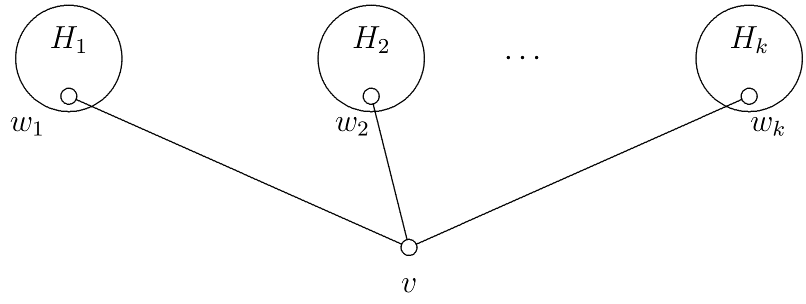


Figure 20: The graph H in the proof of Theorem 3.2.10.

Suppose that for every vertex v of G, $f(G-v) = f(G) = a$. Let v be a vertex of G of degree $\Delta(G)$. We consider two cases.

Case 1. $G-v$ is connected. Then there exists a graph $H' \in \mathcal{CG}^*(n-1; f=a)$ such that $\varepsilon(H') \le \varepsilon(G-v)$. If $f = 2q$ for some integer q, then H' is obtained from q disjoint cliques and $q-1$ edges. Therefore there exists a clique C' of H' of order at most $\lfloor \frac{n-1}{q} \rfloor$. By Corollary 3.2.5, $\deg_G(v) \ge \lceil \frac{n}{q} \rceil - 1$. Note that $\lceil \frac{n}{q} \rceil - 1 = \lfloor \frac{n-1}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-1}{q} \rfloor$ edges from v to C'. Let H be a graph with $V(H) = V(G)$ and $E(H) = (E(H') - E(C')) \cup E(C)$. Then $H \in \mathcal{CG}^*(n; f=a)$ and $\varepsilon(H) \le \varepsilon(G)$, as required.

If $n \ge 4$ and $f = 2q+1$ for some integer q, then H' is obtained from $q+1$ cliques and q edges where one of the cliques is an isolated vertex. Therefore there exists a non-trivial clique C' of H' of order at most $\lfloor \frac{n-2}{q} \rfloor$. By Corollary 3.2.5, $\deg_G(v) \ge \lceil \frac{n-1}{q} \rceil - 1$. Note that $\lceil \frac{n-1}{q} \rceil - 1 = \lfloor \frac{n-2}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-2}{q} \rfloor$ edges from v to C'. Let H be a graph with $V(H) = V(G)$ and $E(H) = (E(H') - E(C')) \cup E(C)$. Then $H \in \mathcal{CG}^*(n; f=a)$ and

$\varepsilon(H) \leq \varepsilon(G)$, as required.

Case 2. $G - v$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the k components of $G - v$ having $n_1, n_2, \ldots, n_k$ vertices, respectively. For each $i \in \{1, 2, \ldots, k\}$, we put $G'_i = \langle V(G_i) \cup \{v\} \rangle$ and $G' = \langle V(G - G_i) \cup \{v\} \rangle$. Suppose that there exists $i, 1 \leq i \leq k$, such that $f(G'_i) = f(G_i) + 1$. Then $f(G'_j) = f(G_j)$ for all $j \neq i$. Then G' is a connected graph of order $n' = n - n_i$ such that $f(G') = f(\bigcup_{i \neq j=1}^{k} G_j) = \sum_{i \neq j=1}^{k} f(G_j)$. For $w \in V(G_i)$, let H^* be a graph with $V(H^*) = V(G_i) \cup V(G')$ and $E(H^*) = E(G_i) \cup E(G') \cup \{vw\}$. Then H^* is a connected graph of order n such that $\varepsilon(H^*) = \varepsilon(G_i) + \varepsilon(G') + 1 \leq \varepsilon(G)$ and $f(H^*) = f(G_i) + f(G') = \sum_{j=1}^{k} f(G_j) = f(G - v) = a$. The structure of such a graph H^* is shown in Figure 32. Then there exist $H_i \in \mathcal{CG}^*(n_i; f = f(G_i))$ and $H' \in \mathcal{CG}^*(n'; f = f(G'))$ such that $\varepsilon(H_i) \leq \varepsilon(G_i)$ and $\varepsilon(H') \leq \varepsilon(G')$. Let $w \in V(H_i)$ and H be a graph with $V(H) = V(H_i) \cup V(H')$ and $E(H) = E(H_i) \cup E(H') \cup \{vw\}$. Then $\varepsilon(H) = \varepsilon(H_i) + \varepsilon(H') + 1 \leq \varepsilon(G_i) + \varepsilon(G') + 1 \leq \varepsilon(G)$ and $f(H) = f(H_i) + f(H') = f(G_i) + f(G') = f(G) = a$. Thus $H \in \mathcal{CG}^*(n; f = a)$ and $\varepsilon(H) \leq \varepsilon(G)$, as required.

Figure 21: The graph H^* in Part 1 of Case 2 of the proof of Theorem 3.2.10.

Suppose that $f(G'_i) = f(G_i)$ for all $i \in \{1, 2, \ldots, k\}$. If G_i is a complete graph for all i, then G'_i is complete. Since G_i is complete, it follows that for any $e \in E(G_i)$, $f(G_i - e) = f(G_i) + 1$. For $w \in V(G_i)$, let H^* be a graph with $V(H^*) = V(G_i - e) \cup V(G')$ and $E(H^*) = E(G_i - e) \cup E(G') \cup \{vw\}$ be a connected graph of order n such that $\varepsilon(H^*) = \varepsilon(G_i - e) + \varepsilon(G') + 1 \leq \varepsilon(G)$ and $f(H^*) = f(G_i - e) + f(G') = f(G_i) + f(G') + 1 = \sum_{j=1}^{k} f(G_j) = f(G)$. By the same argument in the first part of Case 2, there exists $H \in \mathcal{CG}^*(n; f = f(G))$ such that $\varepsilon(H) \leq \varepsilon(G - e) < \varepsilon(G)$, as required.

Assume that there exists $j, 1 \leq j \leq k$, such that G_j is not complete. Then for any $i \in \{1, 2, \dots, k\}$, there exists $H_i \in \mathcal{CG}^*(n_i; f = f(G_i))$ such that $\varepsilon(H_i) \le \varepsilon(G_i)$. For $w_i \in V(H_i)$, let H^* be a graph with $V(H^*) = \bigcup_{i=1}^k V(H_i) \cup \{v\}$ and $E(H^*) = \bigcup_{i=1}^k E(H_i) \cup \{vw_i : i = 1, 2, \dots, k\}$. Then H^* is a connected graph of order n with $\varepsilon(H^*) \le \varepsilon(G)$ and $f(H^*) = \sum_{i=1}^k f(H_i) = \sum_{i=1}^k f(G_i) = \sum_{i=1}^k f(G_i') = f(G) = a$.

Let H_i' be the subgraph $\langle V(H_i) \cup \{v\}\rangle$ of H^*. Then there exists $i, 1 \le i \le k$, such that $f(H_i') = f(H_i) + 1$. By the same argument in the first part of Case 2, there exists $H \in \mathcal{CG}^*(n; f = a)$ such that $\varepsilon(H) \le \varepsilon(H^*) \le \varepsilon(G)$, as required. $\square$

By Theorem 3.2.10 we know that for a graph $G \in \mathcal{CG}(n; f = a)$, there may be many graphs $H \in \mathcal{CG}^*(n; f = a)$. By applying Turán's Theorem, we have the following theorems.

Theorem 3.2.11 *Let n and a be integers satisfying $2 \le a \le n - 1$. Then*

1. $\mathbf{cm}_n(n) = n - 1$,
2. *furthermore, if $4 \le a \le n - 1$, then $\mathbf{cm}_n(a) = \bar{t}(n, q) + q - 1$ if $a = 2q$, and $\mathbf{cm}_n(a) = \bar{t}(n - 1, q) + q$ if $a = 2q + 1$, for some integer q, and*
3. *moreover, $\mathbf{cm}_n(3) = \binom{n-1}{2} + 1$ if $n \ge 3$, and $\mathbf{cm}_n(2) = \binom{n}{2}$ if $n \ge 2$.*

Theorem 3.2.12 *Let n, m and a be integers satisfying $n - 1 \le m \le \binom{n}{2}$. Then*

1. $\mathrm{Min}(f; n, m) = \mathrm{Max}(f; n, m) = n$ *if and only if* $m = n - 1$,
2. $\mathrm{Min}(f; n, m) = \mathrm{Max}(f; n, m) = 2$ *if and only if* $m = \binom{n}{2}$, *and*
3. *if $3 \le a \le n - 1$, then $\mathrm{Min}(f; n, m) = a$ if and only if $\mathbf{cm}_n(a) \le m < \mathbf{cm}_n(a - 1)$.*

3.3 Forests in Biquartic Graphs

Let G and H be bipartite graphs with bi-partitioning sets X and Y. Then G and H are said to have the *same type* if $\deg_G v = \deg_H v$ for all $v \in X \cup Y$. There is an analogous result for a class of bipartite graphs as we shall state in the following theorem.

Theorem 3.3.1 *Let G and H be bipartite graphs with partite sets X and Y. If G and H are of the same type, then there exists a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for every $i = 1, 2, \ldots, t$, G and $G^{\sigma_1\sigma_2\ldots\sigma_i}$ are of the same type and $H = G^{\sigma_1\sigma_2\ldots\sigma_t}$.*

Let G be a bipartite graph with graphical sequence $\mathbf{d}$ and $\mathcal{B}(G)$ be the class of all bipartite graphs of the same type as G. Then by Theorem 3.3.1, the subgraph of the $\Sigma(\mathbf{d})$-graph induced by $\mathcal{B}(G)$ is connected. The following theorem is an immediate consequence of Theorem 2.1.5.

Theorem 3.3.2 *Let G be a bipartite graph. Then f is an interpolation graph parameter over $\mathcal{B}(G)$.*

It is clear that any two r-regular bipartite graphs of order $2n$ are of the same type. Let $\mathcal{B}(r^{2n})$ be the class of all r-regular bipartite graphs of order $2n$. Thus we have the following corollary.

Corollary 3.3.3 *The forest number f is an interpolation graph parameter over $\mathcal{B}(r^{2n})$.*

We will use the following notation.

- $\min(f, \mathcal{B}(r^{2n})) = \min\{f(G) : G \in \mathcal{B}(r^{2n})\}$,
- $\max(f, \mathcal{B}(r^{2n})) = \max\{f(G) : G \in \mathcal{B}(r^{2n})\}$.

In Section 3.1, we found the value of $\min(f, \mathcal{B}(3^{2n}))$. In this section, we find $\max(f, \mathcal{B}(r^{2n}))$ for positive integers r and n with $r \le n$. The problem of finding $\min(f, \mathcal{B}(r^{2n}))$ is more difficult. By using some known results on symmetric designs, we are able to obtain $\min(f, \mathcal{B}(4^{2n}))$. This result is an improvement of Alon's result in [1].

It is easy to see that $\max(f, \mathcal{B}(2^{2n})) = 2n-1$. Let G be an r-regular graph of order $2n$ and F be a maximum induced forest of G. By Theorem 2.2.5, $|F| \le \lfloor \frac{nr-1}{r-1} \rfloor$. Thus $\max(f, \mathcal{B}(r^{2n})) \le \lfloor \frac{nr-1}{r-1} \rfloor$.

Let n and r be positive integers with $3 \le r \le n$. We write $n-1 = (r-1)k + t$, $0 \le t < r-1$. Let $q = r+t$ and $p = r-1$. Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $Y =$

$\{y_0, y_1, \ldots, y_{n-1}\}$, $X_1 = \{x_0, x_1, \ldots, x_{q-1}\}$, $Y_1 = \{y_0, y_1, \ldots, y_{q-1}\}$, and for $1 \le i \le k-1$, let $X_{i+1} = \{x_{q+(i-1)p}, x_{q+(i-1)p+1}, \ldots, x_{q+ip-1}\}$ and $Y_{i+1} = \{y_{q+(i-1)p}, y_{q+(i-1)p+1}, \ldots, y_{q+ip-1}\}$. Let G be a bipartite graph with bi-partitioning sets X and Y. The edge set of G is defined as follows.

1. $E_1 = \bigcup_{j=0}^{q-1}\{x_j y_{j+w} : 1 \le w \le r-1\}$, where the subscripts are taken modulo q.
2. $E_2 = \bigcup_{i=2}^{k} E(K(X_i, Y_i))$ where $K(X_i, Y_i)$ is the complete bipartite graph on bi-partitioning sets X_i and Y_i.
3. $E_3 = \{x_0 y_0, x_q y_{q-1}, x_{q+p} y_{q+p-1}, \ldots, x_{q+(k-2)p} y_{q+(k-2)p-1}\}$.
4. Let $X' = X - \{x_0, x_q, x_{q+p}, \ldots, x_{q+(k-2)p}\}$ and $Y' = Y - \{y_0, y_{q-1}, y_{q+p-1}, \ldots, y_{q+(k-2)p-1}\}$. Let E_4 be a set of $n-k$ independent edges connecting X' to Y' such that $E_4 \cap (E_1 \cup E_2 \cup E_3) = \emptyset$.
5. Let $E = E_1 \cup E_2 \cup E_3 \cup E_4$.

The structure of a 4-regular graph G of order 14 is shown in Figure 33. Thus $G = (X \cup Y, E)$ is an r-regular bipartite graph of order $2n$ with $f(G) = n + k$ and we can conclude the following theorem.

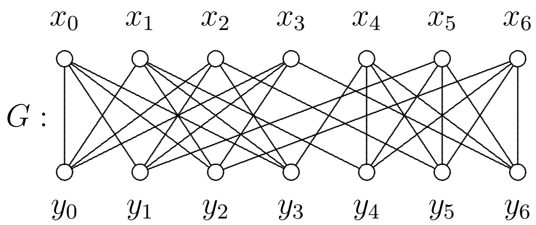


Figure 22: The graph G of order 14 with $f(G) = 9$.

Theorem 3.3.4 $\max(f, \mathcal{B}(r^{2n})) = \lfloor \frac{nr-1}{r-1} \rfloor$.

We consider the problem of determining the maximum order of an r-regular bipartite graph B with partite sets X and Y such that $f(B) = 1 + |X|$.

A balanced incomplete block design BIBD(v, b, r, k, λ) is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a v-element set of elements, called *points*, and $\mathcal{B}$ is a collection of b k-element subsets of $\mathcal{V}$, called *blocks*, such that each element of $\mathcal{V}$ is contained in exactly r blocks and every 2-element subset of $\mathcal{V}$ is contained in exactly λ blocks. The numbers v, b, r, k and λ are *parameters* of the BIBD. Trivial necessary conditions for the existence of a BIBD(v, b, r, k, λ) are

$$vr = bk, \quad \text{and} \quad r(k-1) = \lambda(v-1).$$

A BIBD with $b = v$ (or equivalently $r = k$) is called a *symmetric balanced incomplete block design* SBIBD(v, k, λ) or simply a *symmetric design*. Symmetric designs with $\lambda = 2$ are called *biplanes* of order v. The most fundamental necessary condition on the existence of symmetric designs is due to Bruck, Ryser, and Chowla [11]. Symmetric designs SBIBD$(v, k, 2)$ are known only when $k = 4, 5, 6, 9, 11, 13$. The following are well known facts about symmetric designs SBIBD$(v, k, 2)$.

1. $v = \binom{k}{2} + 1$.
2. Any two blocks intersect in exactly $\lambda = 2$ points.
3. The *dual* incidence structure obtained by interchanging the roles of points and blocks is also a BIBD with the same parameters (hence the term symmetric).
4. SBIBD$(7, 4, 2)$ is unique.

Let $\mathcal{V} = \{1, 2, 3, 4, 5, 6, 7\}$. It is easy to see that an SBIBD$(7, 4, 2)$ contains 7 blocks as follows

$$B_1 = \{1, 2, 3, 5\}, B_2 = \{2, 3, 4, 6\}, B_3 = \{3, 4, 5, 7\}, B_4 = \{4, 5, 6, 1\},$$

$$B_5 = \{5, 6, 7, 2\}, B_6 = \{6, 7, 1, 3\}, \text{ and } B_7 = \{7, 1, 2, 4\}.$$

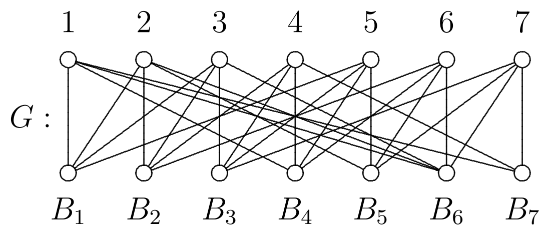


Figure 23: The bipartite graph $B(7, 4, 2)$.

Let $(\mathcal{V}, \mathcal{B})$ be an SBIBD$(v, k, 2)$. A bipartite graph $B(v, k, 2)$ can be defined as a graph with $V(B(v, k, 2)) = \mathcal{V} \cup \mathcal{B}$ and $E(B(v, k, 2)) = \{\{v, B\} : v \in \mathcal{V}, B \in \mathcal{B},\ \text{and } v \in B\}$. It is clear that $B(v, k, 2)$ is a k-regular graph of order $2v$. Thus $f(B(v, k, 2)) \geq 1 + v$. The structure of the graph $B(7, 4, 2)$ is shown in Figure 34.

Lemma 3.3.5 *Let $B = B(7, 4, 2)$ and $G \in \mathcal{B}(4^{14})$ with bi-partitioning sets X and Y. Then*

1. *$f(G) \geq 8$ and $f(B) = 8$.*
2. *$f(G) = 8$ if and only if $G \cong B$.*
3. *If $G \ncong B$, then there exists an induced forest F of G such that $F = \{u, v\} \cup Y$ for some $u, v \in X$.*
4. *For each $e \in E(B)$ there exists an induced forest F_e of $B - e$ such that $F_e = \{u, v\} \cup Y$ for some $u, v \in X$.*
5. *If H is a spanning subgraph of B with $H \ncong B$, then there exists an induced forest F of H such that $F = \{u, v\} \cup Y$ for some $u, v \in X$.*
6. *If H is a spanning subgraph of G with $H \ncong G$, then there exists an induced forest F of H such that $F = \{u, v\} \cup Y$ for some $u, v \in X$.*

Proof. 1. It is clear that $f(G) \geq 8$. Suppose that $f(B) \geq 9$ and let F be an induced forest of B of order 9 and let $F_X = F \cap X$ and $F_Y = F \cap Y$. If $|F_X| = 2$, then $|F_Y| = 7$. Since B is the graph obtained from an SBIBD$(7, 4, 2)$, we have $|N_B(u) \cap N_B(v)| = 2$ for each $u, v \in X$. Thus F contains a cycle. If $F_X = \{u, v, w\} \subseteq X$, then $F_Y = Y - \{y\}$ for some $y \in Y$. Since y is adjacent to at most three vertices in F_X, it follows that the subgraph $\langle F \rangle$ of B contains at least $3 \cdot 4 - 3 = 9$ edges and hence the subgraph $\langle F \rangle$ of B contains a cycle. If $F_X = \{u, v, w, z\} \subseteq X$, then $F_Y = Y - \{p, q\}$ for some $p, q \in Y$. Since there are at most six edges in B connecting F_X to $\{p, q\}$, it follows that the subgraph $\langle F \rangle$ of B contains at least $4 \cdot 4 - 6 = 10$ edges and hence the subgraph $\langle F \rangle$ of B contains a cycle. If $|F_X| \geq 5$, then the result follows by using the dual incidence structure. Thus $f(B) = 8$.

2. Suppose that $f(G) = 8$. Let $X = \{v_1, v_2, \ldots, v_7\}$ and $\lambda_i = |N_G(\{v_1\}) \cap N_G(\{v_i\})|$ for each $i \in \{1, 2, \ldots, 7\}$. Then $\lambda_1 + \lambda_2 + \ldots + \lambda_7$ is the number of edges of the subgraph $\langle X \cap N_G(\{v_1\})\rangle$ of G. We have the following inequality:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_7 \leq 16.$$

Since $\lambda_1 = 4$, we have $\lambda_2 + \ldots + \lambda_7 \leq 12$. Suppose that $G \ncong B$. Then there exists $i \in \{2, 3, \ldots, 7\}$ such that $\lambda_i \geq 3$. Thus there exists $j \in \{2, 3, \ldots, 7\}$ with $i \neq j$ such that $\lambda_j \leq 1$. Therefore $F = \{v_i, v_j\} \cup Y$ is an induced forest of G of order 9. We get a contradiction.

3. Assume that $G \ncong B$. Then by the same argument as 2, there exist $v_i, v_j \in X$ such that $|N_G(\{v_i\}) \cap N_G(\{v_j\})| \leq 1$. Thus $F = \{v_i, v_j\} \cup Y$ is an induced forest of G.

4. Let $e \in E(B)$. Since B is the graph obtained from an SBIBD$(7, 4, 2)$, there exists $u, v \in X$ such that $|N_{B-e}(\{u\}) \cap N_{B-e}(\{v\})| \leq 1$. Then $F_e = \{u, v\} \cap Y$ is an induced forest of $B - e$.

5. Let H be a spanning subgraph of B with $H \ncong B$. Then $\varepsilon(H) < \varepsilon(B)$. By the same argument as in 4, there exists an induced forest F of H such that $F = \{u, v\} \cup Y$ for some $u, v \in X$.

6. Let H be a spanning subgraph of G with $H \ncong G$. If $G \cong B$, then by the same argument as in 3, G contains a forest F such that $F = \{u, v\} \cup Y$ for some $u, v \in X$ as its subgraph. It is clear that F is an induced forest of H. If $G \cong B$, then by the same argument as in 4, H contains a forest F of G such that $F = \{u, v\} \cup Y$ for some $u, v \in X$ as its subgraph. $\square$

In 2001, Alon et al. [1] gave a lower bound of $f(G)$ in terms of the independence number $\alpha(G)$ of G and its maximum degree.

Theorem 3.3.6 *Let G be a connected graph of order n with maximum degree $\Delta \geq 3$. Then $f(G) \geq \alpha(G) + \frac{n-\alpha(G)}{(\Delta-1)^2}$.*

Corollary 3.3.7 *Let G be a bipartite graph of order n with maximum degree $\Delta \geq 3$. Then $f(G) \geq \left(\frac{1}{2} + \frac{1}{2(\Delta-1)^2}\right) n$, and this is sharp for $\Delta = 3$ and $n \equiv 0$ (mod 8).*

By Corollary 3.3.7, we have $f(G) \geq \frac{10}{9} \cdot n$ for any $G \in \mathcal{B}(4^{2n})$, but the best example found in [1] has $f(G) = \frac{7}{6} \cdot n$. Thus $\frac{10}{9} \cdot n \leq \min(f, \mathcal{B}(4^{2n})) \leq \frac{7}{6} \cdot n$.

Lemma 3.3.8 *Let G be an r-regular bipartite graph of order $2n$ with bi-partitioning sets X and Y. If $g(r) = \binom{r}{2} + 1$, then there exists a subset S of X of cardinality $\lceil \frac{n}{g(r)} \rceil$ such that $S \cup Y$ is an induced forest of G.*

Proof. Let $r \geq 3$ and G be an $r-$regular bipartite graph of order $2n$ with bi-partitioning sets X and Y. Suppose that F is an induced forest of G such that $Y \subseteq F$. We put $X' = F \cap X = \{x'_1, x'_2, x'_3, \ldots, x'_t\}$ where t is a maximum number of vertices in X such that for any $i = 1, 2, \ldots, t$, $\left| N_G(\{x'_i\}) \cap \bigcup_{j=1}^{i-1} N_G(\{x'_j\}) \right| \leq 1$. Then $|X - X'| = n - t$ and for any $u \in X - X'$ and $i = 1, 2, \ldots, t$, $|N_G(\{u\}) \cap N_G(\{x'_i\})| \geq 2$. Since there are at least $r(r-1)t$ edges in G connecting vertices in $X - X'$ to vertices in Y, we have $r(r-1)t \geq 2(n-t)$. Thus $t \geq \lceil \frac{n}{g(r)} \rceil$. Thus there exists a subset $S = \{x'_1, x'_2, \ldots, x'_t\}$ of X of cardinality $\lceil \frac{n}{g(r)} \rceil$ such that $S \cup Y$ is an induced forest of G. $\square$

The following corollary is an immediate consequence of Lemma 3.3.8.

Corollary 3.3.9 *If G is a biquartic graph of order $2n$, then $f(G) \geq n + \lceil \frac{n}{7} \rceil$.*

By Corollary 3.3.9, we have $f(G) \geq \frac{8}{7} \cdot n$ for any $G \in \mathcal{B}(4^{2n})$. This means that if we look at the class of biquartic graphs of order $2n$, then the bound can be improved from 55 percent of order to 57 percent. We now construct a class of biquartic graphs to show that $\min(f, \mathcal{B}(4^{2n})) = n + \lceil \frac{n}{7} \rceil$.

Let $H \in \mathcal{B}(4^{10})$ and $K \in \mathcal{B}(4^{12})$. We write $n = 7q + t, 1 \leq t \leq 7$ and construct a biquartic graph of order $2n$ according to the value of t:

1. If $t = 1$, construct a graph G of order $2n$ by taking $q - 1$ copies of $B(7, 4, 2)$ and 2 copies of $K_{4,4}$.

2. If $t = 2$, construct a graph G of order $2n$ by taking $q - 1$ copies of $B(7, 4, 2)$, a copy of $K_{4,4}$ and a copy of H.

3. If $t = 3$, construct a graph G of order $2n$ by taking $q - 1$ copies of $B(7, 4, 2)$ and 2 copies of H.

4. If $t = 4$, construct a graph G of order $2n$ by taking q copies of $B(7, 4, 2)$ and a copy of $K_{4,4}$.

5. If $t = 5$, construct a graph G of order $2n$ by taking q copies of $B(7, 4, 2)$ and a copy of H.

6. If $t = 6$, construct a graph G of order $2n$ by taking q copies of $B(7, 4, 2)$ and a copy of K.

7. If $t = 7$, construct a graph G of order $2n$ by taking $q + 1$ copies of $B(7, 4, 2)$.

It is easy to check that the graphs G constructed above satisfy $f(G) = n + \lceil \frac{n}{7} \rceil$. Figure 35 shows the bipartite graphs G_n of order $2n$ where $n = 11, 12, 13$ such that $f(G) = n + \lceil \frac{n}{7} \rceil$. Thus we have the following theorem.

Theorem 3.3.10 $\min(f, \mathcal{B}(4^{2n})) = n + \lceil \frac{n}{7} \rceil$.

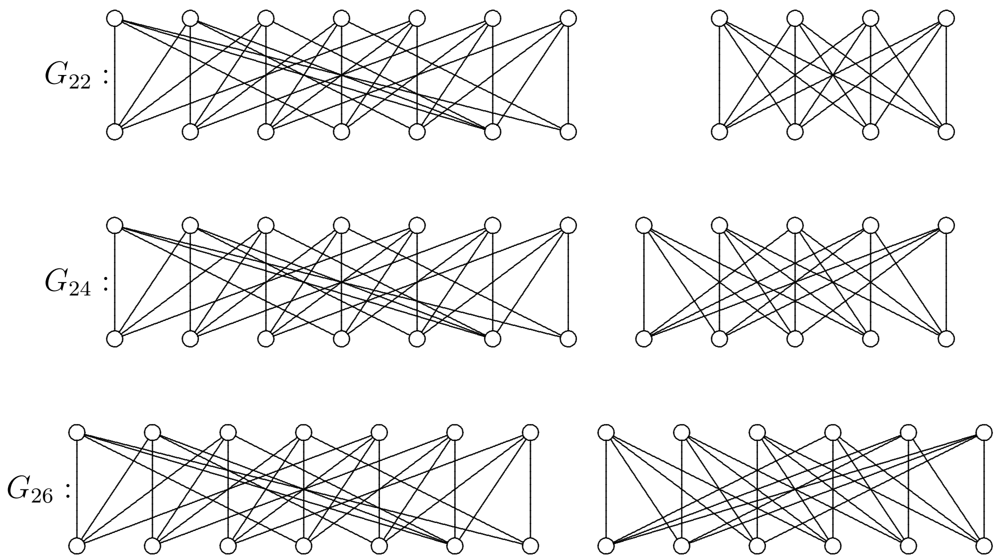


Figure 24: The bipartite graph G_n of order $2n$ where $n = 11, 12, 13$.

3.4 Regular Graphs and their Arboricities

A typical problem in graph theory is one that deals with the decomposition of a graph into various subgraphs possessing some prescribed property. There are ordinarily two problems of this type, one dealing with a decomposition of the vertex set and the other with a decomposition of the edge set. The vertex coloring problem is an example of vertex decomposition while the edge coloring problem is an example of edge decomposition with some additional property. For a graph G, it is always possible to partition $V(G)$ into subsets $V_i, 1 \leq i \leq k$, such that each induced subgraph $\langle V_i \rangle$ contains no cycle. The *vertex arboricity* $a(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. Thus $a(G) = 1$ if and only if G is a forest. For a few classes of graphs, the vertex arboricity is easily determined. For example, $a(C_n) = 2$. $a(K_n) = \lceil \frac{n}{2} \rceil$. The graphs in Figure 36 show that $a(C_5) = 2$, $a(K_5) = 3$, where $P_1 = \{\{v_1\}, \{v_2, v_3, v_4, v_5\}\}$ and $P_2 = \{\{u_1, u_2\}, \{u_3, u_4\}, \{u_5\}\}$ are the partitions of $V(C_5)$ and $V(K_5)$, respectively. Also $a(K_{r,s}) = 1$ if $r = 1$ or $s = 1$, and $a(K_{r,s}) = 2$ otherwise. It is clear that for every graph G of order n, $a(G) \leq \lceil \frac{n}{2} \rceil$.

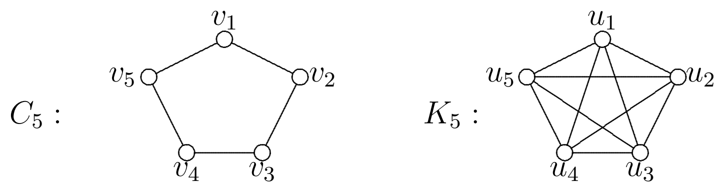


Figure 25: The graphs C_5 and K_5.

We now turn to the second decomposition problem. The *edge arboricity* or simply the *arboricity* $a_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic subgraph. As with vertex arboricity, a nonempty graph has arboricity 1 if and only if it is a forest. Also $a_1(K_n) = \lceil \frac{n}{2} \rceil$.

Let $\pi \in \{a, a_1\}$ and $\mathbf{d}$ be a graphical sequence. Put

$$\pi(\mathbf{d}) := \{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}.$$

The main purpose of this section is to prove the following result.

Theorem 3.4.1 *Let* $\mathbf{d}$ *be a graphical sequence and* $\pi \in \{a, a_1\}$. *Then there exist integers* $x(\pi)$ *and* $y(\pi)$ *such that there exists a graph* $G \in \mathcal{R}(\mathbf{d})$ *with* $\pi(G) = z$ *if* *and only if z is an integer satisfying $x(\pi) \leq z \leq y(\pi)$. That is $\pi(\mathbf{d}) = \{z \in \mathbb{Z} : x(\pi) \leq z \leq y(\pi)\}$.*

The proof of Theorem 3.4.1 follows from Theorem 1.4.2 and the following results.

Lemma 3.4.2 *Let $\pi \in \{a, a_1\}$ and σ be a switching on a graph G. Then $\pi(G^\sigma) \leq \pi(G) + 1$.*

Proof. Let G be a graph and $\sigma = \sigma(a, b; c, d)$ be a switching on G. Then $a(G^\sigma - \{a, c\}) \leq a(G)$ and hence $a(G^\sigma) \leq a(G^\sigma - \{a, c\}) + 1 \leq a(G) + 1$. Since $ac, bd \in E(G^\sigma)$, we have that $a_1(G^\sigma - \{ac, bd\}) \leq a_1(G)$ and hence $a_1(G^\sigma) \leq a_1(G^\sigma - \{ac, bd\}) + 1 \leq a_1(G) + 1$. $\square$

Let $\pi \in \{a, a_1\}$. Then, by Lemma 3.4.2, if $\pi(G) \leq \pi(G^\sigma)$, then we have that $\pi(G) \leq \pi(G^\sigma) \leq \pi(G) + 1$. If $\pi(G) \geq \pi(G^\sigma)$, then by Lemma 3.4.2, we see that $\pi(G) = \pi((G^\sigma)^{\sigma^{-1}}) \leq \pi(G^\sigma) + 1 \leq \pi(G) + 1$, where $\sigma^{-1} = \sigma(a, c; b, d)$. We have the following corollary.

Corollary 3.4.3 *Let G be a graph and σ be a switching on G. If $\pi \in \{a, a_1\}$, then $|\pi(G) - \pi(G^\sigma)| \leq 1$.*

Let $\pi \in \{a, a_1\}$. By Theorem 3.4.1, $\pi(\mathbf{d})$ is uniquely determined by $x(\pi)$ and $y(\pi)$, thus it is reasonable to denote $x(\pi) = \min(\pi, \mathbf{d}) := \min\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$ and $y(\pi) = \max(\pi, \mathbf{d}) := \max\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$. In general, it is not easy to find $\min(\pi, \mathbf{d})$ and $\max(\pi, \mathbf{d})$ for a given graphical sequence $\mathbf{d}$. Thus for simplicity, we will focus on the case where $\mathbf{d} = (r, r, \ldots, r)$ is a constant sequence of length n and in this case we write $\mathbf{d} = r^n$. We first consider the problem of determining $\min(a_1, r^n)$ and $\max(a_1, r^n)$. By applying a famous result of Nash-Williams [26] on edge arboricity as stated in the following theorem, we will see that $\min(a_1, r^n) = \max(a_1, r^n) = \lceil \frac{r+1}{2} \rceil$.

Theorem 3.4.4 *For every nonempty graph G, $a_1(G) = \max\lceil \frac{|E(H)|}{|V(H)|-1} \rceil$, where the maximum is taken over all nontrivial induced subgraphs H of G.*

As a consequence of the above theorem, it follows that $a_1(K_n) = \lceil \frac{n}{2} \rceil$ and

$a_1(K_{r,s}) = \lceil\frac{rs}{r+s-1}\rceil$. Note that $a_1(G) \geq \lceil\frac{nr}{2(n-1)}\rceil$ for any r-regular graph G of order n. The upper bound of a_1 can be obtained in the following lemma. We note further that if $r \in \{0,1\}$, then $a_1(G) = 1$ for all $G \in \mathcal{R}(r^n)$.

Lemma 3.4.5 *If G is an r-regular graph on n vertices and $r \geq 2$, then $a_1(G) \leq \lceil\frac{r+1}{2}\rceil$.*

Proof. Let H be an induced subgraph of G of order m. If $m \geq r+1$, then $\frac{|E(H)|}{|V(H)|-1} \leq \frac{mr}{2(m-1)}$. It follows that $\frac{|E(H)|}{2(|V(H)|-1)} \leq \frac{r+1}{2}$. If $m \leq r$, then $|E(H)| \leq \binom{m}{2}$. It is clear that $\frac{|E(H)|}{2(|V(H)|-1)} \leq \frac{r+1}{2}$. Thus $a_1(G) \leq \frac{r+1}{2}$. □

We have already mentioned that if $r \in \{0,1\}$, then $a(r^n) = a_1(r^n) = \{1\}$. This shows that $\min(a, r^n) = \max(a, r^n) = \min(a_1, r^n) = \max(a_1, r^n) = 1$ if $r \in \{0,1\}$. Let $G \in \mathcal{R}(2^n)$. Then G is a union of cycles and hence not a forest. Thus $a_1(G) \geq 2$. We can easily partition $V(G)$ and $E(G)$ into two subsets in which each subset induces an acyclic subgraph. It follows that $\min(a, 2^n) = \max(a, 2^n) = \min(a_1, 2^n) = \max(a_1, 2^n) = 2$ for all $n \geq 3$. Because of this fact, from now on we will assume that $r \geq 3$ and $n \geq r+1$.

Since each forest on n vertices has at most $n-1$ edges, it follows that $a_1(G) \geq \lceil\frac{nr}{2(n-1)}\rceil$ for every graph $G \in \mathcal{R}(\mathbf{d})$. Consequently, $\min(a_1, r^n) \geq \lceil\frac{nr}{2(n-1)}\rceil$. We have the following theorem.

Theorem 3.4.6 $\min(a_1, r^n) = \max(a_1, r^n) = \lceil\frac{r+1}{2}\rceil$.

Proof. It is clear that $\frac{r}{2} < \frac{nr}{2(n-1)} \leq \frac{(r+1)}{2}$. If r is even, then $\lceil\frac{r+1}{2}\rceil = \frac{r}{2}+1$. It follows that $\lceil\frac{nr}{2(n-1)}\rceil = \lceil\frac{r+1}{2}\rceil$. If r is odd, then $\frac{r}{2} < \lceil\frac{nr}{2(n-1)}\rceil$ and $\frac{r+1}{2}$ is integer. Thus $\lceil\frac{nr}{2(n-1)}\rceil = \lceil\frac{r+1}{2}\rceil$. It follows that $a_1(G) \leq \lceil\frac{r+1}{2}\rceil$ for any r-regular graph on n vertices. Hence, $\min(a_1, r^n) = \max(a_1, r^n) = \lceil\frac{r+1}{2}\rceil$. □

In order to obtain the corresponding extremal results on vertex arboricity we need to introduce some notation on graph construction.

It should be noted once again that an r-regular graph G of order n has $a(G) = 1$ if and only if $r \in \{0,1\}$. Thus $\min(a, r^n) = \max(a, r^n) = 1$ if and only if $r \in \{0,1\}$. Moreover, $\min(a, 2^n) = \max(a, 2^n) = 2$.

Let r and n be integers with $r \geq 3$ and $\mathcal{R}(r^n) \neq \emptyset$. Then

1. If n is even and $n \geq 2r$, then there exists an r-regular bipartite graph G of order n. Therefore $a(G) = 2$.

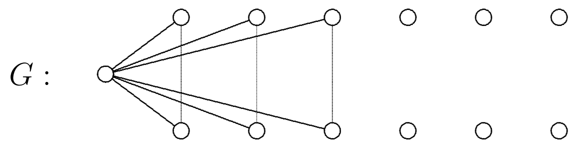
2. If n is odd and $n \geq 2r+1$, then r is even and there exists an r-regular bipartite graph H of order $n-1$. Let F be a set of $\frac{r}{2}$ independent edges of H. Then $H - F$ is a bipartite graph of order n having r vertices of degree $r-1$ and $n-1-r$ vertices of degree r. Let G be a graph obtained from $H-F$ by joining the r vertices of $H-F$ to a new vertex. Therefore G is an r-regular graph of order n having $a(G) = 2$. In Figure 37 (a), G is a 6-regular graph of order 11 having $a(G) = 2$ which is constructed from a 6-regular graph H of order 10.

3. If n is even and $n = 2(r-1)$, then $K_{r-1,r-1}$ is the complete $(r-1)$-regular bipartite graph of order n. Let $X = \{x_1, x_2, \ldots, x_{r-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{r-1}\}$ be the corresponding partite sets of $K_{r-1,r-1}$. Let G be defined by $G = (K_{r-1,r-1} - \{x_2y_2, x_3y_3, \ldots, x_{r-2}y_{r-2}\}) * \{x_1x_2, x_2x_3, \ldots, x_{r-2}x_{r-1}, y_1y_2, y_2y_3, \ldots, y_{r-2}y_{r-1}\}$. Then G is an r-regular graph of order n having $a(G) = 2$ (See Figure 37 (b)).

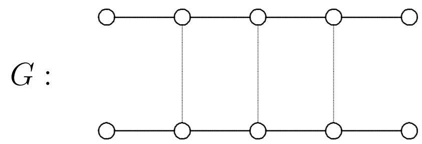
4. If $n = 2r-1$, then r is even. Let X and Y be as described in the previous case and u be a new vertex. Then $H = K(X,Y) * E_1$, where $E_1 = \{ux_1, uy_1, ux_2, uy_2, \ldots ux_{r/2}, uy_{r/2}\}$, is a graph of order $2r-1$ having $r+1$ vertices of degree r and $r-2$ vertices of degree $r-1$. Put $X_1 = \{x_i \in X : \frac{r}{2} < i \leq r-1\}$ and $Y_1 = \{y_i \in Y : \frac{r}{2} < i \leq r-1\}$. Thus $|X_1| = |Y_1| = \frac{r-2}{2}$.

   Case 1 If $\frac{r-2}{2}$ is even, then we can construct an r-regular graph G from H by adding $\frac{r-2}{4}$ independent edges to each of X_1 and Y_1. This construction yields $a(G) = 2$ (See Figure 37 (c)).

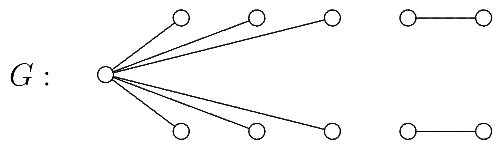
   Case 2 If $\frac{r-2}{2}$ is odd, then $H - x_{r/2}y_{r/2}$ is a graph having $r-1$ vertices of degree r and r vertices of degree $r-1$. Thus it has $\frac{r}{2}$ vertices of degree $r-1$ in X and also $\frac{r}{2}$ vertices in Y. Since $\frac{r}{2}$ is even, we can easily construct an r-regular graph G from $H - x_{r/2}y_{r/2}$ by adding appropriate independent edges so that $a(G) = 2$ (See Figure 37 (d)).



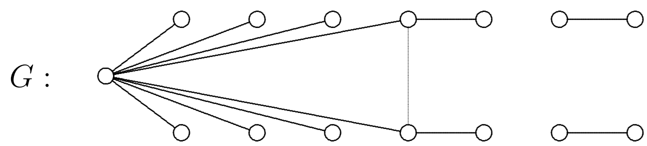
(a) Constructing a 6-regular graph G of order 13 from a 6-regular graph of order 12.



(b) Constructing a 6-regular graph G of order 10 from the 5-regular graph $K_{5,5}$.



(c) Constructing a 6-regular graph G of order 11 from the 5-regular graph $K_{5,5}$.



(d) Constructing a 8-regular graph G of order 15 from the 7-regular graph $K_{7,7}$.

Figure 26: Constructing r-regular graphs from $(r-1)$-regular graphs.

With these observations we easily obtain the following result.

Lemma 3.4.7 $\min(a, r^n) = 2$ for all $n \geq 2r - 2$.

Punnim proved the following result in [33]: Let r and s be integers with $1 \leq s \leq r$. Then

1. $f(G) \leq s + 1$ for all $G \in \mathcal{R}(r^{r+s})$ and

2. there exists a graph $H \in \mathcal{R}(r^{r+s})$ such that $f(H) = s + 1$.

With these observations we have the following result.

Lemma 3.4.8 $\min(a, r^{r+s}) \geq \lceil \frac{r+s}{s+1} \rceil$ if $1 \leq s \leq r - 3$.

It is well known that $a(K_{r+1}) = \lceil \frac{r+1}{2} \rceil$. Thus $\min(a, r^{r+1}) = \max(a, r^{r+1}) = \lceil \frac{r+1}{2} \rceil$. It is also well known that $G = K_{r+2} - F$, where F is a 1-factor of K_{r+2}, is the unique r-regular graph of order $r+2$. Since $E(\overline{G}) = F$, it follows that a subgraph of $\overline{G}$ induced by any four vertices of $V(\overline{G})$ contains at most two edges. Equivalently, a subgraph of G induced by any four vertices must contain a cycle. Thus $a(G) \geq \lceil \frac{r+2}{3} \rceil$. It is easy to show that $a(G) = \lceil \frac{r+2}{3} \rceil$. Thus $\min(a, r^{r+2}) = \max(a, r^{r+2}) = \lceil \frac{r+2}{3} \rceil$.

Since $T_{n,q}$ is the Turán graph of order n containing no clique of order $q+1$, there exists a unique pair of integers x and y such that $n = qx + y,\ 0 \leq y < q$. Note that $T_{n,q}$ is a complete q-partite graph of order n consisting of y partite sets of cardinality $x + 1$ and $q - y$ partite sets of cardinality x. Therefore $T_{n,q}$ is an $(n - x)$-regular graph if and only if $y = 0$. If $y > 0$, then $T_{n,q}$ contains $y(x + 1)$ vertices of degree $n-x-1$ and $(q-y)x$ vertices of degree $n-x$. A slight modification of $T_{n,q}$ yields a regular graph with prescribed arboricity number as in the following theorem.

Theorem 3.4.9 *Let n, x, q and y be positive integers satisfying $n = qx + y,\ 0 < y < q$. Then*

1. *there exists an $(n - x)$-regular graph G of order n such that $a(G) \leq q$ if x is odd,*

2. *there exists an $(n - x)$-regular graph G of order n such that $a(G) \leq q$ if x and y are even, and*

3. *there exists an $(n - x - 1)$-regular graph H of order n such that $a(G) \leq q$ if x is even and y is odd.*

Proof. Note that $T_{n,q}$ is a complete q-partite graph of order n consisting of y partite sets of cardinality $x+1$ and $q-y$ partite sets of cardinality x. Let $V_1, V_2, \ldots, V_y$ be partite sets of $T_{n,q}$ of cardinality $x+1$ and $U_1, U_2, \ldots, U_{q-y}$ be partite sets of $T_{n,q}$

of cardinality x. It is clear that $\deg_{T_{n,q}} v = n-x-1$ if $v \in V_1 \cup V_2 \cup \ldots \cup V_y$ and $\deg_{T_{n,q}} u = n-x$ if $u \in U_1 \cup U_2 \cup \ldots \cup U_{q-y}$. For each $i = 1, 2, \ldots y$, let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i(x+1)}\}$; similarly, for each $j = 1, 2, \ldots, q-y$, let $U_j = \{u_{j1}, u_{j2}, \ldots, u_{jx}\}$.

1. Suppose that x is odd. Since $|V_i| = x+1$ for each $i = 1, 2, \ldots, y$, an $(n-x)$-regular graph G can be obtained from $T_{n,q}$ by adding $\frac{x+1}{2}$ independent edges to each V_i, $1 \le i \le y$. It is clear that $a(G) \le q$.

2. Suppose that x and y are even. Let $H = \bigcup_{i=1}^{y} P(V_i)$ where $P(V_i)$ is a path with V_i as its vertex set and $E_i = \{v_{ik}v_{i(k+1)} : 1 \le k \le x\}$ as its edge set. Let $F_1, F_2, \ldots, F_{y/2}$ be defined by $F_k = \{v_{kt}v_{(k+y/2)t} : 2 \le t \le x\}$ and put $F = \bigcup_{k=1}^{y/2} F_k$. Finally we obtain a graph $G = (T_{n,q} - F) * H$ which is an $(n-x)$-regular graph of order n and $a(G) \le q$.

3. Suppose that x is even and y is odd. Then $n = qx+y$ is odd. If $q-y \ge 2$, then, by Dirac's Theorem [14], a subgraph of $T_{n,q}$ induced by $U_1 \cup U_2 \cup \ldots \cup U_{q-y}$ contains a Hamiltonian cycle C. Since x is even and C is of order $x(q-y)$, it follows that C is of even order. Let F be a set of $\frac{x(q-y)}{2}$ independent edges of C. Then $H = T_{n,q} - F$ is an $(n-x-1)$-regular graph of order n and $a(H) \le q$.

Suppose that $q-y = 1$. Let $F_1 = \{v_{11}u_{11}, v_{12}u_{12}, \ldots, v_{1x}u_{1x}\}$ and $F_2 = \{v_{11}v_{12}, v_{13}v_{14}, \ldots, v_{1(x-1)}v_{1x}\}$. Then $H = (T_{n,q} - F_1) * F_2$ is an $(n-x-1)$-regular graph of order n and $a(H) \le q$. □

Note that if G is an r-regular graph on $r+s$ vertices where $1 \le s \le r$, then $r \ge \frac{r+s}{2}$. It follows, by Dirac's Theorem [14], that G is hamiltonian.

Theorem 3.4.10 Let r and s be integers with $r \ge 3$ and $1 \le s \le r-3$. Then

$$\min(a, r^{r+s}) = \lceil \frac{r+s}{s+1} \rceil.$$

Proof. By Lemma 3.4.8, we have that $\min(a, r^{r+s}) \ge \lceil \frac{r+s}{s+1} \rceil$ if $1 \le s \le r-3$. We have already shown that $\min(a, r^{r+s}) = \lceil \frac{r+s}{s+1} \rceil$ if $s \in \{1, 2\}$. Thus in order to prove this theorem it suffices to construct an r-regular graph G of order $r+s$ such that $a(G) \le \lceil \frac{r+s}{s+1} \rceil$ for $r \ge 3$ and $3 \le s \le r-3$. Suppose that $3 \le s \le r-3$. Let $q = \lceil \frac{r+s}{s+1} \rceil$ and put $r+s = qx+y$, $0 \le y \le q-1$. Then $x \le s+1$ and $x = s+1$ if and only if $y = 0$ and $r+s$ is a multiple of $s+1$.

We first consider the case where $x = s+1$ and therefore $y = 0$ and $q = \frac{r+s}{s+1}$ is an integer. Thus the graph $T_{r+s,q}$ is an $(r-1)$-regular graph of order $r+s$ and $a(T_{r+s,q}) \leq q$. By Theorem 3.4.9, there exists an r-regular graph G of order $r+s$ with $a(G) \leq q$ if s is odd or both s and q are even. If s is even and q is odd, then $r+s = q(s+1)$ is odd and hence r is even which implies s is odd. This is a contradiction.

Now suppose that $y > 0$ and therefore $x \leq s$. Thus $T_{r+s,q}$ contains y partite sets of cardinality $x+1$ and $q-y$ partite sets of cardinality x. By Theorem 3.4.9, there exists an $(r+s-x)$-regular graph G of order $r+s$ with $a(G) \leq q$ if x is odd or both x and y are even, and there exists an $(r+s-x-1)$-regular graph H of order $r+s$ with $a(H) \leq q$ if x is even and y is odd. Since $x \leq s$, it follows that $r+s-x \geq r$. We will consider two cases according to the parity of x and y as in Theorem 3.4.9.

Case 1. If x is odd or both x and y are even, then, by Theorem 3.4.9, there exists an $(r+s-x)$-regular graph G of order $r+s$ with $a(G) \leq q$. Since $x \leq s$, it follows that $r+s-x \geq r$. If $s-x = 0$, then $G \in \mathcal{R}(r^{r+s})$ with $a(G) \leq q$, as required. Suppose that $s-x$ is even and $s-x \geq 2$. By Dirac's Theorem [14], G contains enough edge-disjoint hamiltonian cycles whose removal produces an r-regular graph G' of order $r+s$ and $a(G') \leq a(G) \leq q$, as required.

Suppose $s-x$ is odd. Then either $r+s-x$ or r is odd and therefore $r+s$ is even. Also by Dirac's Theorem [14], G contains a 1-factor F where $G-F$ is an $(r+s-x-1)$-regular graph and $a(G-F) \leq a(G) \leq q$. Since $s-x-1$ is even, the result follows by the same argument as above.

Case 2. If x is even and y is odd, then, by Theorem 3.4.9, there exists an $(r+s-x-1)$-regular graph H of order $r+s$ and $a(H) \leq q$. Since $r+s = qx+y$ is odd, it follows that both $r+s-x-1$ and r are even. Since $x \leq s$, it follows that $r+s-x-1 \geq r-1$. Therefore, it follows by the parity that $r+s-x-1 \geq r$. The result follows easily by the same argument as in Case 1. This completes the proof.

□

Chartrand and Kronk [9] obtained a good upper bound for the vertex arboricity. In particular, they proved the following theorem.

Theorem 3.4.11 *For each graph G, $a(G) \leq 1 + \lfloor \frac{\max\delta(G')}{2} \rfloor$, where the maximum is taken over all induced subgraphs G' of G.*

If we restrict ourselves to the class of regular graphs, we obtain the following corollary to Theorem 3.4.11.

Corollary 3.4.12 *Let G be an r-regular graph. Then $a(G) \leq 1 + \lfloor \frac{r}{2} \rfloor$.*

In general the bound given in Corollary 3.4.12 is not sharp but it is sharp in the class of r-regular graphs of order $n \geq 2r + 2$ as we will prove in the next theorem.

Theorem 3.4.13 *For $n \geq 2r + 2, \max(a, r^n) = 1 + \lfloor \frac{r}{2} \rfloor$.*

Proof. The result follows easily if $r \in \{0, 1, 2\}$. Suppose that $r \geq 3$. We write $n = (r + 1)q + t,\ 0 \leq t \leq r$. Thus $q \geq 2$. Put $G = (q - 1)K_{r+1} \cup H$, where H is an r-regular graph of order $r + t + 1$. Since $a(K_{r+1}) = \lceil \frac{r+1}{2} \rceil = 1 + \lfloor \frac{r}{2} \rfloor$ and $a(H) \leq 1 + \lfloor \frac{r}{2} \rfloor$, it follows that $a(G) = 1 + \lfloor \frac{r}{2} \rfloor$. $\square$

CHAPTER 4

SUMMARY AND OPEN PROBLEMS

In this dissertation, we have found several significant results concerning the forest number of graphs and the arboricity of graphs. More precisely, we found the range of forest numbers in some classes, for example, the class of connected subgraphs of cubic graphs, the class of graphs of order n and size m, and the class of biquartic graphs. We summarize our results in Section 4.1. In Section 4.2, we list some open problems in this direction that we have encountered over the past few years during our study of this graph parameter.

4.1 Summary

This section summarizes our comprehensive work concerning interpolation and extremal results for the graph parameters f and a. The results can be divided into 4 parts. The first part deals with our work on the forest number in some class of cubic graphs of order n. In particular, Punnim [28, 29, 32] proved that f interpolates over $\mathcal{R}(3^n)$ and $\mathcal{CR}(3^n)$ and furthermore, the corresponding extremal results on respective classes of cubic graphs were obtained in [31, 34, 35]. We proved in Section 3.1 that if G runs over the class of connected subgraphs of $\mathcal{CR}(3^n)$, then the values $f(G)$ completely cover a line segment $[a, b]$ of positive integers. Let $\mathcal{J}_1 = \mathcal{CR}(3^n)$ and $\mathcal{J}_2$ be the class of all connected cubic K_4'-free graphs of order n, where K_4' is a graph obtained from K_4 by subdividing one edge. Finally, let $\mathcal{J}_3$ be the class of all connected cubic triangle-free graphs of order n. It is clear that $\mathcal{J}_3 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_1$. Let $X_n = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$. For each $\mathcal{J} \in X_n$, the range of the forest number $\{f(G) : G \in \mathcal{J}\}$ is uniquely determinined by $\min(f, \mathcal{J}) = \min\{f(G) : G \in \mathcal{J}\}$ and $\max(f, \mathcal{J}) = \max\{f(G) : G \in \mathcal{J}\}$. We found the exact values of $\min(f, \mathcal{J})$ and $\max(f, \mathcal{J})$ in Section 3.1 in all situations by the following.

1. $$\max(f, \mathcal{J}) = \begin{cases} \lfloor \frac{3n-2}{4} \rfloor & if\ \mathcal{J} \neq \mathcal{J}_1 - \mathcal{J}_2, \\ \lfloor \frac{3n-4}{4} \rfloor & if\ \mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2, \end{cases}$$

2. $\min(f, \mathcal{J}) = \lceil \frac{2n}{3} \rceil$ if $\mathcal{J} \in \{\mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$,

3. if, $\mathcal{J} \notin \{\mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$, then

$$\min(f, \mathcal{J}) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 \pmod 8, \\ \lceil \frac{5}{8}n \rceil & otherwise. \end{cases}$$

In Section 3.2, the extremal results on f over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$ were obtained in all situations. In 2007, Punnim [36] proved that if G runs over the set $\mathcal{G}(n, m)$, then the values $f(G)$ completely cover a line segment $[a, b]$ of positive intgers. Let $\mathcal{G}(n, m)$ be the set of all graphs of order n and size m and $\mathcal{CG}(n, m)$ be the subset of $\mathcal{G}(n, m)$ consisting of all connected subgraphs. We were able to obtain the extremal results for the forest number in the class $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$ as given below:

1. For an integer $i \in \{1, 2, 3, \ldots, n-2\}$, let $\mathbf{M}_n(n-i) = (i+1)n - \frac{i^2+3i+2}{2}$. Then $\max(f; n, m) = n - i$ if and only if $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$ and $\mathrm{Max}(f; n, m) = n - i$ if and only if $m \geq n-1$ and $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$.

For integers n and a, let $\mathbf{m}_n(a)$ be the minimum number of edges of a graph of order n having the forest number a and $\mathbf{cm}_n(a)$ be the minimum number of edges of a connected graph of order n having the forest number a. Then

2. $\min(f; n, m) = \max(f; n, m) = n$ if and only if $m \in \{0, 1, 2\}$, and

3. $\min(f; n, m) = \max(f; n, m) = \mathrm{Min}(f; n, m) = \mathrm{Max}(f; n, m) = 2$ if and only if $m = \binom{n}{2}$,

4. for $3 \leq a \leq n-1, \min(f; n, m) = a$ if and only if $\mathbf{m}_n(a) \leq m < \mathbf{m}_n(a-1)$, and $\mathrm{Min}(f; n, m) = a$ if and only if $\mathbf{cm}_n(a) \leq m < \mathbf{cm}_n(a-1)$.

Alon [1] proved that if G is a 4-regular bipartite graphs of order $2n$, then $f(G) \geq \frac{10}{9}n$. In Section 3.3 we were able to improve the bound from $\frac{10}{9} \cdot n$ to $\frac{8}{7} \cdot n$. Furthermore, we proved that if G runs over the set $\mathcal{B}(4^{2n})$ of all 4-regular bipartite graphs of order $2n$, then the values of $f(G)$ completely cover a line segment $[a, b]$. We found that $\max(f, \mathcal{B}(r^{2n})) = \lfloor \frac{nr-1}{r-1} \rfloor$ and $\min(f, \mathcal{B}(4^{2n})) = n + \frac{n}{7}$.

There are two types of decompositions of a graph G into forests, namely, vertices or edges. The vertex (edge) arboricity of G denoted by $a(G)$ ($a_1(G)$) is the minimum number of subsets into which the vertex (edge) set of G can be partitioned so that each subset induces an acyclic subgraph. The problem of determining the range of vertex (edge) arboricity in the class of r-regular graphs is considered in Section 3.4. We proved in this section that if $\pi \in \{a, a_1\}$, then there exist integers $x(\pi)$ and $y(\pi)$ such that a graphical sequence $\mathbf{d}$ has a realization G with $\pi(G) = z$ if and only if z is an integer satisfying $x(\pi) \leq z \leq y(\pi)$. Thus for an arbitrary graphical sequence $\mathbf{d}$ and $\pi \in \{a, a_1\}$, the two invariants

$$x(\pi) = \min(\pi, \mathbf{d}) := \min\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\} \text{ and}$$

$$y(\pi) = \max(\pi, \mathbf{d}) := \max\{\pi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

naturally arise and hence $\pi(\mathbf{d}) := \{\pi(G) : G \in \mathcal{R}(\mathbf{d})\} = \{z \in \mathbb{Z} : x(\pi) \leq z \leq y(\pi)\}$. Let $\mathbf{d} = r^n$ be the degree sequence of an r-regular graph of order n. We prove that $a_1(r^n) = \{\lceil \frac{r+1}{2} \rceil\}$. We consider the corresponding extremal problem on vertex arboricity and obtain $\min(a, r^n)$ in all situations and $\max(a, r^n)$ for all $n \geq 2r + 2$, as given below:

1. $\min(a, \mathcal{R}(r^n)) = 2$ for all $n \geq 2r - 2$, and for r and s be integers with $r \geq 3$ and $1 \leq s \leq r - 3, \min(a, \mathcal{R}(r^n)) = \lceil \frac{r+s}{s+1} \rceil$.
2. For $n \geq 2n + 2, \max(a, \mathcal{R}(r^n)) = 1 + \lfloor \frac{n}{2} \rfloor$.

4.2 Open Problems

We have encountered some interesting open problems during our study of the graph parameters f and a.

Based on our results, we give the following open problems:

Open Problem 1 Let $n \geq 5$. What is the minimum value of the forest number in the classes of r-regular bipartite graphs of order $2n$?

Let G be a graph of order n and size m. As mentioned earlier, Theorem 2.2.3 is sharp for $m \leq \frac{3n}{2}$ and $m \equiv 0 \pmod{12}$, as shown by disjoint copies of Q_3.

For a 4-regular graph G, Theorem 2.2.3 gives $f(G) \geq \frac{n}{2}$, but the best example has $f(G) = \frac{4n}{7}$.

For a 5-regular graph G, Theorem 2.2.3 gives $f(G) \geq \frac{3n}{8}$, but the best example has $f(G) \geq \frac{n}{2}$.

In Section 3.4, we improved the bound on f in the class of 4-regular bipartite graphs which obtained the $\min(a, \mathcal{B}(r^n))$. A natural next step of our work is the following problem.

Open Problem 2 What is the maximum value of the vertex arboricity in the class of r-regular graphs of order $n < 2r + 2$?

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