

ENTANGLEMENT OF TWO COUPLED TWO-LEVEL ATOMS



Presented in Partial Fulfillment of the Requirements for the
Master of Science Degree in Physics
at Srinakharinwirot University

April 2011

ENTANGLEMENT OF TWO COUPLED TWO-LEVEL ATOMS



Presented in Partial Fulfillment of the Requirements for the
Master of Science Degree in Physics
at Srinakharinwirot University

April 2011

Copyright 2011 by Srinakharinwirot University

ENTANGLEMENT OF TWO COUPLED TWO-LEVEL ATOMS



Presented in Partial Fulfillment of the Requirements for the
Master of Science Degree in Physics
at Srinakharinwirot University

April 2011

Suttipong Locharoenrat. (2011). *Entanglement of two coupled two-level atoms*. Master thesis, M.Sc.(Physics). Bangkok: Graduate School, Srinakharinwirot University.
Advisor Committee: Dr.Supitch Khemmani, Dr.Jaturong Sukonthachat.

In this thesis, the quantum entanglement of a system of two coupled two-level atoms in thermal equilibrium under an applied laser field has been investigated theoretically. It is shown that, before the laser field is applied, the thermal state is separable (inseparable or entangled) for all temperatures high (low) enough such that the certain inequality holds (is violated). Moreover, a quantitative study shows that the degree of quantum entanglement is zero not only at high temperatures but also approaches zero in the low temperature limit. After the laser field is applied, the degree of quantum entanglement at each temperature oscillates in time between zero and its own maximum value except at the very high temperature where it always vanishes. Furthermore, the degree of quantum entanglement at each time, unlike the situation before the laser field is applied, does not approach zero in the low temperature limit.

Keywords: entanglement, two-level atom, separability, partial transposition, negativity, PPT-criterion, laser field, Rabi's oscillation.

เอนแทงเกิลเมนต์ของคู่ของอะตอมสองระดับ



เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา
ตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต สาขาวิชาฟิสิกส์

เมษายน 2554

สุทธิพงษ์ ไล่เจียรรัตน์. (2554). เอนแทงเกิลเมนต์ของคู่ของอะตอมสองระดับ. ปรินซ์นิพนธ์
วท.ม.(ฟิสิกส์). กรุงเทพฯ: บัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ.
คณะกรรมการควบคุม: ดร.สุพิชญ์ แวมมณี, ดร.จตุรงค์ สุคนธชาติ.

งานวิจัยนี้ ศึกษาในเชิงทฤษฎีเกี่ยวกับควอนตัมเอนแทงเกิลเมนต์ของระบบคู่ของอะตอม
สองระดับในสมดุลความร้อนภายใต้สนามเลเซอร์ การศึกษาพฤติกรรมของระบบนี้พบว่า เมื่อยังไม่มี
สนามเลเซอร์ สถานะเชิงความร้อนจะแยกกันได้ (แยกกันไม่ได้หรือเอนแทงเกิล) ที่ทุก ๆ อุณหภูมิที่
สูง (ต่ำ) พอ ซึ่งสอดคล้อง (ไม่สอดคล้อง) กับสมการที่เหมาะสม การศึกษาในเชิงปริมาณยังบอก
อีกว่า ระดับชั้นควอนตัมเอนแทงเกิลเมนต์จะเป็นศูนย์ ไม่เพียงแต่ที่อุณหภูมิสูง แต่ยังเป็นศูนย์ใน
ลิมิตที่อุณหภูมิต่ำด้วย เมื่อใส่สนามเลเซอร์เข้าไปยังระบบ ระดับชั้นควอนตัมเอนแทงเกิลเมนต์ที่
อุณหภูมิต่ำ ๆ จะกว้างไปมาในเวลาระหว่างค่าศูนย์กับค่าสูงสุด ยกเว้นแต่ที่อุณหภูมิสูงมาก ๆ ซึ่งมี
ค่าเป็นศูนย์เสมอ ยิ่งไปกว่านั้น ระดับชั้นควอนตัมเอนแทงเกิลเมนต์ที่เวลาหนึ่ง ๆ หลังจากใส่สนาม
เลเซอร์แล้ว จะไม่เข้าสู่ศูนย์ในลิมิตที่อุณหภูมิต่ำ ซึ่งไม่เหมือนกับในกรณีที่ยังไม่ใส่สนามเลเซอร์

คำสำคัญ: เอนแทงเกิลเมนต์, อะตอมสองระดับ, การแยกกันได้, การสับเปลี่ยนบางส่วน, เนกาทีวีตี,
เกนท์พีพีที, สนามเลเซอร์, การกวัดแกว่งของราบี

The thesis titled
“Entanglement of Two Coupled Two-Level Atoms”

by

Suttipong Locharoenrat

has approved by The Graduate School as partial fulfillment of the requirements for the
Master of Science degree in Physics at Srinakharinwirot University.

..... Dean of Graduate School
(Assoc. Prof. Dr.Somchai Santiwatanakul)

Thesis Committee

Oral Defense Committee

..... Major-advisor
(Dr.Supitch Khemmani)

..... Chair
(Assist. Prof. Dr.Fuenglada Veerasai)

..... Co-advisor
(Dr.Jaturong Sukonthachat)

..... Committee
(Dr.Supitch Khemmani)

..... Committee
(Dr.Jaturong Sukonthachat)

..... Committee
(Assist. Prof. Dr.Sutee Boonchui)

Acknowledgements

This thesis could not successfully completed without the kindness of advisor's team. First and foremost to my major advisor, Dr.Supitch Khemmani, who gave good advice and be guidance of this thesis since start until successful. My co-advisor, Dr.Jaturong Sukonthachat, for all of kindness and helpful.

I would like to thanks for all of my instructors in department of physics, faculty of science, Srinakharinwirot University, Assist. Prof. Dr.Fuenglada Veerasai and Assist. Prof. Dr.Sutee Boonchui, for all of comment and good suggestion to make my thesis more completed.

I also wish to express my grateful special thanks to Tosayanon's family, for their help and support. A special thanks goes to all my relation and my friends for their kind assistance and encouragement.

Finally, my graduation would not be achieved without best wish from my family, who help me for everything and always gives me greatest love, encouragement and financial support until this study completed.

Suttipong Locharoenrat

TABLE OF CONTENTS

Chapter	Page
1 INTRODUCTION.....	1
Rationale.....	1
Purposes.....	1
Limitation.....	2
Benefits.....	2
2 REVIEW OF THE LITERATURE.....	3
Single system.....	3
Composite system and entanglement.....	7
Hidden-variable and violation of Bell's inequality.....	11
Positive Partial transposition criterion (PPT-criterion).....	13
Necessary and sufficient conditions for separability.....	15
Negativity.....	15
Investigation of the thermal state of two coupled two-level atoms.....	16
3 RESEARCH METHODOLOGY AND RESULTS.....	19
Separability of two coupled two-level atoms in thermal equilibrium.....	19
Two-level atom in a laser field: Rabi's oscillation.....	23
State of two coupled two-level atoms initially in thermal equilibrium under an applied laser field.....	25
The effect of laser field on separability of two coupled two-level atom initially in thermal equilibrium.....	30
4 CONCLUSION AND DISCUSSION.....	38

TABLE OF CONTENTS

Chapter	Page
BIBLIOGRAPHY	40
APPENDIX.....	42
VITAE.....	47



LIST OF FIGURES

Figure	Page
1 Apparatus prepared for the construction of classically correlated state.....	11
2 System of two coupled two-level atoms in thermal equilibrium at absolute temperature T	16
3 Condition for separability in resonance case.....	21
4 Plot of negativity $N(\rho)$ as a function of x at $\omega = 2$ and $g = 1$. Notice that $N(\rho) = 0$ for $0 < x < 0.88$ (separable region) and then goes up for $x > 0.88$ (inseparable or entangled region) to its peak and finally reaches to zero again for large x	22
5 Plot of negativity $N(\rho)$ as a function of x at $\omega = 1.03$ and $g = 1$. Notice that this graph has the same behavior as the one in figure 4, however $\omega = 1.03 < 2$ causes $N(\rho)$ having smaller peak and going to zero at larger x	22
6 Two-level atom having energy gap $\hbar\omega$	23
7 System of two coupled two-level atoms initially in thermal equilibrium under the applied field.....	25
8 The plot of negativity $N(\rho)$ as a function of x and t . At each x , $N(\rho)$ oscillates in time and vanishes at some t	31
9 Different view point of Fig.8.....	32
10 Different view point of Fig.8. Notice that $N(\rho) \rightarrow 0$ for large x at $t = 0$ but $N(\rho) \rightarrow \text{constant} \neq 0$ for large x at all $t > 0$..	32
11 The section of Fig.8 at $t = 10$. Here, the plot is extended to $x = 150$ in order to see that $N(\rho) \rightarrow \text{constant} \neq 0$ for large x . Notice that in the separable region, i.e. $0 < x < 0.88$, $N(\rho) = 0$ (see inset).....	33

LIST OF FIGURES

Figure		Page
12	The section at Fig. 8 at $t = 0.1$. Although $t = 0.1$ is still very small and $N(\rho)$ seems to reach zero for large x , it instead reaches the small constant $\neq 0$ (see inset).....	33
13	The plot of negativity $N(\rho)$ as a function of x and t emphasizing on the separable region i.e. around $x \approx 0.88$. As time goes, one can see the change of states from separable at $t = 0$ to inseparable ($N(\rho) \neq 0$) at certain times and vis ver sa. However, for small (enough) x , $N(\rho) = 0$ for all time t	34
14	Different view point of Fig.11.....	34
15	The section of Fig.13 at $x = 0.7$. Since $x = 0.7 < 0.88$ is in the separable region, $N(\rho) = 0$ at $t = 0$. As time goes, one can see tiny changes of $N(\rho)$ which shows that, at certain times, the entangled states exist but have small degree of entanglement (compared to what happen in Fig.18 at $x = 31$).....	35
16	The plots of negativity $N(\rho)$ as a function of x and t emphasizing on the inseparable region i.e. $x > 0.88$. As time goes, one can see that the system which is initially entangled becomes separable (disentagled) at certain times and vis ver sa.....	35
17	Different view point of Fig.13.....	36
18	The section of Fig.16 at $x = 31$. Since $x = 31 > 0.88$ is in the inseparable region, $N(\rho)$ at $t = 0$ is not equal to zero. However, as time goes, $N(\rho)$ oscillates between high degree of entanglement and lower, some are zero.....	36

Chapter 1

Introduction

1.1 Rationale

Many years ago until now, several attempts have been made in order to construct the high speed communication apparatus by using the well-known pure quantum correlation called “entanglement”. This entanglement enables us, in principle, to create an instantaneous interaction between very far objects such as the famous system of two electrons used to describe the EPR paradox (Einstein; et al. 1935: 777) and the system of two photons or atoms used to create the cryptography and high speed computation called quantum cryptography and quantum computer respectively. Once this kind of system can be constructed, the instantaneous interaction due to entanglement may allow us to make the extremely fast data transportation from one place to another place no matter how far the distance between the two places are, e.g. the technique called quantum teleportation. One of the main problem to create such a communication system is the decoherence due to the contact of the system with its environment which causes the destruction of entanglement, e.g. by the effect of temperature.

In this research, the entanglement of the system of two coupled two-level atoms will be investigated in two aspects by using the definition of separable state given by Werner (Werner. 1989: 4277) and the renowned Positive Partial Transposition criterion (PPT-criterion) given by Peres (Peres. 1996: 1413). The first aspect is to study the coupling and temperature effects on the entanglement of two coupled two-level atoms in thermal equilibrium where the coupling between two atoms is the dipole-dipole interaction. The second aspect is to study the effect of a laser field, which is applied to both atoms, on entanglement of the two coupled two-level atoms initially in the thermal equilibrium, i.e. we aim to search whether the entanglement between two atoms can be controlled by a laser field or not.

1.2 Purposes

1. To study the entanglement of two coupled two-level atoms in thermal equilibrium.

2. To study some effects of laser field on the entanglement of two coupled two-level atoms initially in thermal equilibrium.

3. To pave the way for future applications especially on the field of quantum communication.

1.3 Limitation

In this research, two atoms are considered as they have only two levels and the interaction between them is the electric dipole-dipole interaction.

1.4 Benefits

1. To obtain some new aspects of the entanglement of two coupled two-level atoms in thermal equilibrium.

2. To obtain some new aspects of the entanglement of two coupled two-level atoms initially in thermal equilibrium under applied laser field.

3. To provide the primary system for further applications especially on the field of quantum communication.

Chapter 2

Review of Literature

This chapter consists of the following topics:

- 2.1 Single system
- 2.2 Composite system and entanglement
- 2.3 Hidden-variable and violation of Bell's inequality
- 2.4 Positive Partial Transposition criterion (PPT-criterion)
- 2.5 Necessary and sufficient conditions for separability
- 2.6 Negativity
- 2.7 Investigation of the thermal state of two coupled two-level atoms

2.1 Single system (Sakurai, 1994: 176-179)

The density operator formalism, pioneered by J. von Neumann in 1927, describes physical situations quantitatively with mixed as well as pure ensembles. Our general discussion here is not restricted to spin $\frac{1}{2}$ systems, but for illustrative purposes we return repeatedly to spin $\frac{1}{2}$ systems.

A pure ensemble by definition is a collection of physical systems such that every member is characterized by the same ket $|\alpha\rangle$. In contrast, in a mixed ensemble, a fraction of the members with relative population p_1 are characterized by $|\alpha^{(1)}\rangle$, some other fraction with relative population p_2 , by $|\alpha^{(2)}\rangle$, and so on. Roughly speaking, a mixed ensemble can be viewed as a mixture of pure ensembles, just as the name suggests. The fractional populations are constrained to satisfy the normalization condition

$$\sum_i p_i = 1. \quad (1)$$

As we mentioned previously, $|\alpha^{(1)}\rangle$ and $|\alpha^{(2)}\rangle$ need not be orthogonal. Furthermore, the number of terms in the i sum of (1) need not coincide with the dimensionality N of the ket space; it can easily exceed N . For example, for spin $\frac{1}{2}$ systems with $N=2$, we may consider 40% with spin in the positive z-direction, 30% with spin in the positive x-direction, and the remaining 30% with spin in the negative y-direction.

Suppose we make a measurement on a mixed ensemble of some observable A . We may ask what is the average measured value of A when a large number of measurements are carried out. The answer is given by the ensemble average of A , which is defined by

$$\begin{aligned}\langle A \rangle &\equiv \sum_i p_i \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle \\ &= \sum_i \sum_{a'} p_i \left| \langle a' | \alpha^{(i)} \rangle \right|^2 a' \quad ,\end{aligned}\quad (2)$$

where $|a'\rangle$ is an eigenket of A . Recall that $\langle \alpha^{(i)} | A | \alpha^{(i)} \rangle$ is the usual quantum mechanical expectation value of A taken with respect to state $|\alpha^{(i)}\rangle$. Eq. (2) tells us that these expectation values must further be weighted by the corresponding fractional populations p_i . Notice how probabilistic concepts enter twice; first in $\left| \langle a' | \alpha^{(i)} \rangle \right|^2$ for the quantum-mechanical probability and second, the factor p_i for finding in the ensemble a quantum-mechanical state characterized by $|\alpha^{(i)}\rangle$.

We can now rewrite ensemble average (2) using a more general basis, $\{|b'\rangle\}$:

$$\begin{aligned}\langle A \rangle &= \sum_i p_i \sum_{b'} \sum_{b''} \langle \alpha^{(i)} | b' \rangle \langle b' | A | b'' \rangle \langle b'' | \alpha^{(i)} \rangle \\ &= \sum_{b'} \sum_{b''} \left(\sum_i p_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \right) \langle b' | A | b'' \rangle \quad .\end{aligned}\quad (3)$$

The number of terms in the sum of the $b'(b'')$ is just the dimensionality of the ket space, while the number of terms in the sum of the i depends on how the mixed ensemble is viewed as a mixture of pure ensemble which does not depend on the particular observable A is factored out. This motivates us to define the density operator ρ as follows:

$$\rho \equiv \sum_i p_i |\alpha^{(i)}\rangle \langle \alpha^{(i)}| \quad .\quad (4)$$

The elements of the corresponding density matrix have the following form:

$$\langle b'' | \rho | b' \rangle = \sum_i p_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \quad .\quad (5)$$

The density operator contains all the physically significant information we can possibly obtain about the ensemble in question. Returning to eq. (3), we see that the ensemble average can be written as

$$\begin{aligned}\langle A \rangle &= \sum_{b'} \sum_{b''} \langle b'' | \rho | b' \rangle \langle b' | A | b'' \rangle \\ &= \text{tr}(\rho A) \quad .\end{aligned}\tag{6}$$

Because the trace is independent of representations, $\text{tr}(\rho A)$ can be evaluated using any convenient basis. As a result, eq. (6) is an extremely powerful relation.

There are three properties of the density operator worth recording. First, the density operator is Hermitian, as is evident from eq. (4). Second, the density operator satisfies the normalization condition

$$\begin{aligned}\text{tr}(\rho) &= \sum_i \sum_{b'} p_i \langle b' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \\ &= \sum_i p_i \langle \alpha^{(i)} | \alpha^{(i)} \rangle \\ &= 1 \quad .\end{aligned}\tag{7}$$

Third, the density operator are positive, all eigenvalue are positive ($\rho \geq 0$).

Because of the Hermiticity and the normalization condition, for spin $\frac{1}{2}$ system with dimensionality 2, the density operator or the corresponding density matrix is characterized by three independent real parameters. Four real numbers characterized a $2 \otimes 2$ Hermitian matrix. However, only three are independent because of the normalization condition. The three numbers needed are $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$; the reader may verify that knowledge of these three ensemble averages is sufficient to reconstruct the density operator. The manner in which a mixed ensemble is formed can be rather involved. We may mix pure ensembles characterized by all kinds of $|\alpha^{(i)}\rangle$'s with appropriate p_i 's; yet for spin $\frac{1}{2}$ systems three real numbers completely characterize the ensemble in question. This strongly suggests that a mixed ensemble can be decomposed into pure ensembles in many different ways.

A pure ensemble is specified by $p_i = 1$ for some $|\alpha^{(i)}\rangle$ – with $i = n$, for instance – and $p_i = 0$ for all other conceivable state kets, so the corresponding density operator is written as

$$\rho = |\alpha^{(n)}\rangle\langle\alpha^{(n)}| \quad (8)$$

with no summation. Clearly, the density operator for a pure ensemble is idempotent, that is,

$$\rho^2 = \rho \quad (9)$$

or, equivalently,

$$\rho(\rho - 1) = 0 \quad (10)$$

Thus, for a pure ensemble only, we have

$$\text{tr}(\rho^2) = 1 \quad (11)$$

in addition to eq. (7). The eigenvalues of the density operator for a pure ensemble are zero or one, as can be seen by inserting a complete set of base kets that diagonalizes the density matrix, for a pure ensemble it looks like

$$\rho = \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \\ & & & & & 0 \\ & & & & & & \ddots \\ & & & & & & & 0 \\ 0 & & & & & & & & 0 \end{bmatrix} \quad \text{(diagonal form)} \quad (12)$$

It can be shown that $\text{tr}(\rho^2)$ is maximal when the ensemble is pure; for a mixed ensemble $\text{tr}(\rho^2)$ is a positive number less than one.

Given a density operator, let us see how the corresponding density matrix can be constructed in some specified basis. To this end, we first recall that

$$|\alpha\rangle\langle\alpha| = \sum_{b'} \sum_{b''} |b'\rangle\langle b'| \alpha\rangle\langle\alpha| b''\rangle\langle b''|. \quad (13)$$

This shows that we can form the square matrix corresponding to $|\alpha^{(i)}\rangle\langle\alpha^{(i)}|$ by multiplying the column matrix formed by $\langle b'|\alpha^{(i)}\rangle$ with the row matrix formed by $\langle\alpha^{(i)}|b''\rangle$ which, of course, is equal to $\langle b''|\alpha^{(i)}\rangle^*$. The final step is to sum such square matrices with weighting factors p_i , as indicated in eq. (4). The final form agrees with eq. (5), as expected.

2.2 Composite system and entanglement

Consider the composite system described by the vector (Hilbert) space V consisting of two subsystems described by V_1 and V_2 physically, V of the composite system can be written as $V_1 \otimes V_2$, called the tensor product of V_1 and V_2 defined as follows.

Definition of tensor product (Cohen-Tannoudji; et al. 1977: 154-157)

Let V_1 and V_2 be two vector spaces of dimension N_1 and N_2 respectively. Vectors and operators of these spaces will be assigned an index, (1) or (2), depending on whether they belong to V_1 or V_2 . The vector space V is called the tensor product of V_1 and V_2 :

$$V = V_1 \otimes V_2$$

if there is associated with each pair of vectors, $|\varphi(1)\rangle$ belonging to V_1 and $|\chi(2)\rangle$ belonging to V_2 , a vector of V denote by:

$$|\varphi(1)\rangle \otimes |\chi(2)\rangle$$

which is called the tensor product of $|\varphi(1)\rangle$ and $|\chi(2)\rangle$, this correspondence satisfying the following conditions :

1. It is linear with respect to multiplication by complex numbers

$$[\lambda|\varphi(1)\rangle] \otimes |\chi(2)\rangle = \lambda[|\varphi(1)\rangle \otimes |\chi(2)\rangle]$$

$$|\varphi(1)\rangle \otimes [\mu|\chi(2)\rangle] = \mu[|\varphi(1)\rangle \otimes |\chi(2)\rangle]$$

2. It is distributive with respect to vector addition:

$$|\varphi(1)\rangle \otimes [|\chi_1(2)\rangle + |\chi_2(2)\rangle] = |\varphi(1)\rangle \otimes |\chi_1(2)\rangle + |\varphi(1)\rangle \otimes |\chi_2(2)\rangle$$

$$[\varphi_1(1) + \varphi_2(1)] \otimes |\chi(2)\rangle = |\varphi_1(1)\rangle \otimes |\chi(2)\rangle + |\varphi_2(1)\rangle \otimes |\chi(2)\rangle$$

3. When a basis has been chosen in each of the space V_1 and V_2 , $\{|u_i(1)\rangle\}$ for V_1 and $\{|v_l(2)\rangle\}$ for V_2 , the set of vectors $|u_i(1)\rangle \otimes |v_l(2)\rangle$ constitutes a basis in V . If N_1 and N_2 are finite, the dimension of V is consequently $N_1 N_2$.

Vectors of V

1. Let us first consider a tensor product vector, $|\varphi(1)\rangle \otimes |\chi(2)\rangle$. Whatever $|\varphi(1)\rangle$ and $|\chi(2)\rangle$ may be, they can be expressed in the $\{|u_i(1)\rangle\}$ and $\{|v_l(2)\rangle\}$ bases respectively:

$$|\varphi(1)\rangle = \sum_i a_i |u_i(1)\rangle$$

$$|\chi(2)\rangle = \sum_l b_l |v_l(2)\rangle$$

The expansion of the vector $|\varphi(1)\rangle \otimes |\chi(2)\rangle$ in the $\{|u_i(1)\rangle \otimes |v_l(2)\rangle\}$ basis can be written as:

$$|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,l} a_i b_l |u_i(1)\rangle \otimes |v_l(2)\rangle$$

Therefore, the components of a tensor product vector are the products of the components of the two vectors of the product.

2. There exist in V vectors which are not tensor products of a vector of V_1 by a vector of V_2 . Since $\{|u_i(1)\rangle \otimes |v_l(2)\rangle\}$ by hypothesis constitutes a basis in V , the most general vector of V is expressed by:

$$|\psi\rangle = \sum_{i,l} c_{i,l} |u_i(1)\rangle \otimes |v_l(2)\rangle$$

Given $N_1 N_2$ arbitrary complex numbers $c_{i,l}$, it is not always possible to put them in the form of product, $a_i b_l$, of N_1 numbers a_i and N_2 numbers b_l . Therefore, in general, vector $|\varphi(1)\rangle$ and $|\chi(2)\rangle$ of which $|\psi\rangle$ is the tensor product do not exist. However, an arbitrary vector of V can always be decomposed into a linear combination of tensor product vectors.

The scalar product in V

The existence of scalar product in V_1 and V_2 permits us to define one in V as well. We first define the scalar product of $|\varphi(1)\chi(2)\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$ by $|\varphi'(1)\chi'(2)\rangle = |\varphi'(1)\rangle \otimes |\chi'(2)\rangle$ by setting:

$$\langle \varphi'(1)\chi'(2) | \varphi(1)\chi(2) \rangle = \langle \varphi'(1) | \varphi(1) \rangle \langle \chi'(2) | \chi(2) \rangle$$

For two arbitrary vectors of V , we simply use the fundamental properties of the scalar product, since each of these vectors is a linear combination of tensor product vectors.

Notice, in particular, that the basis $\{|u_i(1)v_l(2)\rangle = |u_i(1)\rangle \otimes |v_l(2)\rangle\}$ is orthonormal if each of the bases $\{|u_i(1)\rangle\}$ and $\{|v_l(2)\rangle\}$ is:

$$\begin{aligned} \langle u_{i'}(1)v_{l'}(2) | u_i(1)v_l(2) \rangle &= \langle u_{i'}(1) | u_i(1) \rangle \langle v_{l'}(2) | v_l(2) \rangle \\ &= \delta_{i'} \delta_{l'} \end{aligned}$$

Tensor product of operator

1. First, consider a linear operator $A(1)$ defined in V_1 . We associate with it a linear operator $\tilde{A}(1)$ acting in V , which we call the extension of $A(1)$ in V , and which is characterized in the following way : when $\tilde{A}(1)$ is applied to a tensor product vector $|\varphi(1)\rangle \otimes |\chi(2)\rangle$, one obtains, by definition:

$$\tilde{A}(1)[|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes |\chi(2)\rangle$$

The hypothesis that $\tilde{A}(1)$ is linear is then sufficient for determining it completely. An arbitrary vector $|\psi\rangle$ of V can be written in the form $|\psi\rangle = \sum_{i,l} c_{i,l} |u_i(1)\rangle \otimes |v_l(2)\rangle$, then gives the action of $\tilde{A}(1)$ on $|\psi\rangle$:

$$\tilde{A}(1)|\psi(1)\rangle = \sum_{i,l} c_{i,l} [A(1)|u_i(1)\rangle] \otimes |v_l(2)\rangle$$

We obtain in an analogous manner the extension $\tilde{B}(2)$ of an operator $B(2)$ initially defined in V_2 .

2. Now let $A(1)$ and $B(2)$ be two linear operators acting respectively in V_1 and V_2 . Their tensor product $A(1) \otimes B(2)$ is the linear operator in V , defined by the following relation which describes its action on the tensor product vectors:

$$[A(1) \otimes B(2)][|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes [B(2)|\chi(2)\rangle]$$

Here also, this definition is sufficient for characterizing $A(1) \otimes B(2)$.

Uncorrelated states (Werner. 1989: 4277),

Consider a composite quantum system described by a Hilbert space $V = V_A \otimes V_B$. An **uncorrelated state** of this system is given by a density operator ρ acting in V of the form $\rho = \rho^A \otimes \rho^B$, where ρ^A and ρ^B are the density operators acting in V_A and V_B respectively. This is equivalent to say that the expectation value $Tr(\rho A \otimes B)$ for the joint measurement of observable A and observable B on the respective subsystem always factorizes,

$$\begin{aligned} Tr(\rho A \otimes B) &= Tr(\rho \cdot A \otimes \mathbf{1}) Tr(\rho \cdot \mathbf{1} \otimes B) \\ &= Tr(\rho^A A) Tr(\rho^B B). \end{aligned} \quad (14)$$

Uncorrelated states can be prepared very easily by using two preparing devices for system A and B , which function independently and yield the states ρ^A and ρ^B respectively. Then the factorization property means that if the measuring devices described by A and B also operate independently, we are simply conducting two separate experiments at the same time and the classical multiplication rule for probability applies.

Correlated states and entanglement (Werner. 1989: 4277)

Suppose that each of the two preparing devices has a switch with settings $i = 1, \dots, n$, and, with setting i , the device produces system in the state ρ_i^A and ρ_i^B . Suppose we have also a random generator, which produces number $i = 1, \dots, n$ with probability p_i . We can combine these three devices into a new preparing apparatus by the following prescription: In each individual experiment, one first draws a random number $i \in \{1, \dots, n\}$. The switches of the two preparing devices are then set according to the result (see figure 1. below). Clearly then, the expectation of a measurement of observables A and B will be

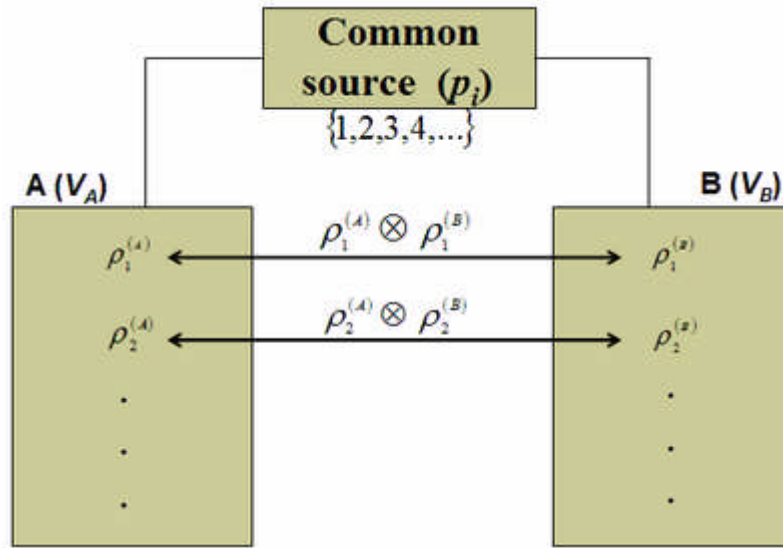


Fig. 1. : Apparatus prepared for the construction of classically correlated state.

$$\sum_i p_i \text{Tr}(\rho_i^A A) \text{Tr}(\rho_i^B B) = \text{Tr}(\rho \cdot A \otimes B)$$

with the density operator

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad (15)$$

The physical source of these correlations is the random generator, which can be chosen as a purely classical device. Therefore, we shall call a density operator (a state) **classically correlated or disentangled** if it can be represented or approximated by density operators of the form eq. (15). States that are not classically correlated is called **EPR correlated or entangled**. Classical correlation does not mean that the state has actually been prepared in the manner described above, but only that its statistical properties can be reproduced by a classical mechanism.

2.3 Hidden-variable and violation of Bell's inequality (Werner. 1989: 4277)

For any set of correlations determined in an experiment one can raise the question whether these correlations can be described within a purely classical "hidden-variable" theory. Such a theory is based on some probability space (Ω, Σ, M) , called the space of

hidden variables, consisting of a σ -algebra Σ of subsets of Ω and a σ -additive normalized measure M on Σ . For any measuring device A with possible outcome ν , one demands the existence of a measurable response function $\omega \mapsto F_A(\nu, \omega) \in \mathfrak{R}$ interpreted as the probability that the outcome ν is obtained in an experiment with known value $\omega \in \Omega$ of the hidden variables. Therefore the response functions must satisfy $F_A(\nu, \omega) \geq 0$ and $\sum_{\nu} F_A(\nu, \omega) = 1$ for every $\omega \in \Omega$, where the sum is over all possible outcomes of the measurement of A . A hidden-variable model for some set of correlations is then given by a probability space and a collection of response functions such that the probability in an experiment with a measuring device A on system A and a device B on system B for obtaining the result ν on A and simultaneously the result μ on B is given by the expression $\int M(d\omega) F_A(\nu, \omega) F_B(\mu, \omega)$.

Any hidden-variable model can be extended to measurements with continuous outcome parameters, and can be modified to a “deterministic model” in which the response functions take only the value 0 and 1. Therefore we shall stay with the above simple definition. The existence of a hidden-variable model is exactly the hypothesis of the usual derivations of Bell’s inequalities, the “locality” of the theory being expressed by the fact that the response function for A is independent of B and vice versa. It is known that while these inequalities are necessary conditions for the existence of a hidden-variable model, they are not sufficient. On the other hand, the set of correlations admitting hidden-variable model is a convex set, and as such is completely described by some set of linear inequalities. We shall refer to any one of these inequalities as a (generalized) Bell’s inequality. Despite some partial results in this direction, no efficient procedure for obtaining all generalized Bell’s inequalities is known.

An interesting question is, then, whether or not the correlations described by a quantum state of a composite system admit a hidden-variable model. To answer this question, let us consider the measuring devices of system A and B represented respectively by observables, i.e. by Hermitian operators $A \in V_A$ and $B \in V_B$ with spectral resolution $A = \sum_{\nu} \alpha_{\nu} P_{\nu}$ and $B = \sum_{\mu} \beta_{\mu} Q_{\mu}$ eigenvalues α_{ν} , β_{μ} and eigenprojection P_{ν} , Q_{μ} . We then say that a state $\rho \in V_A \otimes V_B$ admits a hidden-variable model if there are a probability space (Ω, Σ, M) and response functions, defined for all Hermitian $A = \sum_{\nu} \alpha_{\nu} P_{\nu} \in V_A$ and $B = \sum_{\mu} \beta_{\mu} Q_{\mu} \in V_B$ with discrete spectrum, such that for all A , B , ν and μ ,

$$\int M(d\omega)F_A(v, \omega)F_B(\mu, \omega) = \text{Tr}(W \cdot P_v \otimes Q_\mu) \quad (16)$$

We claim that all classically correlated states admit hidden-variable models, and hence satisfy all Bell inequalities. This can be proven quite simply for convex combinations of products as in eq. (14). We then take $\Omega = \{1, \dots, n\}$, $M(\{r\}) = p_r$, $F_A(v, r) = \text{Tr}(\rho_r^A \cdot P_v)$, and define F_B analogously. Then eq. (15) implies eq. (16). We omit the somewhat technical, but straightforward approximation arguments needed to extend this result to all classically correlated states.

We conclude that every state violating some generalized Bell's inequality, i.e. any state not admitting a hidden-variable model, cannot be classically correlated, i.e. is EPR correlated. The well-known experiments demonstrating a violation of Bell's inequalities can thus be taken as direct experimental evidence for the existence of EPR correlated states. The vital importance of such states for quantum theory is further underlined by the fact that they are automatically generated by an interacting time evolution. To be precise, any unitary time evolution, which takes all classically correlated initial states again to classically correlated states, necessarily factorizes into the product of two separate time evolutions. Consequently, the ground state of an interacting system, which is often especially easy to prepare, is usually not classically correlated. The states of a relativistic quantum-field theory are even more universally EPR correlated, since any state of finite energy violates Bell's inequalities for suitable space like localized observables.

Since any state violating some generalized Bell's inequality is EPR correlated, one might conjecture that the converse holds, i.e. that every state admitting a hidden-variable model is classically correlated. This conjecture is indeed true for pure states of a composite quantum system, given by unit vector in $V_A \otimes V_B$, but is false for general mixed states. In the next section, another criterion called "PPT-criterion", which is claimed to be more sensitive than the violation of Bell's inequality, will be reviewed.

2.4 Positive Partial Transposition criterion (PPT-criterion) (Peres. 1996: 1413)

A striking quantum phenomenon is the inseparability or entanglement of composite quantum systems. Its most famous example is the violation of Bell's inequality, which may be detected if two distant observers, who independently measure subsystems of a composite quantum system, report their results to a common site where that information is analyzed. However, even if Bell's inequality is satisfied by a given composite quantum

system, there is no guarantee that its state can be prepared by two distant observers who receive instructions from a common source. For this to be possible, the density matrix ρ has to be separable i.e. it can be written as

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad (17)$$

where the positive weights p_i satisfy $\sum_i p_i = 1$, and where ρ_i^A and ρ_i^B are density matrices for the two subsystems. A separable system always satisfies Bell's inequality, but the converse is not necessarily true. In the work by Peres (Peres. 1996: 1413) a simple algebraic test is derived, which is a necessary condition for the existence of the decomposition (17) i.e. for the existence of the separability of the state ρ . Peres (Peres. 1996: 1413) also give some examples showing that this criterion is more restrictive than Bell's inequality, or than the α -entropy inequality.

The derivation of this separability condition can be done by writing the density matrix elements explicitly, with all their indices. For example, eq. (17) becomes

$$\rho_{m\mu,n\nu} = \sum_i p_i (\rho_i^A)_{mn} (\rho_i^B)_{\mu\nu} \quad (18)$$

Latin indices refer to the first subsystem, Greek indices to the second one (the subsystems may have different dimensions). Note that this equation can always be satisfied if we replace the quantum density matrices by classical Liouville functions (and the discrete indices are replaced by canonical variables p and q). The reason is that the only constraint that a Liouville function has to satisfy is being non-negative. On the other hand, we want quantum density matrices to have non-negative eigenvalues, rather than non-negative elements, and the latter condition is more difficult to satisfy. Let us now define a new matrix,

$$\sigma_{m\mu,n\nu} \equiv \rho_{n\mu,m\nu} \quad (19)$$

The Latin indices of ρ have been transposed, but not the Greek ones. This is not a unitary transformation but, nevertheless, the σ matrix is Hermitian. When eq. (17) is valid, we have

$$\sigma = \sum_i p_i (\rho_i^A)^T \otimes \rho_i^B \quad (20)$$

Since the transposed matrices $(\rho_i^A)^T \equiv (\rho_i^A)^*$ are non-negative matrices with unit trace, they can also be legitimate density matrices. It follows that none of the eigenvalues of σ is negative. This is a necessary condition for eq. (17) to hold. This condition is called positive partial transpose (PPT) criterion.

Note that the eigenvalues of σ are invariant under separate unitary transformations, U^A and U^B of the bases used by the two observers. In such a case, ρ transforms as

$$\rho \rightarrow (U^A \otimes U^B) \rho (U^A \otimes U^B)^\dagger$$

and we then have

$$\sigma \rightarrow (U^{A^T} \otimes U^B) \sigma (U^{A^T} \otimes U^B)^\dagger$$

which also is unitary transformation, leaving the eigenvalues of σ invariant.

2.5 Necessary and sufficient conditions for separability

Horodeckis (M. Horodecki; et al. 1996: 1) provide necessary and sufficient conditions for the separability of mixed states and find that, for $2 \otimes 2$ and $2 \otimes 3$ systems, the positive of the partial transposition of a state is not only necessary but also sufficient for its separability. However, this is not true for the general systems. We shall use this fact to investigate the entanglement of the system of two coupled two-level atoms, which is the $2 \otimes 2$ system.

2.6 Negativity

Negativity (Vidal; & Werner. 2002: 032314) was introduced by Vidal and Werner in order to measure the degree to which σ (the partial transpose of ρ defined in section 2.4) fails to be positive. Among quantities capable of measuring the so-called “degree of entanglement”, negativity ($N(\rho)$) is the one which is suitable for us to compute the degree of entanglement of our problem. It is defined by

$$N(\rho) = \frac{\|\sigma\|_1 - 1}{2},$$

Where $\|\sigma\|_1 = \text{Tr} \sqrt{\sigma^\dagger \sigma}$ is the trace norm of the partial transpose (of ρ) σ . This negativity is related to its eigenvalues as follows:

$$N(\rho) = \frac{\|\sigma\|_1 - 1}{2} = \frac{\sum_i |\lambda_i| - 1}{2} = \frac{\left(1 + 2 \sum_k |\mu_k|\right) - 1}{2} = \sum_k |\mu_k|$$

Where λ_i is an eigenvalues of σ and $\mu_k = \lambda_k < 0$ is a negative eigenvalues of σ .

Note that if non of its eigenvalues are negative, then $N(\rho) = 0$. However, this implies that there is no entanglement only in the case that PPT-criterion is also the sufficient condition.

2.7 Investigation of the thermal state of two coupled two-level atoms

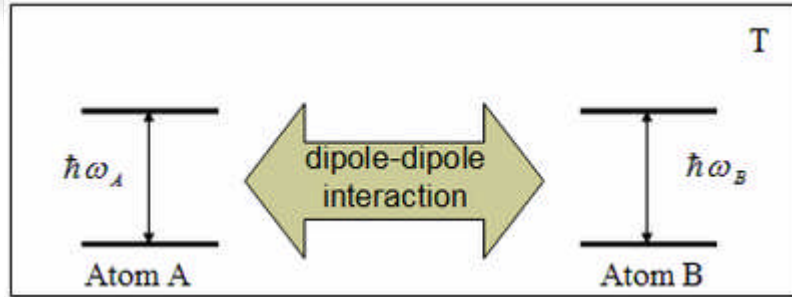


Fig. 2. : System of two coupled two-level atoms in thermal equilibrium at absolute temperature T .

Our system of interest consists of two atoms (see Fig.2. above) which can be described by the following Hamiltonian operator

$$H = \frac{\hbar\omega_A}{2} \sigma_z^A + \frac{\hbar\omega_B}{2} \sigma_z^B + \hbar g (\sigma_-^A \otimes \sigma_+^B + \sigma_+^A \otimes \sigma_-^B), \quad (21)$$

where $\frac{\hbar\omega_A}{2} \sigma_z^A$ represents the energy of atom A,

$\frac{\hbar\omega_B}{2}\sigma_z^B$ represents the energy of atom B ,

$\hbar g(\sigma_-^A \otimes \sigma_+^B + \sigma_+^A \otimes \sigma_-^B)$ represents the dipole-dipole interaction between atom A and atom B (g is the coupling strength),

and σ_z (Pauli spin matrix), σ_{\pm} (Spin flip matrices) are defined by

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Substitute these matrices into Hamiltonian (21), the result is

$$H = \begin{pmatrix} (\omega_A + \omega_B)\frac{\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\delta\hbar}{2} & g\hbar & 0 \\ 0 & g\hbar & -\frac{\delta\hbar}{2} & 0 \\ 0 & 0 & 0 & -(\omega_A + \omega_B)\frac{\hbar}{2} \end{pmatrix}; \quad \delta = \omega_A - \omega_B, \quad (22)$$

Here, we have used the relations $\sigma_{Z,+,-}^A = \sigma_{Z,+,-} \otimes \mathbf{I}_B$ and $\sigma_{Z,+,-}^B = \mathbf{I}_A \otimes \sigma_{Z,+,-}$. Energy eigenvalues and energy eigenvectors can be obtained by considering the characteristic polynomial of Hamiltonian H ,

$$\det(H - E\mathbf{1}) = \begin{vmatrix} (\omega_A + \omega_B)\frac{\hbar}{2} - E & 0 & 0 & 0 \\ 0 & \frac{\delta\hbar}{2} - E & g\hbar & 0 \\ 0 & g\hbar & -\frac{\delta\hbar}{2} - E & 0 \\ 0 & 0 & 0 & -(\omega_A + \omega_B)\frac{\hbar}{2} - E \end{vmatrix} = 0, \quad (23)$$

which yields four energy eigenvalues,

$$\begin{aligned} E_1 &= (\omega_A + \omega_B)\frac{\hbar}{2}, & E_2 &= \frac{\hbar}{2}\sqrt{\delta^2 + 4g^2}, \\ E_3 &= -\frac{\hbar}{2}\sqrt{\delta^2 + 4g^2}, & E_4 &= -(\omega_A + \omega_B)\frac{\hbar}{2}, \end{aligned} \quad (24)$$

and also the corresponding energy eigenvectors,

$$|E_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |E_2\rangle = \begin{pmatrix} 0 \\ \sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \quad |E_3\rangle = \begin{pmatrix} 0 \\ \cos \theta \\ -\sin \theta \\ 0 \end{pmatrix}, \quad |E_4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (25)$$

where $R = \sqrt{\delta^2 + 4g^2}$, $\cos \theta = \frac{R - \delta}{\sqrt{(R - \delta)^2 + 4g^2}}$, $\sin \theta = \frac{2g}{\sqrt{(R - \delta)^2 + 4g^2}}$.

In thermal equilibrium at absolute temperature T , density operator (state) of the system can be written as

$$\rho = \frac{1}{Z} e^{-\beta H}; \quad \beta = \frac{1}{kT}, \quad Z = \text{Tr}(e^{-\beta H}) \geq 0.$$

By using the eigenvectors (25) and eigenvalues (24), the above density operator can be written in the matrix form, called density matrix, as follows:

$$\rho = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} \sum_{i=1}^4 e^{-\beta E_i} |E_i\rangle \langle E_i|$$

$$= \frac{1}{Z} \begin{pmatrix} e^{-\beta \hbar(\omega_A + \omega_B)/2} & 0 & 0 & 0 \\ 0 & e^{-\beta \hbar R/2} \sin^2 \theta + e^{\beta \hbar R/2} \cos^2 \theta & \left(e^{-\beta \hbar R/2} - e^{\beta \hbar R/2} \right) \sin \theta \cos \theta & 0 \\ 0 & \left(e^{-\beta \hbar R/2} - e^{\beta \hbar R/2} \right) \sin \theta \cos \theta & e^{-\beta \hbar R/2} \cos^2 \theta + e^{\beta \hbar R/2} \sin^2 \theta & 0 \\ 0 & 0 & 0 & e^{\beta \hbar(\omega_A + \omega_B)/2} \end{pmatrix}.$$

In the next chapter, we shall use this density operator which represents the state of the system to investigate separability/entanglement via the PPT-criterion and negativity.

Chapter 3

Research Methodology and Results

In this chapter, the method given by Peres (Peres. 1996: 1413) and Vidals & Werner (Vidal; & Werner. 2002: 032314), i.e. the PPT-criterion and negativity, will be used to investigate the separability of the system of two coupled two-level atoms in thermal equilibrium described in section 2.7. Furthermore, the method in quantum optics used to describe the atom-field interaction (Meystre; & Sargent, III. 1998: 66-75) will also be applied to our problem in order to explain the effect of laser field on the entanglement of our system, which is initially in thermal equilibrium, when both atoms are coupled by this laser field.

This chapter consists of the following topics:

- 3.1 Separability of two coupled two-level atoms in thermal equilibrium
- 3.2 Two-level atom in a laser field: Rabi's oscillation
- 3.3 State of two coupled two-level atoms initially in thermal equilibrium under an applied laser field
- 3.4 The effect of laser field on separability of two coupled two-level atoms initially in thermal equilibrium

3.1 Separability of two coupled two-level atoms in thermal equilibrium

In this section, entanglement of the system consisting of two coupled two-level atoms that have the dipole-dipole interaction between them in thermal equilibrium (Locharoenrat; & Khemmani. 2008: 328-334.) is considered by using the definition of separable state given by Werner (Werner. 1989: 4277) and the PPT-criterion given by Peres (Peres. 1996: 1413). Werner said that any system consisting of two subsystem A and B is separable if the density operator of a system can be written or approximated in form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad (26)$$

where $\{p_i\}$ is the probability distribution

$\{\rho_i^A\}$ and $\{\rho_i^B\}$ is the density matrix of subsystem A and B respectively

To apply PPT-criterion, we first need to find the partial transposition of ρ i.e. ρ^{T_A} where the subsystem A is chosen, without loss of generality, to be transposed ($\langle in|\rho^{T_A}|jm\rangle := \langle jn|\rho|im\rangle$). Once the ρ^{T_A} is found the PPT-criterion, which is the necessary condition for separability, states that if ρ is separable, then all of the eigenvalues of ρ^{T_A} must be positive, or equivalently, if at least one of the eigenvalues of ρ^{T_A} is negative, then the state ρ is inseparable or entangled. Fortunately, since our system of interest is the $2 \otimes 2$ system where the PPT-criterion is also a sufficient condition for separability as proved by Horodeckis (M. Horodecki; et al. 1996: 1), the signs of all eigenvalues of ρ^{T_A} thus completely describe the separability of our problem. From the density matrix ρ , one can find ρ^{T_A} and all of its eigenvalues as shown below.

$$\rho^{T_A} = \frac{1}{Z} \begin{pmatrix} e^{-\frac{\beta\hbar(\omega_A + \omega_B)}{2}} & 0 & 0 & \left(e^{-\frac{\beta\hbar R}{2}} - e^{\frac{\beta\hbar R}{2}}\right) \sin\theta \cos\theta \\ 0 & e^{-\frac{\beta\hbar R}{2}} \sin^2\theta + e^{\frac{\beta\hbar R}{2}} \cos^2\theta & 0 & 0 \\ 0 & 0 & e^{-\frac{\beta\hbar R}{2}} \cos^2\theta + e^{\frac{\beta\hbar R}{2}} \sin^2\theta & 0 \\ \left(e^{-\frac{\beta\hbar R}{2}} - e^{\frac{\beta\hbar R}{2}}\right) \sin\theta \cos\theta & 0 & 0 & e^{\frac{\beta\hbar(\omega_A + \omega_B)}{2}} \end{pmatrix} \quad (27)$$

By solveing characteristic polynomial $\det(\rho^{T_A} - \lambda I) = 0$, eigenvalues of ρ^{T_A} can be found as

$$\begin{aligned} \lambda_1 &= \frac{1}{Z} \cosh\left(\frac{\beta\hbar R}{2}\right) - \frac{\delta}{ZR} \sinh\left(\frac{\beta\hbar R}{2}\right) , \\ \lambda_2 &= \frac{1}{Z} \cosh\left(\frac{\beta\hbar(\omega_A + \omega_B)}{2}\right) + \frac{1}{Z} \sqrt{\cosh^2\left(\frac{\beta\hbar(\omega_A + \omega_B)}{2}\right) + \frac{4g^2}{R^2} \sinh^2\left(\frac{\beta\hbar R}{2}\right) - 1} , \\ \lambda_3 &= \frac{1}{Z} \cosh\left(\frac{\beta\hbar(\omega_A + \omega_B)}{2}\right) - \frac{1}{Z} \sqrt{\cosh^2\left(\frac{\beta\hbar(\omega_A + \omega_B)}{2}\right) + \frac{4g^2}{R^2} \sinh^2\left(\frac{\beta\hbar R}{2}\right) - 1} , \\ \lambda_4 &= \frac{1}{Z} \cosh\left(\frac{\beta\hbar R}{2}\right) + \frac{\delta}{ZR} \sinh\left(\frac{\beta\hbar R}{2}\right) . \end{aligned}$$

Notice that λ_1 , λ_2 and λ_4 are always positive, while only the third one, i.e. λ_3 can be positive or negative. Clearly, $\lambda_3 \geq 0$ if and only if $\sinh^2\left(\frac{\hbar R}{2kT}\right) \leq \frac{\delta^2 + 4g^2}{4g^2}$. For the resonance case ($\delta = 0$), the above positive condition for λ_3 reduces to $\sinh^2(\hbar g/kT) \leq 1$ which shows according to Fig.3. how the state of the system is in separable or inseparable (entangled) state depending on the parameter $x = \hbar g/kT$.

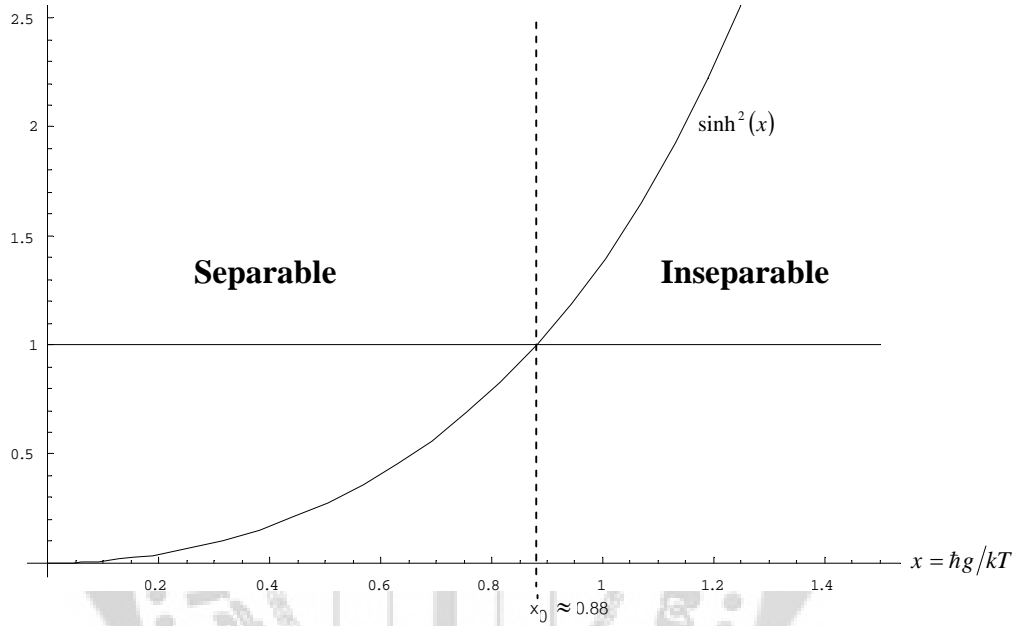


Fig. 3. : Condition for separability in resonance case.

From Fig.3. the intersection of the dash line on x -axis, which is about 0.88, indicates in the resonance case that the system is separable when $\hbar g/kT \leq 0.88$ and is inseparable when $\hbar g/kT > 0.88$. In other words, the system is separable when the temperature is high enough such that the inequality $\hbar g/kT \leq 0.88$ holds and the system is inseparable when the temperature is low enough such that this inequality is violated. Furthermore, a quantitative version of this result can be studied by calculating the negativity as it can tell how much the degree of entanglement is. From section 2.6, the negativity is defined by the sum of an absolute value of negative eigenvalues of partial transposition of ρ . In our case, since only λ_3 (shown under eq. (27)) can be negative, the negativity $N(\rho)$ is just $N(\rho) = |\lambda_3|$. Hence, in resonance case, one simply obtain the following graph.

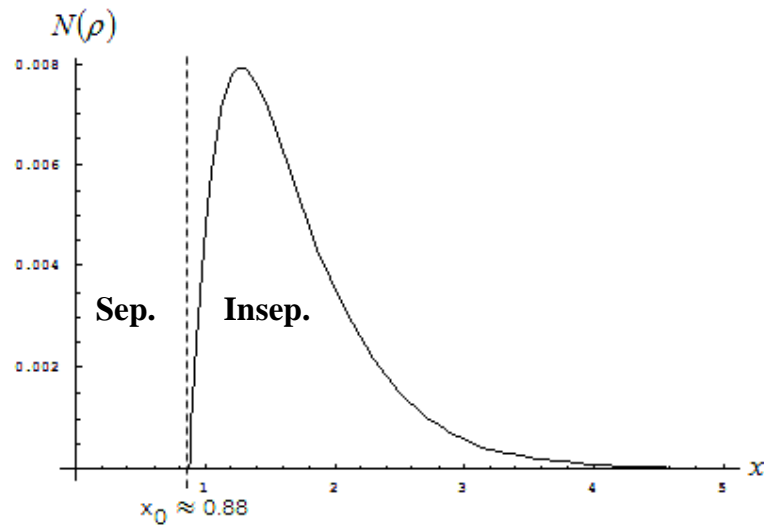


Fig . 4. : Plot of negativity $N(\rho)$ as a function of x at $\omega=2$ and $g=1$. Notice that $N(\rho)=0$ for $0 < x < 0.88$ (separable region) and then goes up for $x > 0.88$ (inseparable on entangled region) to its peak and finally reaches to zero again for large x .

This Fig.4 clearly shows $N(\rho)=0$ in the region $0 < x < 0.88$ (called “separable region”) since all eigenvalues are positive while in the region $x > 0.88$ (called “inseparable region”), $N(\rho) \neq 0$ since $\lambda_3 < 0$. In the inseparable region, $N(\rho)$ reaches its maximum value at $x \approx 1.3$ or at $\hbar g \approx kT$ and reaches zero at large x (low T). For different setting such as $\omega = 1.03$ the graph are shown below.

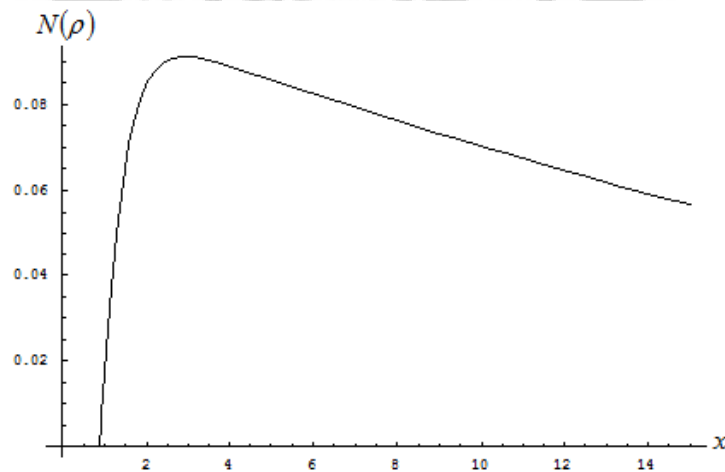


Fig . 5. : Plot of negativity $N(\rho)$ as a function of x at $\omega=1.03$ and $g=1$. Notice that this graph has the same behavior as the one in Fig.4, however $\omega = 1.03 < 2$ causes $N(\rho)$ having smaller peak and going to zero at larger x

Note that, for most realistic settings, $\omega/g \gg 1$ so we will consider only the case $\omega/g > 1$ in this thesis. In the next two sections, we shall consider the effect of laser field on two-level atom and thus on the entanglement of two coupled two-level atoms which is initially in thermal equilibrium.

3.2 Two-level atom in a laser field: Rabi's oscillation

An atom can be interacted with a laser field because an atom composed of charges while a laser field is an electromagnetic wave. In general, this interaction is truly complicated and thus be explained via the multipole moments of increasing order. However, in most practical cases, the wavelength of the field is large compared with the size of an atom so in this “long wave length approximation”, the multipole reduced to an electric dipole so that the Hamiltonian of an atom field can be simply written as $H = H_{atom} - \vec{d} \cdot \vec{E}(t)$, where \vec{d} is an electric dipole operator and $\vec{E}(t) = \hat{x}E_0 \cos(\nu t)$ is an electric field having linear polarization in \hat{x} -direction, amplitude E_0 and frequency ν . If the frequency ν of a field is tuned so that it closes to ω between some two levels of an atom, this atom can then be approximated to have only these two levels. In this case, the Hamiltonian of an atom is simply $H_{atom} = \frac{\hbar\omega}{2}\sigma_z$, where $\sigma_z = |a\rangle\langle a| - |b\rangle\langle b|$ and $\hbar\omega$ is the energy gap of these two level (see Fig.6 below). Here $|a\rangle$ and $|b\rangle$ are upper and lower states respectively. Note that we have chosen the energy zero to be half way between the upper and lower levels.

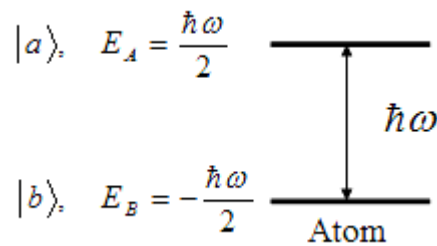


Fig. 6. : Two-level atom having energy gap $\hbar\omega$

Now, in the long wave length approximation (also called dipole approximation) and two-level approximation, Hamiltonian of the system can be written as

$H = \frac{\hbar\omega}{2}\sigma_z - \frac{d_x E_0}{2}\cos(\nu t)$. Furthermore, if the interaction is weak enough, only the term $\frac{e^{-i\nu t}}{2}$ of $\cos(\nu t)$ is enough to be kept. This is called the “rotating-wave approximation”.

Hence, in the case that all these approximations above are satisfied, Hamiltonian of the system can be written as

$$H = \frac{\hbar\omega}{2}\sigma_z - \frac{\wp E_0}{2}(\sigma_+ e^{-i\nu t} + \sigma_- e^{i\nu t}) \quad (28)$$

where $\sigma_+ = |a\rangle\langle b|$, $\sigma_- = |b\rangle\langle a|$, and $\wp = \langle a|d_x|b\rangle$. Note that, without loss of generality, \wp can be set as real and positive.

In order to see the effect of field on the population between two levels, let us consider the general state $|\psi(t)\rangle$ of the system written in the basis $\{|a\rangle, |b\rangle\}$ as

$$|\psi(t)\rangle = C_a(t)e^{\frac{-i\nu t}{2}}|a\rangle + C_b(t)e^{\frac{i\nu t}{2}}|b\rangle. \quad (29)$$

Substitute this equation into Schrödinger equation with Hamiltonian at the form (28), one obtains

$$\frac{d}{dt} \begin{bmatrix} C_a(t) \\ C_b(t) \end{bmatrix} = \frac{i}{2} \begin{bmatrix} -\delta & R_0 \\ R_0 & \delta \end{bmatrix} \begin{bmatrix} C_a(t) \\ C_b(t) \end{bmatrix}$$

where $\delta = \omega - \nu$ is called the “detuning” and $R_0 = \wp E_0 / \hbar$ is called the “Rabi flopping frequency”. In the resonance case i.e. $\delta = 0$ or $\omega = \nu$, this above equation can be solved easily as,

for the initial condition $C_b(0) = 1$ and $C_a(0) = 0$:

$$C_a(t) = i \sin(R_0 t / 2) \text{ and } C_b(t) = \cos(R_0 t / 2),$$

and for the initial condition $C_b(0) = 0$ and $C_a(0) = 1$:

$$C_a(t) = \cos(R_0 t / 2) \text{ and } C_b(t) = \sin(R_0 t / 2). \quad (30)$$

From eq. (30), it is clear that $|C_a(t)|^2$ and $|C_b(t)|^2$ oscillate between values 1 and 0 with frequency R_0 , therefore the probability of finding the atom on the upper or lower level oscillates between two levels with frequency R_0 . This oscillation is called the “Rabi’s oscillation” and that is why R_0 is called Rabi’s flopping frequency.

In the next section, a laser field will be applied on both two-level atoms which are coupled together by a dipole-dipole interaction and initially stay at thermal equilibrium.

3.3 State of two coupled two-level atoms initially in thermal equilibrium under an applied laser field

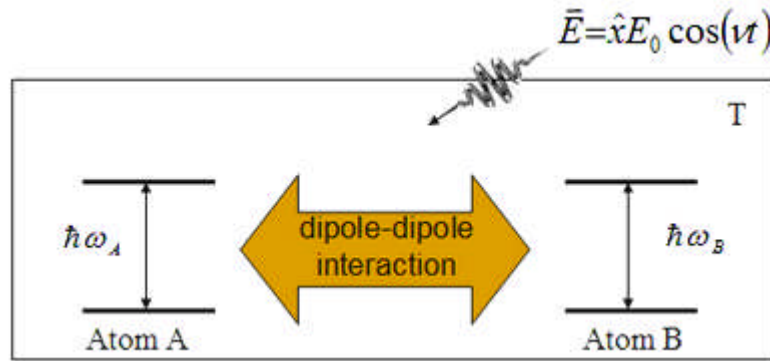


Fig. 7. : System of two coupled two-level atoms initially in thermal equilibrium under the applied field.

Our system of interest now consists of two interacting atoms, each interacts with the laser field, $\vec{E} = \hat{x}E_0 \cos(\nu t)$ (see Fig.6. above). Combine eq. (21) and (28), the Hamiltonian of this system can be written as

$$H = \frac{\hbar\omega_A}{2}\sigma_Z^A + \frac{\hbar\omega_B}{2}\sigma_Z^B + \hbar g(\sigma_-^A \otimes \sigma_+^B + \sigma_+^A \otimes \sigma_-^B) - \frac{1}{2}\wp E_0(\sigma_+^A e^{-i\nu t} + \sigma_-^A e^{i\nu t}) - \frac{1}{2}\wp E_0(\sigma_+^B e^{-i\nu t} + \sigma_-^B e^{i\nu t}) \quad , \quad (31)$$

where $-\frac{1}{2}\wp E_0(\sigma_+^A e^{-i\nu t} + \sigma_-^A e^{i\nu t})$ represents the interaction energy between atom A and field,

and

$-\frac{1}{2}\wp E_0(\sigma_+^B e^{-i\nu t} + \sigma_-^B e^{i\nu t})$ represents the interaction energy between atom B and field.

Here \wp_A and \wp_B are the matrix element of the dipole operator of atom A and B along the polarization of laser field respectively. For simplicity, we now treat these two atoms as identical i.e. we have $\wp_A = \wp_B \equiv \wp$.

Now let us consider the density operator or the state of the system at time t which can be written as

$$\rho(t) = \sum_{i=1}^4 \frac{e^{-\beta E_i}}{Z} |E_i(t)\rangle \langle E_i(t)|, \quad (32)$$

where

$$|E_i(t)\rangle = \sum_{k=1}^4 C_k^{(i)}(t) e^{\frac{-iE_k t}{\hbar}} |E_k\rangle, \quad (33)$$

with E_k and $|E_k\rangle$ defined in eq. (24) and (25).

Substitute $|E_i(t)\rangle$ from eq. (33) into the time-dependent Schrödinger equation, i.e.

$$i\hbar \frac{d}{dt} \sum_{k=1}^4 C_k^{(i)}(t) e^{\frac{-iE_k t}{\hbar}} |E_k\rangle = (H_0 + H_I) \sum_{k=1}^4 C_k^{(i)}(t) e^{\frac{-iE_k t}{\hbar}} |E_k\rangle$$

where H_0 is the first two terms and H_I is the last two term the right hand side of eq. (31) respectively, one obtains the four coupled first order ordinary differential equations of the interaction-picture coefficient $C_k^{(i)}(t)$; $i, k = 1, 2, 3, 4$ as

$$\frac{d}{dt} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix} = \frac{i}{2} R_0 \begin{pmatrix} 0 & (\sin \theta + \cos \theta) e^{-i(\nu + \omega_{21})t} & (\cos \theta - \sin \theta) e^{-i(\nu + \omega_{31})t} & 0 \\ (\sin \theta + \cos \theta) e^{i(\nu - \omega_{12})t} & 0 & 0 & (\sin \theta + \cos \theta) e^{-i(\nu + \omega_{42})t} \\ (\cos \theta - \sin \theta) e^{i(\nu - \omega_{13})t} & 0 & 0 & (\cos \theta - \sin \theta) e^{-i(\nu + \omega_{43})t} \\ 0 & (\sin \theta + \cos \theta) e^{i(\nu - \omega_{24})t} & (\cos \theta - \sin \theta) e^{i(\nu - \omega_{34})t} & 0 \end{pmatrix} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix} \quad (34)$$

with the initial condition $C_k^{(i)}(0) = \delta_{ik}$, where $\omega_{kl} = \frac{E_k - E_l}{\hbar}$, and $R_0 = \frac{\wp E_0}{\hbar} > 0$.

Before going to solve this equation, let us first examine the trivial case where $g = 0$ (no coupling) and $\delta = \Delta = 0$ (resonance), here $\omega_A = \omega_B = \omega$, $\delta = \omega_A - \omega_B$, $\Delta = \nu - \omega$. For

$g=0$, one can verify at the beginning that $|E_1\rangle, |E_2\rangle, |E_3\rangle, |E_4\rangle$ in eq. (25) where $\sin\theta=1$ and $\cos\theta=0$ is the eigenvector of Hamiltonian (22) with the corresponding eigenvalues (24), when $g=0$. Hence, by assigning $\sin\theta$ by 1 and $\cos\theta$ by 0, eq. (34) reduces to

$$\frac{d}{dt} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix} = \frac{i}{2} R_0 \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix}, \quad (35)$$

with the initial condition $C_k^{(i)}(0) = \delta_{ik}$. This differential equation can be solved by diagonalization of the 4x4 matrix in the right hand side of eq. (35), please see the solution in the appendix A. Once all $C_k^{(i)}(t)$ are known, $|E_i(t)\rangle$ in eq. (33) is also known and thus $\rho(t)$ in eq. (32) is then known. Now, let us write down the density operator of subsystem A and B in a single formula as

$$\rho_{A,B}(t) = \sum_{i=1}^2 e^{-\beta E_i^{(A,B)}} |i^{(A,B)}(t)\rangle \langle i^{(A,B)}(t)|,$$

where $|1^{(A,B)}(t)\rangle = \sum_{k=1}^2 C_k^{(\bar{A},\bar{B})}(t) e^{-iE_k^{(A,B)}t/\hbar} |k\rangle$, $|2^{(A,B)}(t)\rangle = \sum_{k=1}^2 C_k^{(A,B)}(t) e^{-iE_k^{(A,B)}t/\hbar} |k\rangle$.

Here, $|1\rangle$ and $|2\rangle$ are $|a\rangle$ and $|b\rangle$ defined in section 3.2 respectively and $E_1^{(A,B)} = \frac{\hbar\omega}{2}$, $E_2^{(A,B)} = -\frac{\hbar\omega}{2}$. $C_k^{(\bar{A},\bar{B})}(t)$ here means the probability amplitude of finding atom A, B in the state $|k\rangle$ ($k=1, 2$) at time t if A, B is initially in the upper state $|1\rangle$. Similarly, $C_k^{(A,B)}(t)$ means the probability amplitude of finding atom A, B in the state $|k\rangle$ at time t if A, B is initially in the lower state $|2\rangle$. Note that since $C_k^{(\bar{A},\bar{B})}(t)$ and $C_k^{(A,B)}(t)$ are all known from eq. (30), one can find the relation between these amplitudes and the amplitude $C_k^{(i)}(t)$ obtained in the appendix A. This relation (see appendix B) thus allow are to verify that $\rho(t) = \rho_A(t) \otimes \rho_B(t)$, as one would expect since, in the case $g=0$, two atoms are independent.

Now, let us consider the very interesting case i.e. $g \neq 0$ with resonance condition $\delta = \Delta = 0$. In this condition, one can see that $\sin \theta = \cos \theta = 1/\sqrt{2}$ (see the definition of $\cos \theta$ and $\sin \theta$ under eq. (25)) and therefore eq. (34) becomes

$$\frac{d}{dt} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix} = \frac{iR_0}{\sqrt{2}} \begin{pmatrix} 0 & e^{-igt} & 0 & 0 \\ e^{igt} & 0 & 0 & e^{igt} \\ 0 & 0 & 0 & 0 \\ 0 & e^{-igt} & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1^{(i)}(t) \\ C_2^{(i)}(t) \\ C_3^{(i)}(t) \\ C_4^{(i)}(t) \end{pmatrix}. \quad (36)$$

Notice that the 4×4 matrix in the right hand side of eq. (36) still depends on time t so in order to apply the diagonalization method, we first make a transformation $C_1^{(i)}(t) = C_a^{(i)}(t)e^{-igt}$, $C_2^{(i)}(t) = C_b^{(i)}(t)$, $C_3^{(i)}(t) = C_c^{(i)}(t)$, $C_4^{(i)}(t) = C_d^{(i)}(t)e^{-igt}$ so that eq. (36) reduces to

$$\frac{d}{dt} \begin{pmatrix} C_a^{(i)}(t) \\ C_b^{(i)}(t) \\ C_c^{(i)}(t) \\ C_d^{(i)}(t) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 2g & \sqrt{2}R_0 & 0 & 0 \\ \sqrt{2}R_0 & 0 & 0 & \sqrt{2}R_0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}R_0 & 0 & 2g \end{pmatrix} \begin{pmatrix} C_a^{(i)}(t) \\ C_b^{(i)}(t) \\ C_c^{(i)}(t) \\ C_d^{(i)}(t) \end{pmatrix}, \quad (37)$$

when the 4×4 matrix is now independent of time t .

Similar to eq. (35), eq. (37) can be solved by diagonalization method to give us the solution (after transforming back to $C_{1,2,3,4}^{(i)}(t)$),

for $i = 1$:

$$C_1^{(1)}(t) = \frac{1}{2} + \frac{1}{2} \left[\cos \frac{Qt}{2} + \frac{ig}{Q} \sin \frac{Qt}{2} \right] e^{-igt/2}, \quad C_2^{(1)}(t) = \frac{i\sqrt{2}R_0}{Q} \sin \frac{Qt}{2} e^{igt/2},$$

$$C_3^{(1)}(t) = 0, \quad C_4^{(1)}(t) = -\frac{1}{2} + \frac{1}{2} \left[\cos \frac{Qt}{2} + \frac{ig}{Q} \sin \frac{Qt}{2} \right] e^{-igt/2},$$

for $i = 2$:

$$C_1^{(2)}(t) = \frac{i\sqrt{2}R_0}{Q} \sin \frac{Qt}{2} e^{-igt/2}, \quad C_2^{(2)}(t) = \left[\cos \frac{Qt}{2} - \frac{ig}{Q} \sin \frac{Qt}{2} \right] e^{igt/2},$$

$$C_3^{(2)}(t) = 0, \quad C_4^{(2)}(t) = \frac{i\sqrt{2}R_0}{Q} \sin \frac{Qt}{2} e^{-igt/2},$$

for $i = 3$:

$$C_1^{(3)}(t) = 0, \quad C_2^{(3)}(t) = 0, \quad C_3^{(3)}(t) = 1, \quad C_4^{(3)}(t) = 0,$$

for $i = 4$:

$$C_1^{(4)}(t) = -\frac{1}{2} + \frac{1}{2} \left[\cos \frac{Qt}{2} + \frac{ig}{Q} \sin \frac{Qt}{2} \right] e^{-igt/2}, \quad C_2^{(4)}(t) = \frac{i\sqrt{2}R_0}{Q} \sin \frac{Qt}{2} e^{igt/2},$$

$$C_3^{(4)}(t) = 0, \quad C_4^{(4)}(t) = \frac{1}{2} + \frac{1}{2} \left[\cos \frac{Qt}{2} + \frac{ig}{Q} \sin \frac{Qt}{2} \right] e^{-igt/2}.$$

Substitute these $C_k^{(i)}(t)$ into eq. (33) to have $|E_i(t)\rangle$ and then substitute $|E_i(t)\rangle$ into eq. (32), one finally obtains

$$\rho(t) = \frac{1}{Z} \begin{pmatrix} [1] & [2] & [2] & [3] \\ [2]^* & [4] & [5] & [6] \\ [2]^* & [5] & [4] & [6] \\ [3]^* & [6]^* & [6]^* & [7] \end{pmatrix}, \quad (38)$$

Where $[1] = \cosh(\beta\hbar\omega) - \left[\cos\left(\frac{Qt}{2}\right) \cos\left(\frac{gt}{2}\right) + \frac{g}{Q} \sin\left(\frac{Qt}{2}\right) \sin\left(\frac{gt}{2}\right) \right] \sinh(\beta\hbar\omega)$
 $+ \frac{2R_0^2}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \left[e^{-\beta\hbar g} - \cosh(\beta\hbar\omega) \right],$

$$[2] = \frac{iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \sinh(\beta\hbar\omega) e^{-\frac{igt}{2}} - \frac{iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \cos\left(\frac{Qt}{2}\right) \cosh(\beta\hbar\omega) e^{-igt}$$

$$+ \left[\frac{iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \cos\left(\frac{Qt}{2}\right) e^{-igt} + \frac{gR_0}{Q^2} \sin^2\left(\frac{Qt}{2}\right) e^{-igt} \right] e^{-\beta\hbar g} + \frac{gR_0}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \cosh(\beta\hbar\omega) e^{-igt},$$

$$[3] = \frac{2R_0^2}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \left[e^{-\beta\hbar g} - \cosh(\beta\hbar\omega) \right] - i \left[\cos\left(\frac{Qt}{2}\right) \sin\left(\frac{gt}{2}\right) - \frac{g}{Q} \sin\left(\frac{Qt}{2}\right) \cos\left(\frac{gt}{2}\right) \right] \sinh(\beta\hbar\omega),$$

$$[4] = \frac{2R_0^2}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \left[\cosh(\beta\hbar\omega) - e^{-\beta\hbar g} \right] + \cosh(\beta\hbar g),$$

$$[5] = \frac{2R_0^2}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \left[\cosh(\beta\hbar\omega) - e^{-\beta\hbar g} \right] - \sinh(\beta\hbar g),$$

$$\begin{aligned}
[6] &= \frac{iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \sinh(\beta\hbar\omega) e^{\frac{igt}{2}} - \frac{iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \cos\left(\frac{Qt}{2}\right) \cosh(\beta\hbar\omega) e^{igt} \\
&+ \left[\frac{-iR_0}{Q} \sin\left(\frac{Qt}{2}\right) \cos\left(\frac{Qt}{2}\right) e^{igt} + \frac{gR_0}{Q^2} \sin^2\left(\frac{Qt}{2}\right) e^{igt} \right] e^{-\beta\hbar g} + \frac{gR_0}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \cosh(\beta\hbar\omega) e^{igt}, \\
[7] &= \cosh(\beta\hbar\omega) + \left[\cos\left(\frac{Qt}{2}\right) \cos\left(\frac{gt}{2}\right) + \frac{g}{Q} \sin\left(\frac{Qt}{2}\right) \sin\left(\frac{gt}{2}\right) \right] \sinh(\beta\hbar\omega) \\
&+ \frac{2R_0^2}{Q^2} \sin^2\left(\frac{Qt}{2}\right) \left[e^{-\beta\hbar g} - \cosh(\beta\hbar\omega) \right].
\end{aligned} \tag{39}$$

Note that $\rho(t)$ in eq. (38) is Hermitian, as it should be, and $\rho(0)$ reduces to ρ obtained in section 2.7 (below eq. (25)), as one might expect because the system is initially in the thermal state.

3.4 The effect of laser field on separability of two coupled two-level atoms initially in thermal equilibrium

By following the same method in section 3.1, we shall first find the partial transpose $\rho^{T_A}(t)$ from eq. (38) and then find its eigenvalues. From eq. (38), ρ^{T_A} reads

$$\rho^{T_A}(t) = \frac{1}{Z} \begin{pmatrix} [1] & [2] & [2]^* & [5] \\ [2]^* & [4] & [3]^* & [6]^* \\ [2] & [3] & [4] & [6] \\ [5] & [6] & [6]^* & [7] \end{pmatrix}. \tag{40}$$

To obtain its eigenvalues, the characteristic equation $|\rho^{T_A}(t) - \lambda I| = 0$ must be solved. Unfortunately, this characteristic equation is very difficult to solve analytically due to the complication of terms [1] – [7]. To proceed, the entanglement will now be studied directly by computing the negativity using simple numerical methods provided by Mathematica program. Note that, for computing purposes, it is more convenience to rewrite the negativity as (Zhu; & Wang. 2008: 343-346) $N(\rho) = \sum_{i=1}^4 \max(0, -\lambda_i)$, where λ_i is an eigenvalues of $\rho^{T_A}(t)$, which is clearly equal to the original definition i.e. the sum of absolute value of the negative eigenvalues.

To study the numerical results of the negativity depending on temperature T and time t , parameters i.e. ω , g , \hbar , R_0 must be set. Notice that ω , g , \hbar always appear together with β in [1] – [7] (see eq. (39)) in the form $\beta\hbar g$ and $\beta\hbar\omega = \beta\hbar g(\omega/g)$. If we let

the dimensionless parameter $x = \beta \hbar g$, then we have $\beta \hbar \omega = (\omega/g)x$. In this notation, it is clear that the behavior of the x dependence on $N(\rho)$ is unchanged if the fraction ω/g is fixed. So, for proposes of programming and illustrating, we set the parameters as $\omega = 1.03$, $g = 1$, $\hbar = 1$ and $R_0 = 1$. As mentioned above, the behavior of the x dependence on $N(\rho)$ is unchanged even if \hbar is not set to be unity or ω and g are set to be other values with fixed ω/g i.e. equal to 1.03 for this setting. Note that variation of fraction ω/g does not change the whole character of the x dependence on $N(\rho)$. Although these setting do not affect much on the x dependence on $N(\rho)$, it affects the t dependence on $N(\rho)$. Since g and $Q = \sqrt{g^2 + 4R_0^2}$ appear in [1] – [7] as frequencies of oscillations in time, values of g and R_0 do affect the oscillating behaviors. Clearly, the setting $g = R_0 = 1$ is not a realistic one but it is still be used here since it causes most resulting graphs easy to be viewed. Surely, the realistic setting causes the change in frequencies but it should not change the whole character of the t dependence on $N(\rho)$.

Now, with these settings, the numerical results representing by many graphs are shown and explained (in Fig. caption) as follows.

Over view & x -dependent section:

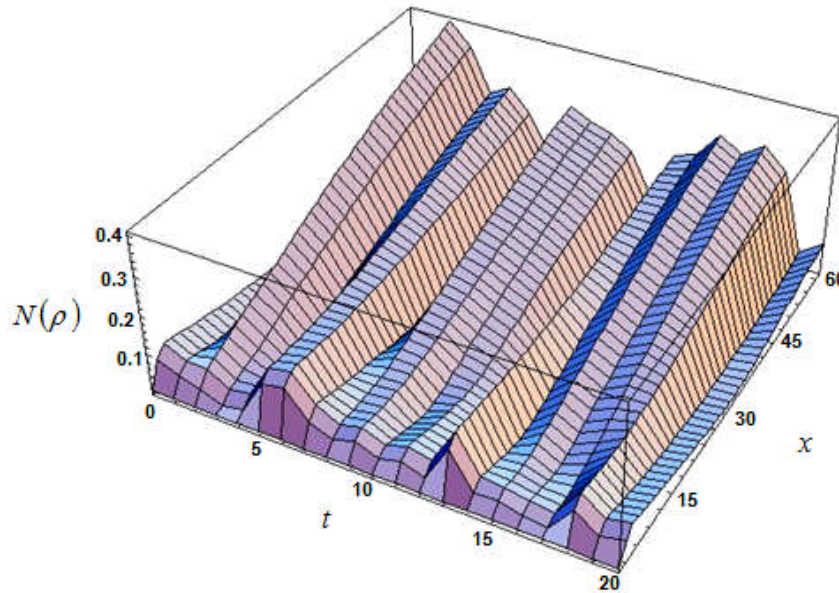


Fig. 8. : The plot of negativity $N(\rho)$ as a function of x and t . At each x , $N(\rho)$ oscillates in time and vanishes at some t (see also in Fig.9 and Fig.10.)

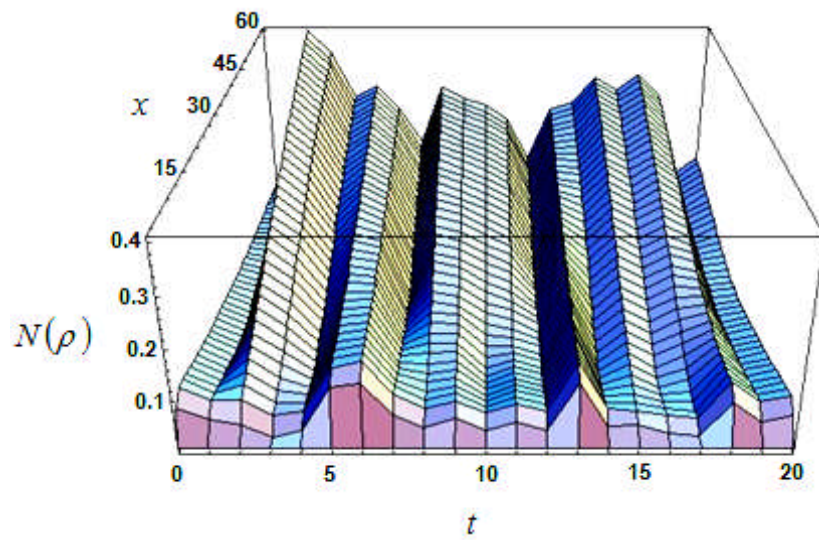


Fig. 9. : Different view point of Fig.8

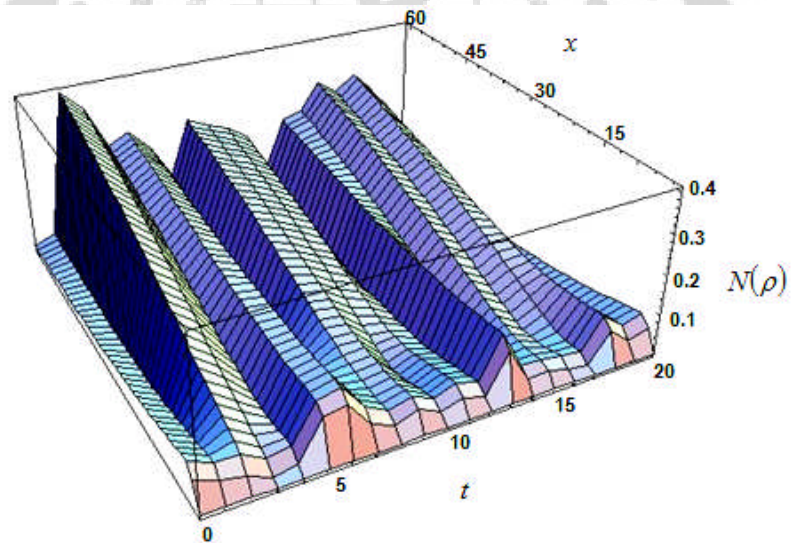


Fig. 10. : Different view point of Fig.8. Notice that $N(\rho) \rightarrow 0$ for large x at $t = 0$ but $N(\rho) \rightarrow \text{constant} \neq 0$ for large x at all $t > 0$.

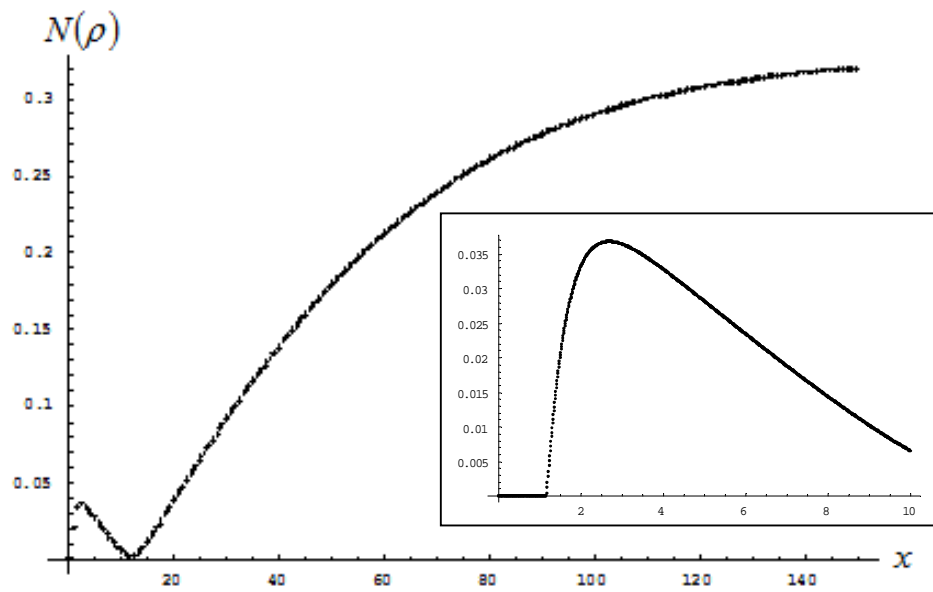


Fig. 11. : The section of Fig.8 at $t = 10$. Here, the plot is extended to $x = 150$ in order to see that $N(\rho) \rightarrow \text{constant} \neq 0$ for large x . Notice that in the separable region, i.e. $0 < x < 0.88$, $N(\rho) = 0$ (see inset).

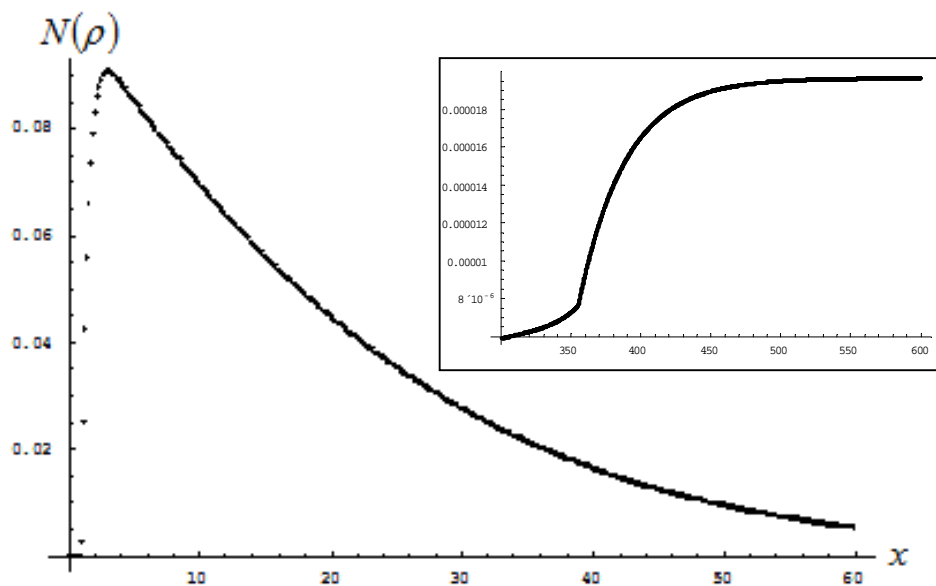


Fig. 12. : The section at Fig.8 at $t = 0.1$. Although $t = 0.1$ is still very small and $N(\rho)$ seems to reach zero for large x , it instead reaches the small constant $\neq 0$ (see inset).

Small x region & t -dependent section:

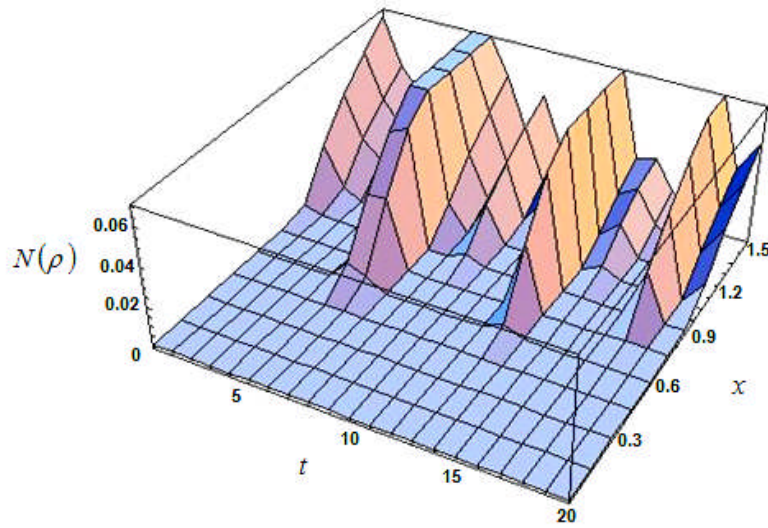


Fig. 13. : The plot of negativity $N(\rho)$ as a function of x and t emphasizing on the separable region i.e. around $x \approx 0.88$. As time goes, one can see the change of states from separable at $t = 0$ to inseparable ($N(\rho) \neq 0$) at certain times and vis ver sa. However, for small (enough) x , $N(\rho) = 0$ for all time t .

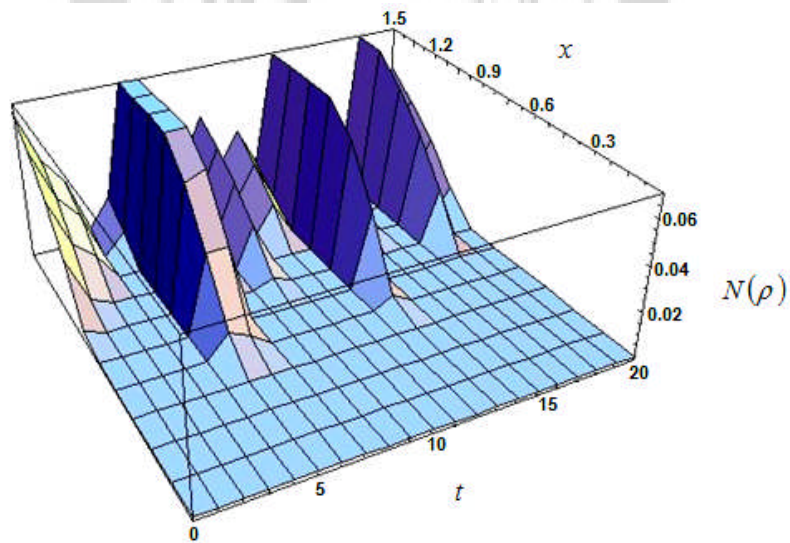


Fig. 14. : Different view point of Fig.13.

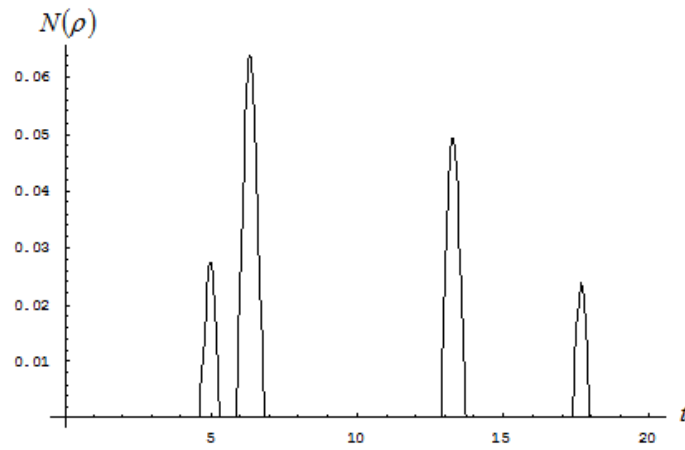


Fig. 15. : The section of Fig.13 at $x = 0.7$. Since $x = 0.7 < 0.88$ is in the separable region, $N(\rho) = 0$ at $t = 0$. As time goes, one can see tiny changes of $N(\rho)$ which shows that, at certain times, the entangled states exist but have small degree of entanglement (compared to what happen in Fig.18 at $x = 31$)

Large x region & t -dependent section:

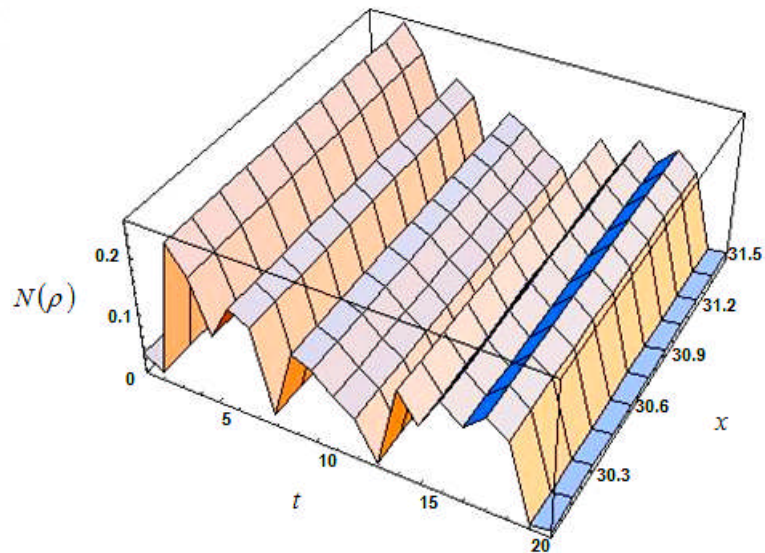


Fig. 16. : The plot of negativity $N(\rho)$ as a function of x and t emphasizing on the inseparable region i.e. $x > 0.88$. As time goes, one can see that the system which is initially entangled becomes separable (disentagled) at certain times and vis ver sa.

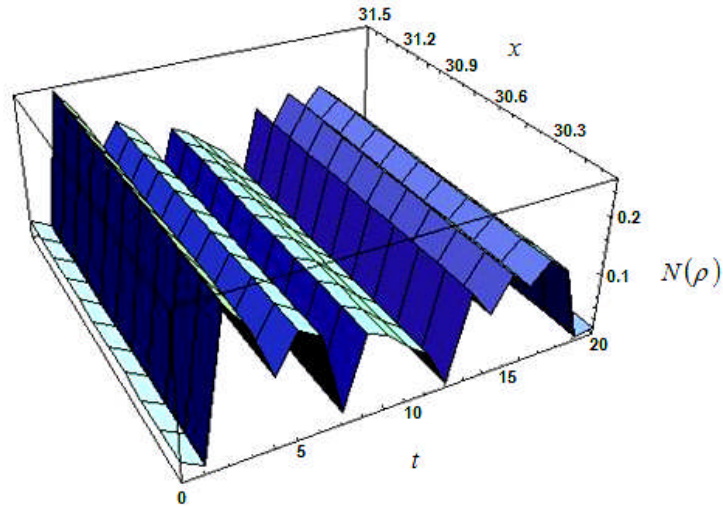


Fig. 17. : Different view point of Fig.16.

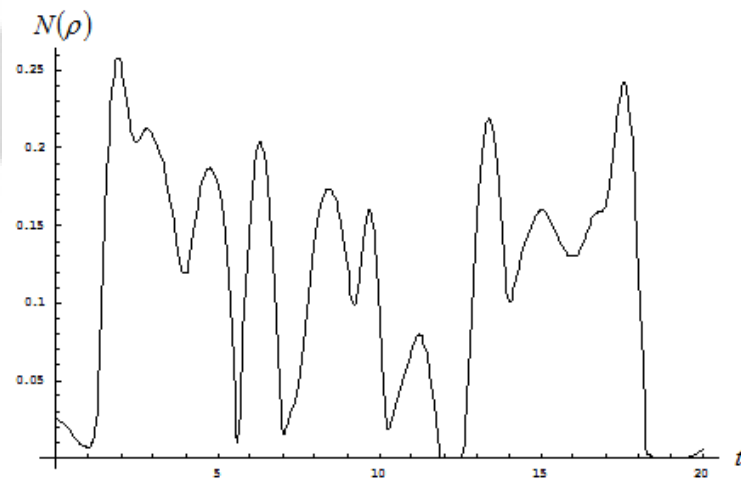


Fig. 18. : The section of Fig.16 at $x = 31$. Since $x = 31 > 0.88$ is in the inseparable region, $N(\rho)$ at $t = 0$ is not equal to zero. However, as time goes, $N(\rho)$ oscillates between high degree of entanglement and lower, some are zero.

From many aspects of $N(\rho)$ as function of x and t above, it is worth to note that for high temperature T (small x), but not too high, the initial vanishing of negativity becomes non-vanishing at certain times so, even it has a small value, one may take advantages of the entanglement at these times for some applications. Moreover, unlike the x -dependent on $N(\rho)$ at $t = 0$, $N(\rho)$ at any time larger than zero does not goes to zero for large x but goes to some constant values depending on each time t .



Chapter 4

Conclusion and Discussion

In this thesis, we have studied some aspects of the system of two coupled two-level atoms, where the coupling is a dipole-dipole type of interaction. In particular, we have obtained the following results:

1. In contact with a heat reservoir at temperature T , we derived the thermal state and then used it to investigate the entanglement between two atoms. By apply the PPT-criterion, which is necessary and sufficient for our $2 \otimes 2$ system, the thermal state was shown to be separable for all temperatures high enough such that the certain inequality holds. In the opposite way, if temperatures are low enough so that this inequality is violated, our system is then inseparable or entangled. The quantitative study of this behavior was also studied by computing the negativity, which is the quantity describing a degree of entanglement. Since in the separable region (region of high T) there is no entanglement, negativity is then zero while in the inseparable region (region of low T), the negativity is not zero. In this inseparable region, the negativity goes to its peak (highest degree of entanglement) when the coupling energy is of the same order as thermal energy and then reaches zero again as T goes to zero.

2. When our system, which is initially in thermal equilibrium, is applied by a laser field, the Rabi's oscillation appearing in each atom causes some interesting situations as follows:

a. In the separable region, for fixed T (not too high), the initial vanishing negativity some times becomes non-vanishing, however, with small degree of entanglement. For too high T , the negativity is always zero.

b. In the inseparable region, for fixed T , the initial non-vanishing negativity oscillators irregularly in time. Some times it goes to high degree of entanglement and some times to zero.

c. For fix time t , if t is very very close to zero, the T – dependent on negativity is almost the same as one at $t = 0$ (initial thermal state) except at very low T where negativity reaches some constants which are not equal to zero. For other times t , T – dependent on negativity is quite different (except at very high T where the negativity is

clearly zero) especially at low T i.e. the negativity goes to some constants (depends on each t) which is “not” zero, as T goes to zero.

It is worth to note that although the system in this thesis is a two coupled two-level atoms, our study and thus all of the results can be used to explain a behavior of any system of two coupled qubits if the mathematical form of their interaction is identical to us. Moreover, since the degree of entanglement varies with time for each temperature T (not too high), it is an interesting question weather one can take advantages of this behavior for a realistic application.





BIBLIOGRAPHY

- Cohen-Tannoudji, C.; Did, B.; & Laloe, F. (1977). *Quantum Mechanics* (V.1). John Wiley & Sons: p. 154-157.
- Einstein, A.; Podolsky, B.; & Rosen, N. (1935). *Can Quantum-Mechanical Description of Physical Reality Considered Complete?*. Phys. Rev. 47, 777.
- Horodecki, M.; Horodecki, P.; & Horodecki, R. (1996). *Separability of mixed states : necessary and sufficient condition*. Phys. Rev. Lett. A 223, 1.
- Locharoenrat, S.; & Khemmani, S. (2008). *Entanglement of Two Coupled Two Level Atoms*, 2nd Academic SWU 1, 328-334.
- Meystre, P.; & Sargent, M., III. (1998). *Elements of Quantum Optics*. Springer-Verlag: p. 66-75.
- Peres, A. (1996). *Separability Criterion for Density Matrices*. Phys. Rev. Lett. 77, 1413.
- Sakurai, J.J. (1994). *Modern Quantum Mechanics* (Revised Edition). Addison-Wesley: p. 176-179.
- Vidal, G.; & Werner, R.F. (2002). *Computable measure of entanglement*. Phys. Rev. A 65, 032314.
- Werner, R.F. (1989). *Quantum states with Einstein-Podolsky-Rosen correlation admitting a hidden-variable model*. Phys. Rev. A 40, 4277.
- Zhu, G.Q.; & Wang, X.G. (2008). *Quantum Dynamics of Bound Entangled States*. Commun. Theor. Phys. 49, 343-346.





APPENDIX A

APPENDIX A

Coefficients in the case $g = 0$

For $i = 1$,

$$C_1^{(1)}(t) = \frac{1 + \cos R_0 t}{2}$$

$$C_2^{(1)}(t) = \frac{i \sin R_0 t}{2}$$

$$C_3^{(1)}(t) = -\frac{i \sin R_0 t}{2}$$

$$C_4^{(1)}(t) = -\left(\frac{1 - \cos R_0 t}{2}\right)$$

For $i = 2$,

$$C_1^{(2)}(t) = \frac{i \sin R_0 t}{2}$$

$$C_2^{(2)}(t) = \frac{1 + \cos R_0 t}{2}$$

$$C_3^{(2)}(t) = \frac{1 - \cos R_0 t}{2}$$

$$C_4^{(2)}(t) = \frac{i \sin R_0 t}{2}$$

For $i = 3$,

$$C_1^{(3)}(t) = -\frac{i \sin R_0 t}{2}$$

$$C_2^{(3)}(t) = \frac{1 - \cos R_0 t}{2}$$

$$C_3^{(3)}(t) = \frac{1 + \cos R_0 t}{2}$$

$$C_4^{(3)}(t) = -\frac{i \sin R_0 t}{2}$$

For $i = 4$,

$$C_1^{(4)}(t) = -\left(\frac{1 - \cos R_0 t}{2}\right)$$

$$C_2^{(4)}(t) = \frac{i \sin R_0 t}{2}$$

$$C_3^{(4)}(t) = -\frac{i \sin R_0 t}{2}$$

$$C_4^{(4)}(t) = \frac{1 + \cos R_0 t}{2}$$



APPENDIX B

APPENDIX B

Relation between coefficients

From eq. (30), we have

$$C_1^{(\bar{A},\bar{B})}(t) = \cos \frac{1}{2} R_0 t$$

$$C_1^{(\underline{A},\underline{B})}(t) = i \sin \frac{1}{2} R_0 t$$

$$C_2^{(\bar{A},\bar{B})}(t) = i \sin \frac{1}{2} R_0 t$$

$$C_2^{(\underline{A},\underline{B})}(t) = \cos \frac{1}{2} R_0 t$$

Compare these coefficient to those in appendix A, one obtains

$$C_1^{(1)}(t) = C_1^{(\bar{A})}(t) C_1^{(\bar{B})}(t)$$

$$C_2^{(1)}(t) = C_1^{(\bar{A})}(t) C_2^{(\bar{B})}(t)$$

$$C_3^{(1)}(t) = -C_2^{(\bar{A})}(t) C_1^{(\bar{B})}(t)$$

$$C_4^{(1)}(t) = C_2^{(\bar{A})}(t) C_2^{(\bar{B})}(t)$$

$$C_1^{(2)}(t) = C_1^{(\bar{A})}(t) C_1^{(\underline{B})}(t)$$

$$C_2^{(2)}(t) = C_1^{(\bar{A})}(t) C_2^{(\underline{B})}(t)$$

$$C_3^{(2)}(t) = -C_2^{(\bar{A})}(t) C_1^{(\underline{B})}(t)$$

$$C_4^{(2)}(t) = C_2^{(\bar{A})}(t) C_2^{(\underline{B})}(t)$$

$$C_1^{(3)}(t) = -C_1^{(\underline{A})}(t) C_1^{(\bar{B})}(t)$$

$$C_2^{(3)}(t) = -C_1^{(\underline{A})}(t) C_2^{(\bar{B})}(t)$$

$$C_3^{(3)}(t) = C_2^{(\underline{A})}(t) C_1^{(\bar{B})}(t)$$

$$C_4^{(3)}(t) = -C_2^{(\underline{A})}(t) C_2^{(\bar{B})}(t)$$

$$C_1^{(4)}(t) = C_1^{(\underline{A})}(t) C_1^{(\underline{B})}(t)$$

$$C_2^{(4)}(t) = C_1^{(\underline{A})}(t) C_2^{(\underline{B})}(t)$$

$$C_3^{(4)}(t) = -C_2^{(\underline{A})}(t) C_1^{(\underline{B})}(t)$$

$$C_4^{(4)}(t) = C_2^{(\underline{A})}(t) C_2^{(\underline{B})}(t)$$



VITAE

Name: Mr. Suttipong Locharoenrat
Date of Birth: March 8, 1984
Place of Birth: Surin, Thailand
Address: Deluxe mansion (Room 405) 999 Soi Shuttiporn, Prachasongkrock Rd. Dindeang Bangkok 10400

Current Position: Instructor
Work Place: Saint Dominic school, Ratchathewi, Bangkok

Educational Background:

2002 High school, Sirindhorn School, Thailand
 2006 B.Sc. (Physics), Srinakharinwirot University, Thailand
 2011 M.Sc. (Physics), Srinakharinwirot University, Thailand

Conference Presentations:

2008 Oral presentation for "Entanglement of two coupled two-level atoms" at 2nd Academic SWU, held at Srinakharinwirot University (January 31 – February 1), Bangkok, Thailand.

Publication:

2008 Suttipong Locharoenrat and Supitch Khemmani, *Entanglement of two coupled two-level atoms*, 2nd Academic SWU **1** (2008) 328-334.